ON TWO LEMMAS OF BROWN AND SHEPP HAVING APPLICATION TO SUM SETS AND FRACTALS

C. E. M. PEARCE¹ and J. E. PEČARIĆ²

(Received 3 January 1993; revised 20 September 1993)

Abstract

An improvement is made to two results of Brown and Shepp which are useful in calculations with fractal sets.

1. Introduction

Recently there has been a resurgence the study of sum sets. They have, inter alia, application to fractals, which can often be attractors or Markov attractors of iterated function systems (see the seminal paper of Barnsley and Demko [1]). Measure properties of sum sets are important in the study of dynamical systems (see, for example, Newhouse [5] and Palis and Takens [6]). The calculation of associated Hausdorff dimensions and Hausdorff measures and other properties can be delicate. In [3], G. Brown and L. Shepp provided two key lemmas which have proved valuable in making available a number of simple calculations in this area.

We say two positive numbers \( s \) and \( t \) are conjugate if \( s^{-1} + t^{-1} = 1 \). By \( \|f\|_p \) we denote the \( L^p \) norm of a real-valued function \( f \). Assuming the relevant quantities exist, the results of Brown and Shepp alluded to are as follows.

(i) Suppose \( s_1 < s_0 < s_2 \) and let \( s_i \) be conjugate to \( t_i \) (\( i = 0, 1, 2 \)). Then

\[
\|f\|_{s_0} \|g\|_{s_0} \leq \max \{\|f\|_{s_1}, \|g\|_{t_1}, \|f\|_{s_2}, \|g\|_{t_2}\}.
\]

(ii) Suppose that, for \( i = 0, 1, 2 \), we have \( s_i, t_i \geq 1 \) and \( as_i^{-1} + bt_i^{-1} = 1 \) for positive constants \( a, b \). If \( s_1 \leq s_0 \leq s_2 \), then

\[
M_{s_0}(x, u)M_{t_0}(y, v) \leq \max_{i=1,2} M_{s_i}(x, u)M_{t_i}(y, v)
\]
and, if further \( a : b = \log n : \log m \), then

\[
S_{s_0}(x)S_{s_0}(y) \leq \max_{i=1,2} S_{s_i}(x)S_{s_i}(y),
\]

(2)

where the mean of order \( t \) is

\[
M_t(x, u) = \left[ \sum_{i=1}^{n} u_i x_i^t \right]^{1/t}
\]

and the sum of order \( t \) is

\[
S_t(x) = \left[ \sum_{i=1}^{n} x_i^t \right]^{1/t}.
\]

It is implicit in these statements that \( x = (x_i)_1^n, u = (u_i)_1^n, y = (y_i)_1^n, v = (v_i)_1^n \) have positive entries, that \( \sum u_i = 1 = \sum v_i \) and that \( m \) may differ from \( n \). Various applications of (ii) are given by Brown [2] and Brown and Shepp [3].

2. Results

We now proceed to some useful extensions of these results.

THEOREM 1. Suppose \( s_1 \leq s_0 \leq s_2 \) with \( s_i \geq a, t_i \geq b \) and \( a s_i^{-1} + b t_i^{-1} = 1 \) \((i = 0, 1, 2), a, b > 0 \). Then assuming the relevant quantities exist,

\[
\|f\|_{s_0} \|g\|_{s_0} \leq \max \{\|f\|_{t_1} \|g\|_{t_1}, \|f\|_{s_2} \|g\|_{s_2}\}.
\]

PROOF. As in [3] choose \( \alpha_1, \alpha_2 \) positive and such that \( \alpha_1 + \alpha_2 = 1 \) and \( s_0 = \alpha_1 s_1 + \alpha_2 s_2 \).

By the Hölder inequality

\[
\|f\|_{s_0}^{s_0} = \|f\|_{\alpha_1 s_1 + \alpha_2 s_2}^{\alpha_1 s_1 + \alpha_2 s_2} \leq \|f\|_{s_1}^{\alpha_1 s_1} \|f\|_{s_2}^{\alpha_2 s_2}.
\]

(3)

If we choose

\[
\beta_i = \alpha_i \frac{s_i}{s_0} \frac{t_0}{t_i} \quad (i = 1, 2)
\]

then

\[
\beta_1 + \beta_2 = \left(\alpha_1 \frac{s_1}{t_1} + \alpha_2 \frac{s_2}{t_2}\right) \frac{t_0}{s_0}
\]

\[
= \frac{b}{s_0 - a} \left(\alpha_1 \frac{s_1 - a}{b} + \alpha_2 \frac{s_2 - a}{b}\right)
\]

\[
= \frac{\alpha_1 s_1 + \alpha_2 s_2 - a(\alpha_1 + \alpha_2)}{b} \frac{1}{s_0 - a}
\]

(4)
and

\[ \beta_1 t_1 + \beta_2 t_2 = \frac{\alpha_1 s_1 + \alpha_2 s_2}{s_0} t_0 = t_0, \]

so again by Hölder’s inequality

\[ \|g\|_t^0 = \|g\|_{\beta_1 t_1 + \beta_2 t_2} \leq \|g\|_{t_1} \|g\|_{t_2}. \]

Since

\[ \beta_i t_i = \frac{\alpha_i s_i}{s_0} \quad (i = 1, 2), \]

(3) and (6) may be combined to provide

\[ \|f\|_{s_0} \|g\|_t^0 \leq \left[ \|f\|_{s_1} \|g\|_{n} \right]^{\alpha_1 s_1/s_0} \left[ \|f\|_{s_2} \|g\|_{t_2} \right]^{\alpha_2 s_2/s_0}. \]

As

\[ \frac{\alpha_1 s_1}{s_0} + \frac{\alpha_2 s_2}{s_0} = 1, \]

the right-hand side of (8) is a weighted geometric mean and the result follows.

**THEOREM 2.** Let \( x = (x_i)^n, u = (u_i)^n, y = (y_i)^n, v = (v_i)^n \) be sequences of positive numbers and let \( s_i, t_i \ (i = 0, 1, 2) \) satisfy the conditions of Theorem 1. Then

\[ S_{n}^{[s_0]}(x, u)S_{m}^{[t_0]}(y, v) \leq \max_{i=1, 2} S_{n}^{[s_i]}(x, u)S_{m}^{[t_i]}(y, v), \]

where

\[ S_{n}^{[t]}(x, u) = \left[ \sum_{i=1}^{n} x_{i}^{t} \right]^{1/t}. \]

**PROOF.** We proceed as in Theorem 1 but use the discrete Hölder inequality. With \( \alpha_1, \alpha_2 \) as in Theorem 1 we have

\[ \sum_{i=1}^{n} u_{i} x_{i}^{s_0} = \sum_{i=1}^{n} u_{i} x_{i}^{\alpha_1 s_1 + \alpha_2 s_2} \leq \left[ \sum_{i=1}^{n} u_{i} x_{i}^{s_1} \right]^{\alpha_1} \left[ \sum_{i=1}^{n} u_{i} x_{i}^{s_2} \right]^{\alpha_2} \]

by the weighted form of the Hölder inequality [4, p. 136 Theorem 1(c)]. That is,

\[ S_{n}^{[s_0]}(x, u)^{s_0} \leq S_{n}^{[s_1]}(x, u)^{\alpha_1 s_1} S_{n}^{[s_2]}(x, u)^{\alpha_2 s_2}. \]

If \( \beta_1, \beta_2 \) are chosen as in Theorem 1, (4) and (5) hold as before and Hölder’s inequality again gives

\[ \sum_{i=1}^{m} v_{i} y_{i}^{t_0} = \sum_{i=1}^{m} v_{i} y_{i}^{\beta_1 t_1 + \beta_2 t_2} \leq \left[ \sum_{i=1}^{m} v_{i} y_{i}^{t_1} \right]^{\beta_1} \left[ \sum_{i=1}^{m} v_{i} y_{i}^{t_2} \right]^{\beta_2}, \]
so that
\[ S_m^{[1]}(y, v)^{\alpha_1} \leq S_m^{[1]}(y, v)^{\beta_1} S_m^{[2]}(y, v)^{\beta_2}. \] (11)

Combining (10) and (11) gives, via (7), that
\[ S_n^{[0]}(x, u) S_m^{[0]}(y, v) \leq \left[ S_n^{[1]}(x, u) S_m^{[1]}(y, v) \right]^{\alpha_1 / \alpha_0} \left[ S_n^{[2]}(x, u) S_m^{[2]}(y, v) \right]^{\alpha_2 / \alpha_0}, \]
and the desired result follows as before.

**REMARK 1.** If \( \sum u_i = 1 = \sum v_i \), we have (1) from (9) but with the wider supposition that \( s_i \geq a \) and \( t_i \geq b \) in place of \( s_i \geq 1, t_i \geq 1 \). Moreover, if \( u_i = 1 \) \( (i = 1, \ldots, n) \), \( v_i = 1 \) \( (i = 1, \ldots, m) \), then we have (2) without the requirement that \( a : b = \log n : \log m \).

**REMARK 2.** The proof of Theorem 1 depends only on Hölder’s inequality and certain convexity properties of the (real) exponents. Consequently the proof is valid in a general measure space. Indeed, \( f \) and \( g \) can even be taken from different measure spaces. From this viewpoint, Theorem 2 becomes a special case of the more general form of Theorem 1, since the expression \( S_n^{[1]}(x, u) \) is then the usual norm \( \| \cdot \| \), with respect to the obvious discrete measure.

**Acknowledgement**

The authors wish to thank the referee for helpful comments on an earlier version of this paper.

**References**