J. Austral. Math. Soc. (Series A) 40 (1986), 253–260

AUTOMORPHISM ORBITS OF FINITE GROUPS

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(Received 1 February 1984; revised 12 November 1984)

Communicated by D. E. Taylor

Abstract

Let G be a finite group and let $\mathrm{Aut}(G)$ be its automorphism group. Then G is called a k-orbit group if G has k orbits (equivalence classes) under the action of $\mathrm{Aut}(G)$. (For $g, h \in G$, we have $g \sim h$ if $g^\alpha = h$ for some $\alpha \in \mathrm{Aut}(G)$.) It is shown that if G is a k-orbit group, then $k \leqslant |G|/p + 1$, where p is the least prime dividing the order of G. The 3-orbit groups which are not of prime-power order are classified. It is shown that A_5 is the only insoluble 4-orbit group, and a structure theorem is proved about soluble 4-orbit groups.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 20 F 28; secondary 20 E 36.

Let G be a finite group and let $\mathrm{Aut}(G)$ be its automorphism group. Then G is partitioned into equivalence classes under the action of $\mathrm{Aut}(G)$—we say that g, h are equivalent if $g^\alpha = h$ for some $\alpha \in \mathrm{Aut}(G)$. The equivalence classes are called *automorphism orbits*. We call G a *k-orbit group* if it has k automorphism orbits. The identity constitutes the only 1-orbit group, and it is easy to see that the (finite) 2-orbit groups are precisely the elementary abelian groups of prime-power order. In this paper, we prove that if G is a k-orbit group, then $k \leqslant 1 + |G|/p$ where p is the least prime divisor of $|G|$. We also completely classify the 3-orbit groups which are not of prime-power order and the insoluble 4-orbit groups. We also prove a structure theorem about 4-orbit groups which are not of prime-power order.

NOTATION. The notation is standard (compare Huppert [1]) with the following additions: $\pi(G)$ denotes the set of prime divisors of $|G|$, $\mathrm{Syl}_p(G)$ denotes the set of Sylow p-subgroups of G, and N char G denotes that N is a characteristic subgroup of G.

We begin with

THEOREM 1. *Let G be a finite non-abelian group and let p be the least prime dividing $|G|$. Then G has at most $|G|/p$ automorphism orbits.*

PROOF. Let r be the number of conjugacy classes of G and k the number of automorphism orbits. Then $k \leqslant r$. Also G has r irreducible complex characters of which $|G/G'|$ are linear and the rest have degree at least p. Hence

$$|G| \geqslant |G/G'| + (r - |G/G'|)p^2$$

(since the squares of the degrees of the characters all equal $|G|$). Hence

$$r \leqslant \frac{1}{p^2}\left[|G| + (p^2 - 1)|G/G'|\right].$$

If $|G'| > p$, then $r \leqslant p^{-2}[|G| + (p-1)|G|] = |G|/p$, which proves the result. Hence we may assume that $|G'| = p$. In particular, $G' \leqslant Z(G)$. If A is an abelian direct factor of G, say $G = A \times B$, then G has at most $|A|m$ automorphism orbits, where m is the number of automorphism orbits of B. Hence, by induction, we may assume that G has no abelian direct factors, and hence that G is indecomposable.

So we now have: G is an indecomposable p-group, and $|G'| = p$. Let G/G' have type $(r_1, \ldots, r_m)$, i.e. let G/G' be a direct product of r_1 cyclic groups of order p, r_2 of order $p^2, \ldots, r_m$ of order p^m. Let $Z_i = \Omega_i(Z(G)) = \{z \in Z(G) \mid z^{p^i} = 1\}$. Theorem 1 of Sanders [2] states that the group $C(G)$ of central automorphisms of G has order $\prod_{i=1}^{m}|Z_i|^{r_i}$. Hence $|C(G)| \geqslant \prod_{i=1}^{m}|Z_1|^{r_i} = |Z_1|^d$, where $d = r_1 + \cdots + r_m$ is the minimal number of generators of G/G' and hence of G.

We call an element $g \in G$ *small* if

$$|\{g^\sigma \mid \sigma \in C(G)\}| < |Z_1|,$$

and we call g *nearly small* if

$$|\{g^\sigma \mid \sigma \in C(G)\}| \leqslant |Z_1|.$$

Let S be the subgroup generated by the small elements. Suppose first that $S = G$. Let $\{s_1, \ldots, s_d\}$ be a set of small elements which generate G. Let b be the d-tuple $(s_1, \ldots, s_d)$ and, for $\sigma \in C(G)$, let $b^\sigma = (s_1^\sigma, \ldots, s_d^\sigma)$. Note that for σ, $\tau \in C(G)$, we have $b^\sigma = b^\tau$ if and only if $\sigma\tau^{-1}$ fixes each one of $s_1, \ldots, s_d$, and thus if and only if $\sigma = \tau$, since $\{s_1, \ldots, s_d\}$ generates G. Hence $|\{b^\sigma \mid \sigma \in C(G)\}| = |C(G)|$. However, since s_i is small, $|\{s_i^\sigma \mid \sigma \in C(G)\}| < |Z_1|$, and hence $|\{b^\sigma \mid \sigma \in C(G)\}| < |Z_1|^d$. This contradicts Sanders' Theorem 1 [2]. Hence $S \neq G$.

Suppose that $|Z_1| \geqslant p^2$. Then we have

$$\begin{aligned} k &\leqslant |Z(G)| + \frac{1}{p}|S - Z(G)| + \frac{1}{p^2}|G - S| \\ &\leqslant |G|\left[\frac{1}{p^2} + \frac{1}{p}\cdot\frac{p-1}{p^2} + \frac{p-1}{p^3}\right] \\ &= |G|\left[\frac{3p-2}{p^3}\right] \\ &\leqslant |G|/p. \end{aligned}$$

Hence we may assume that $|Z_1| = p$, and thus that $Z(G)$ is cyclic.

Suppose next that G/G' is not elementary abelian. Let T be the subgroup generated by the nearly small elements of G. Arguing as above, we find that $|C(G)| \leqslant |Z_1|^d$, and this contradicts Sanders' theorem unless $Z(G) = Z_1$. Hence $|Z(G)| > p$ implies that $T \neq G$, and then we obtain as above, that

$$k \leqslant |Z(G)| + \frac{1}{p}|T - Z(G)| + \frac{1}{p^2}|G - T| \leqslant |G|/p.$$

Hence we may assume that $|Z(G)| = p$. But $|G'| = p$ implies that $[x, y]^p = 1$ for all x, $y \in G$, and thus that $[x, y^p] = 1$, whence $y^p \in Z(G)$. Hence $\Phi(G) \leqslant Z(G) = G'$, and G/G' is elementary abelian.

So the proof is reduced to a consideration of the following situation: G/G' is elementary abelian and $Z(G)$ is cyclic. Suppose that M is a maximal subgroup of G and that $G = MZ(G)$. Let $\sigma \in \mathrm{Aut}(M)$ and let $\sigma_0 \in \mathrm{Aut}(Z(G))$ be such that $\sigma_0|_{Z(G)\cap M} = \sigma|_{Z(G)\cap M}$. Extend σ_0 to G by setting $\sigma_0(m) = \sigma(m)$ for all $m \in M$, and by making σ_0 multiplicative. Then $\sigma_0 \in \mathrm{Aut}(G)$, and so the theorem holds for G if it holds for M. Hence we may assume that $M = G$, and thus that $Z(G) = \Phi(G)$ has order p. But then G is an extraspecial p-group. So $\mathrm{Aut}(G)$ is known (Winter [4]). Suppose $|G| = p^{2n+1}$. Then $Z(G)$ is composed of two automorphism orbits (by the Theorem of [4]), and each $g \in G \setminus Z(G)$ is conjugate to gz for all $z \in Z(G)$. Hence the number of automorphism orbits of G is at most $2 + p^{-1}(p^{2n+1} - p) = p^{2n} + 1$ with equality if and only if elements in distinct cosets of $Z(G)$ belong to distinct automorphism orbits. However, by [4, (3c), page 161], the automorphisms of G which act trivially on $G/Z(G)$ are precisely the group of inner automorphisms. Since G has an outer automorphism we conclude that there exist g_1, $g_2 \in G$ such that $g_1Z(G) \neq g_2Z(G)$, but such that g_1, g_2 lie in the same automorphism orbit. (We remark here that $\mathrm{Aut}(G)$ does not transitively permute the non-trivial elements of $G/Z(G)$ in the case where G has exponent $p^2 (p > 2)$ by the Theorem of [4].) Hence G has at most $p^{2n} = |G|/p$ automorphism orbits, and the proof is complete.

COROLLARY. *Let $G \neq 1$ be a finite group. Then G has at most $1 + |G|/p$ automorphism orbits, where p is the least prime dividing $|G|$.*

PROOF. Using Theorem 1, we may assume that G is abelian. Thus G is the direct product of cyclic groups of prime power orders. Since the cyclic group $C(q^k)$ of prime power order q^k has exactly $k+1$ automorphism orbits, and since the direct product of r copies of $C(q^k)$ also has exactly $k+1$ automorphism orbits, the result follows immediately except in the case $G = \mathbf{Z}_2 \times \mathbf{Z}_4$. But in this case, G has only 4 automorphism orbits. This completes the proof.

REMARK. We note that the bound in Theorem 1 is attained by the dihedral group of order 8.

We now consider 3-orbit groups.

THEOREM 2. *Let G be a finite group which is not of prime-power order. The following are equivalent:*

(1) *G is a 3-orbit group;*

(2) *$|G| = p^n q$, and G has a normal elementary abelian Sylow p-subgroup P, for some primes p, q, and for some integer $n \geqslant 1$. Furthermore, p is a primitive root* mod q *(i.e. $q-1$ is the least natural number e with $p^e \equiv 1 \bmod q$). Let Q be a Sylow q-subgroup of G. Then P, regarded as a* GF$(p)[Q]$*-module, is a direct sum of $t \geqslant 1$ copies of the (unique) irreducible* GF$(p)[Q]$*-module of dimension $q-1$. In particular $|P| = p^{t(q-1)}$.*

PROOF. (1) Assume that G is a 3-orbit group and that G is not of prime-power order. Then $|G| = p^a q^b$ for some primes p, q and integers $a \geqslant 1$, $b \geqslant 1$. So G is soluble. We may thus assume that $O_p(G) \neq 1$. Since $O_p(G)$ char G, and since G is a 3-orbit group, we thus find that $P = O_p(G)$ is a Sylow p-subgroup of G. Also, since $\Omega_1(Z(P))$ char G, P is elementary abelian. Let $Q \in \mathrm{Syl}_q(G)$. Since G is a 3-orbit group, it has no element of order pq. Hence Q acts fixed-point-freely on P, so that Q is cyclic, or $q = 2$ and Q is (generalized) quaternion [1, V(8.15)]. Since Q must have exponent q, we thus obtain $|Q| = q$.

We now regard P as a GF$(p)[Q]$-module. We write the operation in P as addition and the action of Q (by conjugation) as multiplication. By Maschke's theorem, $P = P_1 \oplus \cdots \oplus P_r$, where the P_i are irreducible GF$(p)[Q]$-modules. Also by Huppert [1, II(3.10), page 166], $|P_i| = p^e$, where e is the order of p mod q (i.e. e is the least natural number with $p^e \equiv 1 \bmod q$). If $Q = \langle \alpha \rangle$, we may choose a basis for P_i so that α is represented by the companion matrix of its

https://doi.org/10.1017/S1446788700027221 Published online by Cambridge University Press

minimal polynomial $m_i(\lambda)$ on P_i. Hence P_i is determined up to GF$(p)[Q]$-isomorphism by the minimal polynomial $m_i(\lambda)$ of α on P_i. We now claim that $P_1, \ldots, P_r$ are all isomorphic modules. For suppose that P_1 is not isomorphic to P_2. Then $m_1(\lambda) \neq m_2(\lambda)$. Let $0 \neq u_i \in P_i$ $(i = 1, 2)$. Since G is a 3-orbit group, there exists $\sigma \in \mathrm{Aut}(G)$ with $u_1^\sigma = u_1 + u_2$. Now $\alpha^{\sigma^{-1}} = \alpha^k w$ for some $w \in P$, and for some $k \geqslant 1$ with $(k, q) = 1$. Let $g(\lambda)$ be the minimal polynomial of α^k on P_1. Note that $\deg g = p^e$. Consider

$$u_1^\sigma g(\alpha) = u_1^\sigma\left(g(\alpha^{\sigma^{-1}})\right)^\sigma$$
$$= u_1^\sigma\left(g(\alpha^k)\right)^\sigma \qquad \text{(since } P \text{ is abelian)}$$
$$= \left[u_1 g(\alpha^k)\right]^\sigma = 0.$$

Hence $(u_1 + u_2)g(\alpha) = 0$. But the minimal polynomial of α on $u_1 + u_2$ is $m_1(\lambda)m_2(\lambda)$ (since $m_1 \neq m_2$ implies $(m_1, m_2) = 1$). Since $\deg g = \deg m_i$, we have a contradiction. So all the P_i are isomorphic GF$(p)[Q]$-modules.

Next, for any i with $(i, q) = 1$, there exists $\tau \in \mathrm{Aut}(G)$ with $\alpha^\tau = \alpha^i$. Let $0 \neq u \in P$. Then the minimal polynomial $m(\alpha)$ such that $um(\alpha) = 0$ is also the minimal polynomial of α on P. But

$$0 = um(\alpha) = [um(\alpha)]^\tau = u^\tau m(\alpha^\tau) = u^\tau m(\alpha^i) \quad \text{(since } P \text{ is abelian)}.$$

Hence $m(\alpha^i) = 0$, and thus $m(\lambda)$ divides $m(\lambda^i)$ $(i = 1, 2, \ldots, q-1)$. Hence, if w is a root of $m(x)$ in the algebraic closure of GF(p), so also is w^i. Therefore $m(\lambda)$ is divisible by the cyclotomic polynomial $\Phi_{q-1}(\lambda)$. Hence $e \geqslant q - 1$, so that $e = q - 1$, and the result follows.

(2) Assume that G satisfies (2). Then Q has no fixed point on P, so G has elements of order 1, p, q only. We first show that any two elements of order p are conjugate. Note that P is a homogeneous GF$(p)[Q]$-module, so if $0 \neq u \in P$, then $P_0 = \{uf(\alpha) | f(x) \in \mathrm{GF}(p)[x]\}$ is an irreducible GF$(p)[Q]$-submodule of order p^{q-1} (using the same notation as above). Let $0 \neq v \in P$ and let $P_1 = \{vf(\alpha) | f(x) \in \mathrm{GF}(p)[x]\}$. If $P_0 = P_1$, then we can write $P = P_0 \oplus P_2$ as GF$(p)[Q]$-modules. But then a routine calculation shows that the map σ defined by $\alpha^\sigma = \alpha$, $u^\sigma = v$, and $w^\sigma = w$ $(w \in P_2)$ extends to an automorphism of G.

If $P_0 \neq P_1$, then we may write $P = P_0 \oplus P_1 \oplus P_2$ as GF$(p)[Q]$-modules. Again a routine calculation shows that the map τ defined by $\alpha^\tau = \alpha$, $u^\tau = v$, $v^\tau = u$, and $w^\tau = w$ $(w \in P_2)$ extends to an automorphism τ of G.

We must now show that the elements of order q form a single orbit under Aut(G). Since P transitively permutes the Sylow q-subgroups it suffices, by Sylow's theorem, to show that, for all i with $1 \leqslant i \leqslant q - 1$, there exists $\theta \in \mathrm{Aut}(G)$ with $\alpha^\theta = \alpha^i$.

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Let $P = P_1 \oplus \cdots \oplus P_t$ as irreducible $\mathrm{GF}(p)[Q]$-modules and let $0 \neq u_j \in P_j$. Then each $u \in P_j$ is uniquely expressible as $u_j f(\alpha)$ for some $f(\lambda) \in \mathrm{GF}(p)[\lambda]$ with $\deg f < q - 1$. Define a map θ by $\alpha^\theta = \alpha^i$, $u_j^\theta = u_j$, and $(u_j f(\alpha))^\theta = u_j f(\alpha^i)$.

Note that if $0 \neq u \in P$ is such that $ug(\alpha) = 0$ for some $g(x) \in \mathrm{GF}(p)[x]$, then $\Phi_{q-1}(\lambda)$ divides $g(\lambda)$, and hence it also divides $g(\lambda^i)$. So $ug(\alpha^i) = 0$. This proves that the natural extension of θ to P is well defined. Thus, by a routine calculation, we see that θ extends to an automorphism of G. This proves that G is a 3-orbit group, and so the proof of the theorem is complete.

We next consider 4-orbit groups.

THEOREM 3. *Let G be an insoluble 4-orbit group. Then $G \cong A_5$.*

PROOF. Since G is insoluble, it follows that $|\pi(G)| \geqslant 3$, and hence, since G is a 4-orbit group, that $|\pi(G)| = 3$. By the Feit-Thompson theorem, $|G|$ is even. So we may write $\pi(G) = \{2, p, r\}$. Since G is a 4-orbit group, the only possible orders for elements of G are 1, 2, p, r. In particular, G has an elementary abelian Sylow 2-subgroup. Let N be a minimal characteristic subgroup of G. Then N is the direct product of isomorphic simple groups. Also G/N is at most a 3-orbit group. Hence G/N is soluble, and thus N is not soluble. So N is a 4-orbit group. Since N char G, and since G is a 4-orbit group, we must have $N = G$. Since all elements of G-$\{1\}$ have prime order, N is simple. Hence G is simple.

By Walter's classification theorem [3], G is one of the following groups:

(1) $\mathrm{PSL}_2(q)$, $q \equiv \pm 3 \bmod 8$;

(2) $\mathrm{SL}_2(2^n)$, for some $n \geqslant 2$;

(3) a group of Ree type;

(4) the small Janko group J_1.For q odd, $\mathrm{PSL}_2(q)$ has cyclic subgroups of order $(q \pm 1)/2$, and thus, since $|\pi(G)| = 3$, we must have $(q \pm 1)/2 = 2$ in case (1). Thus $q = 5$ and $G \cong A_5$. Again $\mathrm{SL}_2(2^n)$ has cyclic subgroups of order $2^n \pm 1$, and so again, since $2^{2n} - 1 \equiv 0 \bmod 3$, we have $2^n - 1 = 3$. This leads in case (2) to $n = 4$ and $G = \mathrm{SL}_2(4) \cong A_5$. Groups which satisfy (3) or (4) have elements of nonprime order, and also their orders are divisible by more than three primes. So (3) and (4) are impossible for G. This proves the theorem.

We now consider the soluble case.

THEOREM 4. *Let G be a finite soluble 4-orbit group which is not of prime-power order. Then $|G| = p^a q^b$, and G has a normal Sylow p-subgroup P for some primes p, q. Let Q be a Sylow q-subgroup of G. Then one of the following holds:*

(1) *Q acts fixed-point-freelyy on P, $|Q| = q$, and P is a 2-orbit or 3-orbit group;*

(2) *P is elementary abelian, and Q is cyclic of order q^2 or quaternion of order 8;*

(3) *$G = P \times Q$, where P, Q are elementary abelian.*

PROOF. Let $p \in \pi(G)$ be such that $O_p(G) \neq 1$. Suppose first that $|\pi(G)| = 3$. Let $\pi(G) = \{p, q, r\}$ and suppose that $O_q(G/O_p(G)) \neq 1$. Let $Q \in \mathrm{Syl}_q(G)$, and let $R \in \mathrm{Syl}_r(G)$. Let $1 \neq x \in R$ and let $N = QO_p(G)$. Since the only possible orders for elements of G are 1, p, q, r, and since $N \triangleleft G$, it follows that x acts fixed-point-freely on N. So, by Thompson's theorem on fixed-point-free automorphisms of prime order [1, V(8.14)], N is nilpotent. But then G has an element of order pq. This is a contradiction. Hence $|\pi(G)| = 2$.

Let $\pi(G) = \{p, q\}$, let $P \in \mathrm{Syl}_p(G)$, and let $Q \in \mathrm{Syl}_q(G)$. Suppose that P is not normal in G. Then since $\Omega_1(Z(O_p(G)))$ is characteristic in G, and since G is a 4-orbit group, $O_p(G)$ is elementary abelian. Furthermore, $QO_p(G)$ and $G/O_p(G)$ are 3-orbit groups. By Theorem 2, firstly Q is cyclic of order q, and secondly Q is elementary abelian of order $q^{t(p-1)}$ (for some $t \geqslant 1$) and $|P/O_p(G)| = p$. Hence $t = p - 1 = 1$, so that $p = 2$ and $G/O_p(G)$ is dihedral of order $2q$. The only possible orders for the elements in the four automorphism orbits of G are 1, 2, q, 2 or 1, 2, q, 4. The first possibility implies that P is abelian and hence that $P \leqslant C_G(O_p(G)) = O_p(G)$. Hence P has exponent 4, and each element of $P - O_p(G)$ has order 4. Now $QO_p(G)$ is characteristic in G, and by the Frattini argument, we have $G = O_p(G)N_G(Q)$. Hence $N_G(Q)$ contains a 2-element $\beta \notin O_2(G)$. Let $Q = \langle \alpha \rangle$. Then $[\alpha, \beta^2] \in Q$, and $1 \neq \beta^2 \in O_2(G)$. Hence $[\alpha, \beta^2] = 1$, and $\alpha\beta^2$ is an element of order $2q$. This is a contradiction. So P is normal in G, as required.

Next, if P is not elementary abelian, then P is a 3-orbit group. Hence every element outside P has order q and acts fixed-point-freely on P. Thus $|Q| = q$. Suppose then that P is elementary abelian. If the automorphism orbits of G are represented by elements of orders 1, p, p, q, then again conclusion (1) holds. Suppose these orders are 1, p, q, q or 1, p, q, q^2. Then Q acts fixed-point-freely on P, so either Q is cyclic of order q or q^2, or Q is a (generalized) quaternion group. Since Q has exponent at most q^2, this implies that if Q is non-cyclic, then Q must be quaternion of order 8.

The only remaining possibility is that G has elements of orders 1, p, q, pq. Then Q is elementary abelian. For each maximal subgroup A of Q, let C_A be the centralizer in P of A. If $O_q(G) \neq 1$, then $O_p(G) \times O_q(G)$ is a characteristic 4-orbit subgroup of G. So it equals G, and (3) holds.

Suppose then that $O_q(G) = 1$, so that Q acts faithfully on P. By Maschke's theorem, P is the direct sum of irreducible $\mathrm{GF}(p)[Q]$-modules. Let $W \leqslant P$ be an irreducible $\mathrm{GF}(p)[Q]$-module. If K is the kernel of Q on W, then Q/K is cyclic, and thus K is a maximal subgroup of Q. Since the set of nonidentity elements of P forms one automorphism orbit, it follows that, for each $1 \neq x \in P$, $C_Q(x)$ has index q in Q. Also, the elements of order q form one automorphism orbit, so that $|C_P(y)| = p^c$ for all $1 \neq y \in Q$ (for some $c \geqslant 0$ independent of y). Let $|P| = p^a$, and let $|Q| = q^b$. We now count the number of elements of G of order pq. Note that if $w \in G$ has order pq, then $w = xy$, where x has order p, y has order q, and $xy = yx$; moreover, this representation is unique. Now each $1 \neq y \in Q$ has $|P|/p^c$ conjugates (which belong to distinct Sylow q-subgroups), and y commutes with $p^c - 1$ elements of order p. Hence the number of elements of order pq is

$$(q^b - 1)p^a(p^c - 1)/p^c = p^{a-c}(p^c - 1)(q^b - 1).$$

On the other hand, each $1 \neq x \in P$ commutes with the $|Q|/q = q^{b-1}$ elements of Q. Each of these except for the identity has p^{a-c} conjugates (which belong to distinct Sylow q-subgroups). Hence the number of elements of order pq is

$$(p^a - 1)(q^{b-1} - 1)p^{a-c}.$$

A comparison of the two counts yields

$$(p^c - 1)(q^b - 1) = (p^a - 1)(q^{b-1} - 1).$$

If $b > 1$, we thus have

$$(p^a - 1)/(p^c - 1) = (q^b - 1)/(q^{b-1} - 1) = m, \text{ say.}$$

Now $q - 1 < m < q$, and $p^{a-c} - 1 < m < p^{a-c}$, so that $q - 1 < p^{a-c} < m + 1 < q + 1$. Thus $p^{a-c} = q$, which gives a contradiction. Hence $b = 1$, as asserted. The proof is complete.

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https://doi.org/10.1017/S1446788700027221 Published online by Cambridge University Press