A GENERALIZATION OF CLIFFORD ALGEBRAS

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Let **K** be a field which contains a primitive nth root of unity ω if n is odd and a primitive $2n$th root of unity ζ such that $\zeta^2 = \omega$ if n is even.

Define $C_{p,q}^{(n)}$ to be the polynomial algebra generated over **K** by the set $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}\}$ subject to the relations

$$e_i^n = +1 \quad \text{for} \quad i = 1, \ldots, p;$$

$$e_i^n = -1 \quad \text{for} \quad i = p+1, \ldots, p+q;$$

$$\text{and } e_i e_j = \omega e_j e_i \quad \text{for} \quad 1 \leqq i < j \leqq p+q.$$

$C_{p,q}^{(n)}$ is called a *generalized Clifford algebra*. Our aim in this paper is to find the structure of $C_{p,q}^{(n)}$ for all values of p, q and n. This has already been accomplished for the special cases $p > 0$, $q = 0$ and $p = 0$, $q > 0$ by A. O. Morris in [**1**] and [**2**].

Let $\mathbf{K}(n)$ denote the full matrix algebra of $n \times n$ matrices over **K**. We first prove

LEMMA 1. *There exists an algebra isomorphism*

$$C_{1,1}^{(n)} \cong \mathbf{K}(n).$$

Proof. Define, for $i, j = 1, \ldots, n$,

$$E_{ij} = \frac{1}{n}\left\{\sum_{p=0}^{n-1} \omega^{(i-1)p} e_1^p\right\}(-e_2)^{j-i}.$$

As in [**1**], it can be easily proved that

$$E_{ij} E_{kl} = \delta_{jk} E_{il}.$$

Let $S = \{E_{ij} \mid i, j = 1, \ldots, n\}$ and put $S_x = \{E_{ij} \mid j - i \equiv x (\operatorname{mod} n)\}$; then we have $S = \bigcup_{x=0}^{n-1} S_x$, $S_x \cap S_y = \emptyset$ if $x \neq y$.

Since ω is a primitive nth root of unity, we have

$$\det[\omega^{(i-1)(j-1)}] = \prod_{0 \leqq i < j \leqq n-1} (\omega^i - \omega^j) \neq 0.$$

Thus each S_x $(x = 0, \ldots, n-1)$ is a linearly independent set over **K**, and therefore so is S.

Also $(C_{1,1}^{(n)} : \mathbf{K}) = n^2 = (\mathbf{K}(n) : \mathbf{K})$ and so the set S is a **K**-basis for $C_{1,1}^{(n)}$, giving us the required isomorphism.

The next result will enable us to compute inductively the algebras $C_{p,q}^{(n)}$ for any p, q and n.

LEMMA 2. *There exist algebra isomorphisms*

(i) $C_{p,q}^{(n)} \cong C_{1,1}^{(n)} \otimes_{\mathbf{K}} C_{p-1,q-1}^{(n)}$,

(ii) $C_{p,0}^{(n)} \otimes_{\mathbf{K}} C_{0,2}^{(n)} \cong C_{0,p+2}^{(n)}$,

(iii) $C_{0,q}^{(n)} \otimes_{\mathbf{K}} C_{2,0}^{(n)} \cong C_{q+2,0}^{(n)}$.

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Proof. (ii) and (iii) have been proved in [**2**, Theorem 4]. For the proof of (i), define

$$f = e_1^{n-1} e_{p+1}^{1-n}.$$

We have

$$\begin{aligned} f^n &= (e_1^{n-1} e_{p+1}^{1-n})^n \\ &= \omega^{-\frac{1}{2}n(n-1)^2(1-n)} e_1^{n(n-1)} e_{p+1}^{n(1-n)} \\ &= \omega^{\frac{1}{2}n(n-1)^3} 1 . (-1)^{1-n}. \end{aligned}$$

If $n = 2d$ is even, then

$$\begin{aligned} f^n &= \omega^{d(n-1)^3}(-1) \\ &= -\omega^{d(n^3-3n^2+3n-1)} \\ &= -\omega^{-d} \\ &= -\omega^d. \end{aligned}$$

But $\omega^{2d} = 1$, $\omega^d \neq 1$ by the definition of the primitive nth root of unity ω, and so

$$0 = \frac{\omega^{2d}-1}{\omega^d - 1} = \omega^d + 1.$$

Hence we have $f^n = 1$ in the case that $n = 2d$ is even.
Similarly, if $n = 2d+1$ is odd

$$f^n = \omega^{n(2d)^3/2} = \omega^{4nd^3} = 1.$$

Hence, in either case, we have $f^n = 1$. Also, for $i = 1$ or $p+1$, we have

$$e_i f = \omega f e_i.$$

Next we define a mapping ϕ from $C_{p,q}^{(n)}$ into $C_{1,1}^{(n)} \otimes_{\mathbf{K}} C_{p-1,q-1}^{(n)}$ by

$$\phi(e_i) = \begin{cases} e_i \otimes 1 & \text{if } i = 1 \text{ or } p+1, \\ f \otimes e_i & \text{if } i = 2, \ldots, p \text{ or } i = p+2, \ldots, p+q. \end{cases}$$

We have $\phi(e_i)^n = 1$ for $i = 1, \ldots, p$ and $\phi(e_i)^n = -1$ for $i = p+1, \ldots, p+q$. Therefore ϕ maps identity onto identity. Since $e_i f = \omega f e_i$ for $i = 1$ or $p+1$ and using the defining relations of $C_{p,q}^{(n)}$, we can easily verify that

$$\phi(e_i)\phi(e_j) = \omega\phi(e_j)\phi(e_i)$$

for $1 \leqq i < j \leqq p+q$.

Thus, since ϕ maps basis elements of $C_{p,q}^{(n)}$ onto basis elements of $C_{1,1}^{(n)} \otimes_{\mathbf{K}} C_{p-1,q-1}^{(n)}$ and

$$\begin{aligned} (C_{p,q}^{(n)} : \mathbf{K}) &= n^{p+q} \\ &= (C_{1,1}^{(n)} \otimes_{\mathbf{K}} C_{p-1,q-1}^{(n)} : \mathbf{K}), \end{aligned}$$

we see that ϕ is an isomorphism, as required.

If A is an algebra over $\mathbf{K}$, denote a direct sum of n copies of A by ${}^nA$, i.e.

$${}^nA = A \oplus A \oplus \ldots \oplus A \quad (n \text{ copies}).$$

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The following lemma is [**2**, Theorem 2].

LEMMA 3. *Let* **K** *be a field which contains a primitive nth root of unity* ω *if n is odd and a primitive 2nth root of unity* ζ, *such that* $\zeta^2 = \omega$, *if n is even. Then*

(i) $C^{(n)}_{1,0} \cong C^{(n)}_{0,1} \cong {}^n\mathbf{K}$,

(ii) $C^{(n)}_{2,0} \cong C^{(n)}_{0,2} \cong \mathbf{K}(n)$.

Thus we have the following theorem.

THEOREM 4. *If* **K** *is a field containing a primitive nth root of unity* ω *if n is odd and a primitive 2nth root of unity* ζ, *such that* $\zeta^2 = \omega$, *if n is even, then*

(i) $C^{(n)}_{p,q} \cong \mathbf{K}(n^{\lambda})$ *if* $p+q = 2\lambda$ *is even and*

(ii) $C^{(n)}_{p,q} \cong {}^n\mathbf{K}(n^{\lambda})$ *if* $p+q = 2\lambda+1$ *is odd.*

Proof. The proof of both parts of the theorem is carried out by a simple inductive argument using Lemmas 1, 2 and 3.

From now on we shall assume that **K** does not contain a primitive 2*n*th root of unity ζ such that $\zeta^2 = \omega$.

We now define, as in [**2**], **C** to be the quadratic field $\mathbf{K}(\sqrt{\omega})$, and **H** to be the generalized quaternion algebra regarded as the polynomial algebra over **K** generated by x, y subject to the relations

$$x^2 = y^2 = \omega^{-1}.1, \quad xy = -yx.$$

For completeness, we now state two lemmas which are proved in [**2**].

LEMMA 5. *Let* **C** *and* **H** *be defined as above; then there exist isomorphisms*

(i) $\mathbf{C} \otimes_{\mathbf{K}} \mathbf{C} \cong \mathbf{C} \oplus \mathbf{C}$,

(ii) $\mathbf{H} \otimes_{\mathbf{K}} \mathbf{C} \cong \mathbf{C}(2)$,

(iii) $\mathbf{H} \otimes_{\mathbf{K}} \mathbf{H} \cong \mathbf{K}(4)$.

Proof. This is proved in [**2**, Lemma 1].

LEMMA 6. *Let* **K** *be a field which contains a primitive nth root of unity* ω *but not a primitive 2nth root of unity* ζ *such that* $\zeta^2 = \omega$. *Then*

(i) $C^{(n)}_{1,0} \cong {}^n\mathbf{K}$;

(ii) $C^{(n)}_{0,1} \cong \begin{cases} {}^n\mathbf{K} & \text{if } n \text{ is odd,} \\ {}^\nu\mathbf{C} & \text{if } n = 2\nu \text{ is even;} \end{cases}$

(iii) $C^{(n)}_{2,0} \cong \mathbf{K}(n)$;

(iv) $C^{(n)}_{0,2} \cong \begin{cases} \mathbf{K}(n) & \text{if } n \text{ is odd or } n = 2\nu, \text{ where } \nu \text{ is even,} \\ \mathbf{H}(\nu) & \text{if } n = 2\nu, \text{ where } \nu \text{ is odd;} \end{cases}$

(v) $C^{(n)}_{1,1} \cong \mathbf{K}(n)$.

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Proof. (i), (ii), (iii) and (iv) are proved in Theorem 3 of [**2**].

The proof of (v) is exactly the same as in Lemma 1 since the proof did not depend on the existence of a primitive $2n$th root of unity ζ such that $\zeta^2 = \omega$.

We are now in a position to prove

THEOREM 7. *If* $\mathbf{K}$ *is a field which contains a primitive nth root of unity* ω *but not a primitive 2nth root of unity* ζ *such that* $\zeta^2 = \omega$, *then for n odd we have*

(i) $C_{p,q}^{(n)} \cong \mathbf{K}(n^\lambda)$ *if* $p+q = 2\lambda$ *is even,*

(ii) $C_{p,q}^{(n)} \cong {}^n\mathbf{K}(n^\lambda)$ *if* $p+q = 2\lambda+1$ *is odd.*

Proof. The theorem is proved by a simple inductive argument using Lemmas 1, 2 and 6.

We give the next two results in tabular form.

THEOREM 8. *If* $\mathbf{K}$ *is a field as given in Theorem* 7, *then, for* $n = 2v$, *where v is even,* $C_{p,q}^{(n)}$ *is given by the table*

$p+q/-p+q=$	-4	-3	-2	-1	0	1	2	3	4
0					$\mathbf{K}$				
1				${}^n\mathbf{K}$		${}^v\mathbf{C}$			
2			$\mathbf{K}(n)$		$\mathbf{K}(n)$		$\mathbf{K}(n)$		
3		${}^v\mathbf{C}(n)$		${}^n\mathbf{K}(n)$		${}^v\mathbf{C}(n)$		${}^n\mathbf{K}(n)$	
4	$\mathbf{K}(n^2)$		$\mathbf{K}(n^2)$		$\mathbf{K}(n^2)$		$\mathbf{K}(n^2)$		$\mathbf{K}(n^2)$

Proof. These results follow from Lemmas 2 and 6. For example,

$$\begin{aligned} C_{2,1}^{(n)} &\cong C_{1,1}^{(n)} \otimes_{\mathbf{K}} C_{1,0}^{(n)}, && \text{by Lemma 2(i),} \\ &\cong \mathbf{K}(n) \otimes_{\mathbf{K}} {}^n\mathbf{K}, && \text{by Lemma 6,} \\ &\cong {}^n\mathbf{K}(n) \end{aligned}$$

and

$$\begin{aligned} C_{3,1}^{(n)} &\cong C_{1,1}^{(n)} \otimes_{\mathbf{K}} C_{2,0}^{(n)}, && \text{by Lemma 2(i),} \\ &\cong \mathbf{K}(n) \otimes_{\mathbf{K}} \mathbf{K}(n), && \text{by Lemma 6,} \\ &\cong \mathbf{K}(n^2); \end{aligned}$$

whereas we have

$$\begin{aligned} C_{1,2}^{(n)} &\cong C_{1,1}^{(n)} \otimes_{\mathbf{K}} C_{0,1}^{(n)}, && \text{by Lemma 2(i),} \\ &\cong \mathbf{K}(n) \otimes_{\mathbf{K}} \mathbf{C}, && \text{by Lemma 6,} \\ &\cong {}^v\mathbf{C}(n) \end{aligned}$$

and

$$\begin{aligned} C_{1,3}^{(n)} &\cong C_{1,1}^{(n)} \otimes_{\mathbf{K}} C_{0,2}^{(n)}, && \text{by Lemma 2(i),} \\ &\cong \mathbf{K}(n) \otimes_{\mathbf{K}} \mathbf{K}(n), && \text{by Lemma 6,} \\ &\cong \mathbf{K}(n^2). \end{aligned}$$

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The remaining entries in the table are obtained in exactly the same way.

THEOREM 9. *If* $\mathbf{K}$ *is a field as given in Theorem 7 and* $n = 2v$, *where* v *is odd, then* $C_{p,q}^{(n)}$ *is given by the table*

$p+q$ / $-p+q =$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
0									$\mathbf{K}$								
1								${}^{n}\mathbf{K}$		${}^{v}\mathbf{C}$							
2							$\mathbf{K}(n)$		$\mathbf{K}(n)$		$\mathbf{H}(v)$						
3						${}^{v}\mathbf{C}(n)$		${}^{n}\mathbf{K}(n)$		${}^{v}\mathbf{C}(n)$		${}^{n}\mathbf{H}(v)$					
4					$\mathbf{H}(nv)$		$\mathbf{K}(n^2)$		$\mathbf{K}(n^2)$		$\mathbf{H}(nv)$		$\mathbf{H}(nv)$				
5				${}^{n}\mathbf{H}(nv)$		${}^{v}\mathbf{C}(n^2)$		${}^{n}\mathbf{K}(n^2)$		${}^{v}\mathbf{C}(n^2)$		${}^{n}\mathbf{H}(nv)$		${}^{v}\mathbf{C}(n^2)$			
6			$\mathbf{H}(n^2v)$		$\mathbf{H}(n^2v)$		$\mathbf{K}(n^3)$		$\mathbf{K}(n^3)$		$\mathbf{H}(n^2v)$		$\mathbf{H}(n^2v)$		$\mathbf{K}(n^3)$		
7		${}^{v}\mathbf{C}(n^3)$		${}^{n}\mathbf{H}(n^2v)$		${}^{v}\mathbf{C}(n^3)$		${}^{n}\mathbf{K}(n^3)$		${}^{v}\mathbf{C}(n^3)$		${}^{n}\mathbf{H}(n^2v)$		${}^{v}\mathbf{C}(n^3)$		${}^{n}\mathbf{K}(n^3)$	
8	$\mathbf{K}(n^4)$		$\mathbf{H}(n^3v)$		$\mathbf{H}(n^3v)$		$\mathbf{K}(n^4)$		$\mathbf{K}(n^4)$		$\mathbf{H}(n^3v)$		$\mathbf{H}(n^3v)$		$\mathbf{K}(n^4)$		$\mathbf{K}(n^4)$

Proof. The theorem follows from Lemmas 2 and 6. We give a couple of examples; the remaining entries in the table are obtained in the same way. For example,

$$\begin{aligned} C_{3,1}^{(n)} &\cong C_{1,1}^{(n)} \otimes_{\mathbf{K}} C_{2,0}^{(n)}, && \text{by Lemma 2(i)},\\ &\cong \mathbf{K}(n) \otimes_{\mathbf{K}} \mathbf{K}(n), && \text{by Lemma 6}, \end{aligned}$$

and

$$\begin{aligned} C_{1,3}^{(n)} &\cong C_{1,1}^{(n)} \otimes_{\mathbf{K}} C_{0,2}^{(n)}, && \text{by Lemma 2(i)},\\ &\cong \mathbf{K}(n) \otimes_{\mathbf{K}} \mathbf{H}(v), && \text{by Lemma 6},\\ &\cong \mathbf{H}(nv). \end{aligned}$$

We note that the table in Theorem 8 is of periodicity 4 and the table in Theorem 9 is of periodicity 8. These tables have been obtained for the special case $n = 2$ in Porteous [**3**].

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