A GENERALIZATION OF CLIFFORD ALGEBRAS

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Let \( K \) be a field which contains a primitive \( n \)th root of unity \( \omega \) if \( n \) is odd and a primitive 2\( n \)th root of unity \( \zeta \) such that \( \zeta^2 = \omega \) if \( n \) is even.

Define \( C_{p,q}^{(n)} \) to be the polynomial algebra generated over \( K \) by the set \( \{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}\} \) subject to the relations

\[
\begin{align*}
e_i^n &= +1 \quad \text{for } i = 1, \ldots, p; \\
e_i^n &= -1 \quad \text{for } i = p + 1, \ldots, p + q; \\
and \quad e_i e_j &= \omega e_j e_i \quad \text{for } 1 \leq i < j \leq p + q.
\end{align*}
\]

\( C_{p,q}^{(n)} \) is called a generalized Clifford algebra. Our aim in this paper is to find the structure of \( C_{p,q}^{(n)} \) for all values of \( p, q \) and \( n \). This has already been accomplished for the special cases \( p > 0, q = 0 \) and \( p = 0, q > 0 \) by A. O. Morris in [1] and [2].

Let \( K(n) \) denote the full matrix algebra of \( n \times n \) matrices over \( K \). We first prove

**Lemma 1.** There exists an algebra isomorphism

\[ C_{1,1}^{(n)} \cong K(n). \]

**Proof.** Define, for \( i, j = 1, \ldots, n \),

\[ E_{ij} = -\frac{1}{n} \left\{ \sum_{\ell=0}^{n-1} \omega^{i-1} p \ell \right\} \left( -e_2 \right)^{-i}. \]

As in [1], it can be easily proved that

\[ E_{ij} E_{kl} = \delta_{jk} E_{il}. \]

Let \( S = \{E_{ij} \mid i, j = 1, \ldots, n\} \) and put \( S_x = \{E_{ij} \mid j - i \equiv x \text{(mod } n\text{)}\} \); then we have \( S = \bigcup_{x=0}^{n-1} S_x \).

Since \( \omega \) is a primitive \( n \)th root of unity, we have

\[ \det \left[ \omega^{(i-1)(j-1)} \right] = \prod_{0 \leq i < j \leq n-1} (\omega^i - \omega^j) \neq 0. \]

Thus each \( S_x (x = 0, \ldots, n-1) \) is a linearly independent set over \( K \), and therefore so is \( S \).

Also \( (C_{1,1}^{(n)} : K) = n^2 = (K(n) : K) \) and so the set \( S \) is a \( K \)-basis for \( C_{1,1}^{(n)} \), giving us the required isomorphism.

The next result will enable us to compute inductively the algebras \( C_{p,q}^{(n)} \) for any \( p, q \) and \( n \).

**Lemma 2.** There exist algebra isomorphisms

(i) \[ C_{p,q}^{(n)} \cong C_{1,1}^{(n)} \otimes_K C_{p-1,q-1}^{(n)}; \]

(ii) \[ C_{p,0}^{(n)} \otimes_K C_{0,2}^{(n)} \cong C_{p+2,0}^{(n)}; \]

(iii) \[ C_{n,q}^{(n)} \otimes_K C_{2,0}^{(n)} \cong C_{n+2,0}^{(n)}. \]
Proof. (ii) and (iii) have been proved in [2, Theorem 4]. For the proof of (i), define

\[ f = e_1^{n-1} e_{p+1}^1. \]

We have

\[ f^n = (e_1^{n-1} e_{p+1}^1)^n = \omega^{\frac{1}{2}n(n-1)} e_1^{n(n-1)} e_{p+1}^{n(n-1)} = \omega^{\frac{n(n-1)}{2}} 1. \]

If \( n = 2d \) is even, then

\[ f^n = \omega^{d(n-1)^2} (-1) = -\omega^{d(n^2 - 3n - 1)} = -\omega^d. \]

But \( \omega^2d = 1, \omega^d \neq 1 \) by the definition of the primitive \( n \)th root of unity \( \omega \), and so

\[ 0 = \frac{\omega^{2d} - 1}{\omega^d - 1} = \omega^d + 1. \]

Hence we have \( f^n = 1 \) in the case that \( n = 2d \) is even.

Similarly, if \( n = 2d+1 \) is odd

\[ f^n = \omega^{n(2d)^3/2} = \omega^{4nd^3} = 1. \]

Hence, in either case, we have \( f^n = 1 \). Also, for \( i = 1 \) or \( p+1 \), we have

\[ e_i f = \omega fe_i. \]

Next we define a mapping \( \phi \) from \( C_{p,q}^{(n)} \) into \( C_{1,1}^{(n)} \otimes_K C_{p-1,q-1}^{(n)} \) by

\[ \phi(e_i) = \begin{cases} e_1 \otimes 1 & \text{if } i = 1 \text{ or } p+1, \\ f \otimes e_i & \text{if } i = 2, \ldots, p \text{ or } i = p+2, \ldots, p+q. \end{cases} \]

We have \( \phi(e_i)^n = 1 \) for \( i = 1, \ldots, p \) and \( \phi(e_i)^n = -1 \) for \( i = p+1, \ldots, p+q \). Therefore \( \phi \) maps identity onto identity. Since \( e_i f = \omega fe_i \) for \( i = 1 \) or \( p+1 \) and using the defining relations of \( C_{p,q}^{(n)} \), we can easily verify that

\[ \phi(e_i)\phi(e_j) = \omega \phi(e_j)\phi(e_i) \]

for \( 1 \leq i < j \leq p+q \).

Thus, since \( \phi \) maps basis elements of \( C_{p,q}^{(n)} \) onto basis elements of \( C_{1,1}^{(n)} \otimes_K C_{p-1,q-1}^{(n)} \) and

\[ (C_{p,q}^{(n)}, K) = n^{p+q} = (C_{1,1}^{(n)} \otimes_K C_{p-1,q-1}^{(n)}, K), \]

we see that \( \phi \) is an isomorphism, as required.

If \( A \) is an algebra over \( K \), denote a direct sum of \( n \) copies of \( A \) by \( ^nA \), i.e.

\[ ^nA = A \oplus A \oplus \ldots \oplus A \quad (n \text{ copies}). \]
LEMMA 3. Let $\mathbb{K}$ be a field which contains a primitive $n$th root of unity $\omega$ if $n$ is odd and a primitive $2n$th root of unity $\zeta$, such that $\zeta^2 = \omega$, if $n$ is even. Then

(i) $C_{1,0}^{(n)} \cong C_{0,1}^{(n)} \cong \mathbb{K}$,
(ii) $C_{2,0}^{(n)} \cong C_{0,2}^{(n)} \cong \mathbb{K}(n)$.

Thus we have the following theorem.

THEOREM 4. If $\mathbb{K}$ is a field containing a primitive $n$th root of unity $\omega$ if $n$ is odd and a primitive $2n$th root of unity $\zeta$, such that $\zeta^2 = \omega$, if $n$ is even, then

(i) $C_{p,q}^{(n)} \cong \mathbb{K}(n^4)$ if $p + q = 2\lambda$ is even and
(ii) $C_{p,q}^{(n)} \cong \mathbb{K}(n^4)$ if $p + q = 2\lambda + 1$ is odd.

Proof. The proof of both parts of the theorem is carried out by a simple inductive argument using Lemmas 1, 2 and 3.

From now on we shall assume that $\mathbb{K}$ does not contain a primitive $2n$th root of unity $\zeta$ such that $\zeta^2 = \omega$.

We now define, as in [2], $\mathbb{C}$ to be the quadratic field $\mathbb{K}(\sqrt{\omega})$, and $\mathbb{H}$ to be the generalized quaternion algebra regarded as the polynomial algebra over $\mathbb{K}$ generated by $x, y$ subject to the relations

$x^2 = y^2 = \omega^{-1}, 1, \quad xy = -yx.$

For completeness, we now state two lemmas which are proved in [2].

LEMMA 5. Let $\mathbb{C}$ and $\mathbb{H}$ be defined as above; then there exist isomorphisms

(i) $\mathbb{C} \otimes_{\mathbb{K}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C},$
(ii) $\mathbb{H} \otimes_{\mathbb{K}} \mathbb{C} \cong \mathbb{C}(2),$
(iii) $\mathbb{H} \otimes_{\mathbb{K}} \mathbb{H} \cong \mathbb{K}(4).$

Proof. This is proved in [2, Lemma 1].

LEMMA 6. Let $\mathbb{K}$ be a field which contains a primitive $n$th root of unity $\omega$ but not a primitive $2n$th root of unity $\zeta$ such that $\zeta^2 = \omega$. Then

(i) $C_{1,0}^{(n)} \cong \mathbb{K}$;
(ii) $C_{0,1}^{(n)} \cong \begin{cases} \mathbb{K} & \text{if } n \text{ is odd,} \\ \mathbb{C} & \text{if } n = 2\nu \text{ is even;} \end{cases}$
(iii) $C_{2,0}^{(n)} \cong \mathbb{K}(n)$;
(iv) $C_{0,2}^{(n)} \cong \begin{cases} \mathbb{K}(n) & \text{if } n \text{ is odd or } n = 2\nu, \text{ where } \nu \text{ is even,} \\ \mathbb{H}(\nu) & \text{if } n = 2\nu, \text{ where } \nu \text{ is odd;} \end{cases}$
(v) $C_{1,1}^{(n)} \cong \mathbb{K}(n)$. 

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Proof. (i), (ii), (iii) and (iv) are proved in Theorem 3 of [2].

The proof of (v) is exactly the same as in Lemma 1 since the proof did not depend on the existence of a primitive 2nth root of unity \( \zeta \) such that \( \zeta^2 = \omega \).

We are now in a position to prove

**Theorem 7.** If \( K \) is a field which contains a primitive nth root of unity \( \omega \) but not a primitive 2nth root of unity \( \zeta \) such that \( \zeta^2 = \omega \), then for \( n \) odd we have

\[
\begin{align*}
(i) \quad C_{p,q}^{(n)} & \cong K(n^2) \quad \text{if } p + q = 2\lambda \text{ is even}, \\
(ii) \quad C_{p,q}^{(n)} & \cong \text{"} K(n^2) \quad \text{if } p + q = 2\lambda + 1 \text{ is odd}.
\end{align*}
\]

Proof. The theorem is proved by a simple inductive argument using Lemmas 1, 2 and 6.

We give the next two results in tabular form.

**Theorem 8.** If \( K \) is a field as given in Theorem 7, then, for \( n = 2v \), where \( v \) is even, \( C_{p,q}^{(n)} \) is given by the table

<table>
<thead>
<tr>
<th>( p + q )</th>
<th>( -p + q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( K )</td>
</tr>
<tr>
<td>1</td>
<td>( \text{&quot;} K ) \quad \text{&quot;} C )</td>
</tr>
<tr>
<td>2</td>
<td>( K(n) ) \quad ( K(n) ) \quad ( K(n) )</td>
</tr>
<tr>
<td>3</td>
<td>( \text{&quot;} C(n) ) \quad ( \text{&quot;} K(n) ) \quad ( \text{&quot;} C(n) ) \quad ( \text{&quot;} K(n) )</td>
</tr>
<tr>
<td>4</td>
<td>( K(n^2) ) \quad ( K(n^2) ) \quad ( K(n^2) ) \quad ( K(n^2) )</td>
</tr>
</tbody>
</table>

Proof. These results follow from Lemmas 2 and 6. For example,

\[
\begin{align*}
C_{2,1}^{(n)} & \cong C_{1,1}^{(n)} \otimes_K C_{1,0}^{(n)}, \quad \text{by Lemma 2(i)}, \\
& \cong K(n) \otimes_K \text{"} K, \quad \text{by Lemma 6}, \\
& \cong \text{"} K(n)
\end{align*}
\]

and

\[
\begin{align*}
C_{2,1}^{(n)} & \cong C_{1,1}^{(n)} \otimes_K C_{2,0}^{(n)}, \quad \text{by Lemma 2(i)}, \\
& \cong K(n) \otimes_K K(n), \quad \text{by Lemma 6}, \\
& \cong K(n^2);
\end{align*}
\]

whereas we have

\[
\begin{align*}
C_{1,2}^{(n)} & \cong C_{1,1}^{(n)} \otimes_K C_{0,1}^{(n)}, \quad \text{by Lemma 2(i)}, \\
& \cong K(n) \otimes_K \text{"} C, \quad \text{by Lemma 6}, \\
& \cong \text{"} C(n)
\end{align*}
\]

and

\[
\begin{align*}
C_{1,3}^{(n)} & \cong C_{1,1}^{(n)} \otimes_K C_{0,2}^{(n)}, \quad \text{by Lemma 2(i)}, \\
& \cong K(n) \otimes_K K(n), \quad \text{by Lemma 6}, \\
& \cong K(n^2).
\end{align*}
\]
The remaining entries in the table are obtained in exactly the same way.

**Theorem 9.** If $K$ is a field as given in Theorem 7 and $n = 2v$, where $v$ is odd, then $C_{p,q}^{(n)}$ is given by the table

\[
p+q/-p+q = -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
\]

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>K</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$^aK$</td>
<td>$^bC$</td>
</tr>
<tr>
<td>2</td>
<td>$^aC(n)$</td>
<td>$^aK(n)$</td>
</tr>
<tr>
<td>3</td>
<td>$^bC(n)$</td>
<td>$^aK(n)$</td>
</tr>
<tr>
<td>4</td>
<td>H($nv$)</td>
<td>$^aK(n^2)$</td>
</tr>
<tr>
<td>5</td>
<td>$^bH(nv)$</td>
<td>$^aC(n^2)$</td>
</tr>
<tr>
<td>6</td>
<td>H($n^2v$)</td>
<td>$^aH(n^2v)$</td>
</tr>
<tr>
<td>7</td>
<td>$^bC(n^3)$</td>
<td>$^aH(n^2v)$</td>
</tr>
<tr>
<td>8</td>
<td>K($n^4$)</td>
<td>H($n^3v$)</td>
</tr>
</tbody>
</table>

**Proof.** The theorem follows from Lemmas 2 and 6. We give a couple of examples; the remaining entries in the table are obtained in the same way. For example,

\[
C_{3,1}^{(n)} \cong C_{1,1}^{(n)} \otimes_K C_{2,0}^{(n)}, \quad \text{by Lemma 2(i)},
\]

\[
\cong K(n) \otimes_K K(n), \quad \text{by Lemma 6},
\]

and

\[
C_{1,3}^{(n)} \cong C_{1,1}^{(n)} \otimes_K C_{0,2}^{(n)}, \quad \text{by Lemma 2(i)},
\]

\[
\cong K(n) \otimes_K H(v), \quad \text{by Lemma 6},
\]

\[
\cong H(nv).
\]

We note that the table in Theorem 8 is of periodicity 4 and the table in Theorem 9 is of periodicity 8. These tables have been obtained for the special case $n = 2$ in Porteous [3].

**References**