MULTIPLICATION OPERATORS
AND DYNAMICAL SYSTEMS

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Abstract

Let \( X \) be a completely regular Hausdorff space, let \( V \) be a system of weights on \( X \) and let \( T \) be a locally convex Hausdorff topological vector space. Then \( CV_b(X, T) \) is a locally convex space of vector-valued continuous functions with a topology generated by seminorms which are weighted analogues of the supremum norm. In the present paper we characterize multiplication operators on the space \( CV_b(X, T) \) induced by operator-valued mappings and then obtain a (linear) dynamical system on this weighted function space.

Keywords and phrases: system of weights, locally convex spaces, multiplication operators, dynamical systems.

Introduction

Let \( X \) be a non-empty set, let \( T \) be a topological algebra and let \( L(X, T) \) be the linear space of all functions from \( X \) to \( T \). Let \( F(X, T) \) be a topological vector subspace of \( L(X, T) \). Let \( \psi \) be a mapping on \( X \) such that \( \psi f \in L(X, T) \) whenever \( f \in F(X, T) \). This gives rise to a linear transformation \( M_\psi : F(X, T) \to L(X, T) \) defined as \( M_\psi f = \psi f \), where the product of functions is defined pointwise. In case \( M_\psi \) takes \( F(X, T) \) into itself and is continuous, it is called a multiplication operator on \( F(X, T) \) induced by the mapping \( \psi \).

This paper is a continuation of our earlier paper [8] in which we have studied multiplication operators on weighted spaces of vector-valued con-
tinuous functions induced by scalar-valued and vector-valued mappings. In the present paper we concentrate on the study of multiplication operators on weighted spaces of vector-valued mappings induced by operator-valued mappings and then we endeavor to study a (linear) dynamical system on these function spaces.

**Preliminaries**

Let $X$ be a completely regular Hausdorff space, let $T$ be a Hausdorff locally convex topological vector space over $\mathbb{C}$ and let $C(X, T)$ be the vector space of all continuous functions from $X$ into $T$. By $cs(T)$ we mean the set of all continuous seminorms on $T$, and $B(T)$ denotes the set of all continuous linear operators on $T$. By a system of weights we mean a set $V$ of non-negative upper-semicontinuous functions on $X$ such that, given any $x \in X$, there is some $v \in V$ for which $v(x) > 0$ and for every pair $u, v \in V$ and $\alpha > 0$, there exists $w \in V$ so that $\alpha u \leq w$ and $\alpha v \leq w$ (point wise on $X$).

Now we consider the following vector space of vector-valued continuous functions:

$$CV_b(X, T) = \{ f \in C(X, T) : v f(X) \text{ is bounded in } T \text{ for all } v \in V \}.$$  

Now, let $v \in V$, $q \in cs(T)$ and $f \in C(X, T)$. If we put $\| f \|_{v,q} = \sup \{ v(x) q(f(x)) : x \in X \}$, then $\| \cdot \|_{v,q}$ is a seminorm on $CV_b(X, T)$ and the family $\{ \| \cdot \|_{v,q} : v \in V, q \in cs(T) \}$ defines a locally convex topology on $CV_b(X, T)$.

In case $T = \mathbb{C}$, we shall omit $T$ from our notation and write $CV_b(X)$ in place of $CV_b(X, \mathbb{C})$. We also write $\| \cdot \|_v$ in place of $\| \cdot \|_{v,q}$ for each $v \in V$, where $q(z) = |z|$, $z \in \mathbb{C}$. We shall denote by $B_{v,q}$ the closed unit ball corresponding to the seminorm $\| \cdot \|_{v,q}$. In case $T = (\mathcal{T}, q)$, any normed linear space, we simply write $B_v$. We refer to the papers of Bierstedt [1, 2] and Prolla [7] for more details and examples of these function spaces.

Let $\mathcal{F}$ be the family of all bounded subsets of $T$ and let $M \in \mathcal{F}$ and $p \in cs(T)$. If we define the function

$$S_{M,p} : B(T) \to \mathbb{R}^+$$

as

$$S_{M,p}(A) = \sup \{ p(A(y)) : y \in M \}$$

then $S_{M,p}$ is a seminorm on $B(T)$ and the family $\{ S_{M,p} : M \in \mathcal{F}, p \in cs(T) \}$ defines a locally convex topology on $B(T)$ which we call the topology of uniform convergence on bounded sets and denote by $\mathcal{U}$. Thus $(B(T), \mathcal{U})$ is a locally convex topological vector space of continuous linear operators on
For more details of these topologies on the spaces of linear operators we refer to Grothendieck [4] and Kothe [5].

2. Functions inducing multiplication operators

Throughout this section we will work under the following modest requirements, while developing our characterisation of an operator-valued mapping $\psi : X \to B(T)$ which induces a multiplication operator on $CV_b(X, T)$:

(2.a) $X$ is a completely regular Hausdorff space;
(2.b) $T$ is a Hausdorff locally convex topological vector space;
(2.c) $V$ is a system of weights on $X$.

In the following theorem we characterise operator-valued mappings which induce multiplication operators on $CV_b(X, T)$.

2.1. THEOREM. Let $\psi : X \to B(T)$ be an operator-valued continuous function. Then $M_\psi : CV_b(X, T) \to CV_b(X, T)$ is a multiplication operator if and only if for every $v \in V$ and $p \in cs(T)$, there exist $u \in V$ and $q \in cs(T)$ such that $v(x)p(\psi(x)y) \leq u(x)q(y)$, for every $x \in X$ and $y \in T$.

PROOF. First, let us suppose that for every $v \in V$ and $p \in cs(T)$, there exist $u \in V$ and $q \in cs(T)$ such that

$v(x)p(\psi(x)y) \leq u(x)q(y)$, for every $x \in X$ and $y \in T$.

Then we shall show that $M_\psi$ is a continuous linear operator on $CV_b(X, T)$. First of all, we show that $M_\psi$ is an into map. Let $\{x_\alpha : \alpha \in \Delta\}$ be a net in $X$ such that $x_\alpha \to x$. To show that $\psi(x_\alpha)f(x_\alpha) \to \psi(x)f(x)$ in $T$, it suffices to show that for every $p \in cs(T)$ and $\varepsilon > 0$, there exists $\alpha_0 \in \Delta$ such that

$p(\psi(x_\alpha)f(x_\alpha) - \psi(x)f(x)) < \varepsilon$, for every $\alpha \geq \alpha_0$.

Now,

(i) $p(\psi(x_\alpha)f(x_\alpha) - \psi(x)f(x)) \leq p[(\psi(x_\alpha) - \psi(x))(f(x_\alpha))] + p[\psi(x)(f(x_\alpha) - f(x))]$.

Since the set $\{f(x_\alpha) : \alpha \in \Delta\}$ is bounded in $T$, for every $p \in cs(T)$ and $\varepsilon > 0$, there exists $\alpha_1 \in \Delta$ such that

(ii) $p[(\psi(x_\alpha) - \psi(x))(f(x_\alpha))] < \varepsilon/2$, for every $\alpha \geq \alpha_1$.

Again, since $\psi(x)$ is a continuous linear operator on $T$, for every $p \in cs(T)$ and $\varepsilon > 0$, there exists a neighbourhood $W$ of the origin in $T$ such that
\[ p(y(x)y) < \epsilon/2 \text{ for every } y \in W. \] Since \( f \) is continuous, there exists \( \alpha_2 \in \Delta \) such that \( f(x_\alpha) - f(x) \in W \), for \( \alpha \geq \alpha_2 \) and consequently

(iii) \[ p[y(x)f(x_\alpha) - f(x)] < \epsilon/2, \quad \text{for every } \alpha \geq \alpha_2. \]

Let \( \alpha_0 \in \Delta \) be such that \( \alpha_1 \leq \alpha_0 \) and \( \alpha_2 \leq \alpha_0 \). Then from (ii) and (iii) it follows that

\[ p(y(x)f(x_\alpha) - y(x)f(x)) \leq \epsilon, \quad \text{for every } \alpha \geq \alpha_0. \]

This proves the continuity of \( \psi f \). Further, let \( v \in V, \ p \in \text{cs}(T) \) and \( f \in CV_b(X, T) \). The

\[ \|\psi f\|_{v,p} = \sup\{v(x)p(\psi(x)f(x)) : x \in X\} \leq \sup\{u(x)q(f(x)) : x \in X\} < \infty. \]

This implies that \( \psi f \in CV_b(X, T) \). Clearly \( M_\psi \) is linear on \( CV_b(X, T) \).

In order to prove the continuity of \( M_\psi \) on \( CV_b(X, T) \), it is enough to show that \( M_\psi \) is continuous at the origin. For this, suppose \( \{f_\alpha\} \) is a net in \( CV_b(X, T) \) such that \( \|f_\alpha\|_{v,p} \to 0 \), for every \( v \in V \) and \( p \in \text{cs}(T) \).

\[ \|M_\psi f_\alpha\|_{v,p} = \sup\{v(x)p(\psi(x)f_\alpha(x)) : x \in X\} \leq \sup\{u(x)q(f_\alpha(x)) : x \in X\} = \|f_\alpha\|_{u,q} \to 0. \]

This proves the continuity of \( M_\psi \) at the origin and hence \( M_\psi \) is continuous on \( CV_b(X, T) \).

Conversely, suppose \( M_\psi \) is a continuous linear operator on \( CV_b(X, T) \). We shall show that for every \( v \in V \) and \( p \in \text{cs}(T) \), there exist \( u \in V \) and \( q \in \text{cs}(T) \) such that

\[ v(x)p(\psi(x)y) \leq u(x)q(y), \quad \text{for every } x \in X \text{ and } y \in T. \]

Let \( v \in V \) and \( p \in \text{cs}(T) \). Since \( M_\psi \) is continuous at the origin, there exist \( u \in V \) and \( q \in \text{cs}(T) \) such that \( M_\psi(B_{u,q}) \subseteq B_{v,p} \). We claim that

\[ v(x)p(\psi(x)y) \leq 2u(x)q(y), \quad \text{for every } x \in X \text{ and } y \in T. \]

Take \( x_0 \in X \), \( y_0 \in T \) and set \( u(x_0)q(y_0) = \epsilon \). In case \( \epsilon > 0 \), the set

\[ G = \{x \in X : u(x)q(y_0) < 2\epsilon\} \]

is an open neighbourhood of \( x_0 \). Thus, according to [6, Lemma 2], there exists \( f \in CV_b(X) \) such that \( 0 \leq f \leq 1 \), \( f(x_0) = 1 \) and \( f(X - G) = 0 \). Define \( g(x) = f(x)y_0 \), for every \( x \in X \). Then clearly \( g \in CV_b(X, T) \) and for every \( p \in \text{cs}(T), \ 0 \leq (p \circ g) \leq p(y_0), \ (p \circ g)(x_0) = p(y_0) \) and \( (p \circ g)(X - G) = 0 \). Let \( h = (2u(x_0)q(y_0))^{-1}g \). Then clearly \( h \in B_{u,q} \) and this yields that \( \psi h \in B_{v,p} \). Hence \( v(x)p(\psi(x)h(x)) \leq 1 \), for every \( x \in X \). From this, it follows that

\[ v(x)p(\psi(x)g(x)) \leq 2u(x_0)q(y_0), \quad \text{for every } x \in X. \]

This implies that

\[ v(x_0)p(\psi(x_0)y_0) \leq 2u(x_0)q(y_0). \]
On the other hand, suppose $u(x_0)q(y_0) = 0$. Then the following three cases arise:

(i) $u(x_0) = 0$, $q(y_0) \neq 0$;
(ii) $u(x_0) \neq 0$, $q(y_0) = 0$;
(iii) $u(x_0) = 0$, $q(y_0) = 0$.

Let us suppose that (i) holds and let $v(x)\psi(x)y_0 > 0$. Put $\varepsilon = v(x)\psi(x)y_0/2$. Then $G = \{ x \in X : \psi(x) < \varepsilon \}$ is an open neighbourhood of $x_0$ and hence again by [6, Lemma 2], there exists $f \in CV_b(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - G) = 0$. Again, define $g(x) = f(x)y_0$, for every $x \in X$. Then clearly $g \in CV_b(X, T)$ and for every $p \in cs(T)$, $0 \leq (p \circ g) \leq p(y_0)$, $(p \circ g)(x_0) = p(y_0)$ and $(p \circ g)(X - G) = 0$. Consider $h = e^{-1}g$. Then $h \in B_{u,q}$ and therefore $\psi h \in B_{v,p}$. Hence $v(x)\psi(x)h(x)) \leq 1$ for every $x \in X$. This implies that

$$v(x)\psi(x)g(x)) \leq \frac{v(x)\psi(x)y_0}{2}, \quad \text{for every } x \in X.$$  

From this, it follows that

$$v(x)\psi(x)y_0 \leq \frac{v(x)\psi(x)y_0}{2}$$

which is impossible and hence in this case our claim is established.

Case (ii). Suppose $u(x_0) \neq 0$, $q(y_0) = 0$ and $v(x)\psi(x)y_0 > 0$. Put $\varepsilon = v(x)\psi(x)y_0/2$. Then $G = \{ x \in X : u(x) < \varepsilon + u(x_0) \}$ is an open neighbourhood of $x_0$ and therefore by [6, Lemma 2], there exists $f \in CV_b(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - G) = 0$. Define $g(x) = f(x)y_0$, for every $x \in X$. Then clearly $g \in CV_b(X, T)$ and for every $p \in cs(T)$, $0 \leq (p \circ g) \leq p(y_0)$, $(p \circ g)(x_0) = p(y_0)$ and $(p \circ g)(X - G) = 0$. Consider $h = e^{-1}g$. Then $h \in B_{u,q}$ and this yields that $\psi h \in B_{v,p}$. This implies that $v(x)\psi(x)h(x)) \leq 1$, for every $x \in X$. From this, it follows that

$$v(x)\psi(x)g(x)) \leq \frac{v(x)\psi(x)y_0}{2}, \quad \text{for every } x \in X.$$  

Further, it implies that

$$v(x)\psi(x)y_0 \leq \frac{v(x)\psi(x)y_0}{2}$$

which is impossible and hence in this case too our claim is established.

Case (iii). Finally, suppose $u(x_0) = 0$ and $q(y_0) = 0$. Let $v(x)\psi(x)y_0 > 0$ and put $\varepsilon = v(x)\psi(x)y_0/2$. Then $G = \{ x \in X : u(x) < \varepsilon \}$ is an open neighbourhood of $x_0$ and again by [6, Lemma 2], there exists $f \in CV_b(X)$ such that $0 \leq f \leq 1$, $f(x_0) = 1$ and $f(X - G) = 0$. Define $g(x) = f(x)y_0$, for every $x \in X$. Then clearly $g \in CV_b(X, T)$ and for every
Consider $h = e^{-1} g$. Then $h \in B_{u,q}$ and this implies that $\psi h \in B_{v,p}$. Hence $v(x)p(\psi(x)h(x)) \leq 1$, for every $x \in X$. From this, it follows that

$$v(x)p(\psi(x)g(x)) \leq \frac{v(x_0)p(\psi(x_0)y_0)}{2}, \quad \text{for every } x \in X.$$ 

Further, it implies that

$$v(x_0)p(\psi(x_0)y_0) \leq \frac{v(x_0)p(\psi(x_0)y_0)}{2},$$

which is a contradiction and with this our claim is established. This completes the proof of the theorem.

2.2 REMARK (i). Every constant map $\psi: X \to B(T)$ induces a multiplication operator on $CV_b(X, T)$. For, if we define $\psi: X \to B(T)$ as $\psi(x) = A$, for every $x \in X$ where $A$ is any continuous linear operator on $T$. Let $v \in V$, and $p \in cs(T)$. Since $A$ is a continuous linear operator, there exist $m > 0$ and $q \in cs(T)$ such that

$$p(Ay) \leq mq(y), \quad \text{for every } y \in T.$$ 

This implies that $p(\psi(x)y) \leq mq(y)$, for every $x \in X$ and $y \in T$. Further, it follows that

$$v(x)p(\psi(x)y) \leq mv(x)q(y) \quad (\text{for every } x \in X \text{ and } y \in T)$$

$$\leq u(x)q(y) \quad (\text{for every } x \in X \text{ and } y \in T).$$

Hence by Theorem 2.1, $M_\psi$ is a multiplication operator on $CV_b(X, T)$.

(ii) Let $X$ be a completely regular Hausdorff space and let $T = Y$ be any Banach space. Then every continuous bounded operator-valued mapping induces a multiplication operator on $CV_b(X, Y)$. For, let $\psi: X \to B(Y)$ be a bounded operator-valued mapping. Then there exists $m > 0$ such that $\|\psi(x)\| \leq m$, for every $x \in X$, Let $v \in V$, $x \in X$ and $y \in Y$. Then

$$v(x)\|\psi(x)y\| \leq v(x)\|\psi(x)\|\|y\| \leq mv(x)\|y\|$$

$$\leq u(x)\|y\| \quad (\text{for every } x \in X \text{ and } y \in Y).$$

Hence by Theorem 2.1, $M_\psi$ is a multiplication operator on $CV_b(X, Y)$.

If $T = Y$ is any Banach space and $V$ is the system of weights generated by the characteristic functions of all compact subsets, then it turns out that every continuous operator-valued mapping induces a multiplication operator on $CV_b(X, Y)$. This we shall establish in the following proposition.

2.3 PROPOSITION. Let $X$ be a completely regular Hausdorff space and let

$$V = \{\lambda \chi_K : \lambda \geq 0, \ K \subset X \text{ and } K \text{ is a compact set}\}.$$
Then every continuous mapping $\psi: X \to B(Y)$, induces a multiplication operator $M_\psi$ on $CV_b(X, Y)$.

**Proof.** In order to show that $M_\psi$ is a continuous linear operator on $CV_b(X, Y)$, in the light of Theorem 2.1 it is enough to show that for every $v \in V$, there exists $u \in V$ such that

$$v(x)\|\psi(x)y\| \leq u(x)\|y\|, \quad \text{for every } x \in X \text{ and } y \in Y.$$ If $v \in V$, then $v = \lambda \chi_K$, for some compact subset $K$ of $X$. Since $\psi: X \to B(Y)$ is continuous, $\psi(K)$ is a compact subset in $B(Y)$. Let $m = \text{Sup}\{\|\psi(x)\|: x \in K\}$. Put $u(x) = \lambda m \chi_K(x)$. Then $u \in V$. Let $x \in K$ and $y \in Y$. Then

$$\|\psi(x)y\| \leq \|\psi(x)\|\|y\| \leq m\|y\|.$$ From this, it follows that

$$\lambda \chi_K(x)\|\psi(x)y\| \leq \lambda \chi_K(x)m\|y\|.$$ This implies that

$$v(x)\|\psi(x)y\| \leq u(x)\|y\|, \quad \text{for every } x \in K \text{ and } y \in Y.$$ On the other hand, if $x \in X \setminus K$, then obviously

$$v(x)\|\psi(x)y\| \leq u(x)\|y\|.$$ Thus $v(x)\|\psi(x)y\| \leq u(x)\|y\|$, for every $x \in X$, $y \in Y$ and hence $M_\psi$ is a multiplication operator on $CV_b(X, Y)$. This completes the proof of the theorem.

**2.4 Remark (i).** From the above proposition, we note that if $\psi: X \to B(T)$ where $T$ is any Banach space, is an unbounded continuous operator-valued mapping, even then $\psi$ gives rise to a multiplication operator $M_\psi$ on $CV_b(X, T)$, where $V$ is the system of weights generated by the characteristic functions of all compact subsets of $X$.

(ii) In the above proposition, if we replace the system of weights

$$V = \{\lambda \chi_K: \lambda \geq 0, \ K \subset X \text{ and } K \text{ is a compact set}\}$$

by $C_c^+(X)$, the set of all positive continuous functions having compact supports, even then the conclusion holds.

**2.5 Corollary.** Let $X$ have the discrete topology and

$$V = \{\lambda \chi_K: \lambda \geq 0, \ K \subset X \text{ and } K \text{ is a compact set}\}.$$
Then every map \( \psi: X \to B(T) \), where \( T \) is a Banach space, induces a multiplication operator \( M_\psi \) on \( CV_b(X, T) \).

Now, we shall give certain examples of operator-valued mappings which induce and do not induce multiplication operators on \( CV_b(X, T) \).

2.6 Example. Let \( X = \mathbb{N} \) with discrete topology and let \( T = l^2 \), the Hilbert space of all square summable sequences of complex numbers. If we define \( \psi: \mathbb{N} \to B(l^2) \) by \( \psi(n) = U^n \), where \( U \) is the unilateral shift operator on \( l^2 \), then

\[
\|\psi(n)\| = \|U^n\| \leq \|U\|^n \leq 1, \quad \text{for every } n \in \mathbb{N}.
\]

This shows that \( \psi \) is a bounded operator-valued mapping and hence by Remark 2.2 (ii), \( M_\psi \) is a multiplication operator on \( CV_b(X, T) \).

2.7 Example. Let \( X = \mathbb{N} \), with discrete topology and \( T = \mathbb{R}^2 \), the real Banach space. Define \( \psi: \mathbb{N} \to B(\mathbb{R}^2) \) by \( \psi(n) = P^n \), where \( P \) is a projection operator on \( \mathbb{R}^2 \). Then \( \|\psi(n)\| = \|P^n\| \leq \|P\|^n \leq 1 \). This implies that \( \psi \) is a bounded operator-valued mapping and hence by Remark 2.2(ii), \( M_\psi \) is a multiplication operator on \( CV_b(X, T) \).

2.8 Example. Let \( X = \mathbb{N} \) be the set of natural numbers with discrete topology and let \( V = K^+(\mathbb{N}) \), the system of all positive constant weights. Let \( T = C_b(\mathbb{N}) = l^\infty \) be the Banach space of all bounded sequences of complex numbers and \( B(l^\infty) \), the Banach algebra of bounded operators on \( l^\infty \). Define \( \psi: \mathbb{N} \to B(l^\infty) \) as \( \psi(n) = C(\phi_n) \), where \( C(\phi): l^\infty \to l^\infty \) is the composition operator induced by a map \( \phi: \mathbb{N} \to \mathbb{N} \). Then it can be seen that for every \( v \in V \), there exists \( u \in V \) such that

\[
v(n)\|\psi(n)f\| \leq u(n)\|f\|, \quad \text{for every } n \in \mathbb{N} \text{ and } f \in l^\infty
\]

and hence by Theorem 2.1, \( M_\psi \) is a multiplication operator on \( CV_b(X, T) \).

2.9 Example. Let \( X = \mathbb{N} \), the set of natural numbers with discrete topology, \( T = l^2 \) and let \( B(l^2) \) be the Banach space of bounded linear operators on \( l^2 \). Let \( v(n) = n \), for every \( n \in \mathbb{N} \). Then \( V = \{\lambda v: \lambda \geq 0\} \) is a system of weights on \( \mathbb{N} \). Let us define \( \psi: \mathbb{N} \to B(l^2) \) as \( \psi(n) = A^n \), where \( A \) is the multiplication operator on \( l^2 \) induced by the constant function \( 2 \), that is, \( A: l^2 \to l^2 \) is defined as

\[
A(x_1, x_2, \ldots) = 2(x_1, x_2\ldots).
\]

Then clearly one can check that

\[
v(n)\|\psi(n)x\| \leq u(n)\|x\|.
\]

Thus \( \psi \) does not induce a multiplication operator \( M_\psi \) on \( CV_b(\mathbb{N}, l^2) \). In fact \( M_\psi \) is not even an into map. For, take \( f: \mathbb{N} \to l^2 \) as \( f(n) = 1/n^2 \). Then
obviously $f \in CV_b(N, l^2)$ but $\psi f(n) = \psi(n)f(n) = A^n(1/n^2) = 2^n/n^2 \to \infty$ as $n \to \infty$ and therefore $\psi f \notin CV_b(N, l^2)$. In this example, if we take $V$ as the system of positive constant weights on $N$, even then $\psi$ does not induce a multiplication operator $M_\psi$ on $CV_b(N, l^2)$. If fact, if $f(n) = 1/n$, then $f \in CV_b(N, l^2)$ but $\psi f \notin CV_b(N, l^2)$.

3. Dynamical systems induced by multiplication operators

Throughout this section we shall take $X$ to be the real line $R$ (with the usual topology) and $T$ to be a Banach space. We shall denote by $B(T)$, the Banach algebra of all bounded linear operators on $T$ and by $F_b(R)$, the set of all continuous bounded functions on $R$. Let $V$ be a system of weights on $R$. Then clearly $CV_b(R, T)$ is a locally convex Hausdorff topological vector space with the weighted topology defined in the last section. Now let $U$ be a countable set of non-negative upper semicontinuous functions on $R$ such that $W = \{\lambda u: \lambda \geq 0, \ u \in U\}$ is a system of weights on $R$ with $W \approx V$. Then one can easily prove that the weighted space $CV_b(R, T)$ is metrizable. In case $T = C$, the metrizable weighted space $CV_b(R)$ is a special case of the result proved by Summers [10, Theorem 2.10].

Now, fix $g \in F_b(R)$ and $A \in B(T)$. For each $t \in R$, we define $\psi_t: R \to B(T)$ as $\psi_t(w) = e^{tg(w)}A$, for every $w \in R$. We can easily see that $\psi_t$ is a bounded operator-valued mapping from $R \to B(T)$ and hence by Remark 2.2(ii), $\psi_t$ induces a multiplication operator $M_{\psi_t}$ on the weighted metrizable locally convex Hausdorff space $CV_b(R, T)$.

3.1 Theorem. Let $g \in F_b(R)$, $A \in B(T)$ and let $\Pi_{A, g}: R \times CV_b(R, T) \to C(R, T)$ be the function defined by $\Pi_{A, g}(t, f) = M_{\psi_t}f$ for $t \in R$ and $f \in CV_b(R, T)$. Then $\Pi_{A, g}$ is a dynamical system on $CV_b(R, T)$.

Proof. Since $M_{\psi_t}$ is a multiplication operator on $CV_b(R, T)$ for every $t \in R$, we can conclude that $\Pi_{A, g}(t, f)$ belongs to $CV_b(R, T)$ whenever $t \in R$ and $f \in CV_b(R, T)$. Thus $\Pi_{A, g}$ is a function from $R \times CV_b(R, T)$ to $CV_b(R, T)$. It can be easily seen that $\Pi_{A, g}(0, f) = f$, and

$$\Pi_{A, g}(t + s, f) = \Pi_{A, g}(t, \Pi_{A, g}(s, f))$$

for all $t, s \in R$ and $f \in CV_b(R, T)$.

In order to show that $\Pi_{A, g}$ is a dynamical system on $CV_b(R, T)$, it is enough to show that $\Pi_{A, g}$ is a separately continuous map since joint continuity follows from [3, Theorem 1]. Let us first prove the continuity of
\[\Pi_{A,g}(t_n, f) \to \Pi_{A,g}(t, f)\] in \(CV_b(\mathbb{R}, T)\). Let \(v \in V\). Then
\[
\|\Pi_{A,g}(t_n, f) - \Pi_{A,g}(t, f)\|_v = \|\psi_{t_n}f - \psi_tf\|_v
= \sup\{v(w)\|\psi_{t_n}(w) - \psi_t(w)(f(w))\| : w \in \mathbb{R}\}
= \sup\{v(w)\|e^{t_n A}(f(w)) - e^{tA}(f(w))\| : w \in \mathbb{R}\}
\leq \sup\{v(w)\|e^{t_n A}(f(w)) - e^{tA}(f(w))\| : w \in \mathbb{R}\}
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\leq \sup\{v(w)\|e^{t_n A}(f(w)) - e^{tA}(f(w))\| : w \in \mathbb{R}\}
\leq \|f\|_v \to 0\] as \(|t_n - t| \to 0\).

This proves the continuity of \(\Pi_{A,g}\) in the first argument. Now, we shall prove the continuity of \(\Pi_{A,g}\) in the second argument. Let \(\{f_{\alpha}\}\) be a net in \(CV_b(\mathbb{R}, T)\) such that \(f_{\alpha} \to f\) in \(CV_b(\mathbb{R}, T)\). Then \(\|f_{\alpha} - f\|_v \to 0\) for every \(v \in V\). We shall show that
\[\Pi_{A,g}(t, f_{\alpha}) \to \Pi_{A,g}(t, f)\] in \(CV_b(\mathbb{R}, T)\).

For this, let \(v \in V\). Then
\[
\|\Pi_{A,g}(t, f_{\alpha}) - \Pi_{A,g}(t, f)\|_v = \|\psi_tf_{\alpha} - \psi_tf\|_v
= \sup\{v(w)\|\psi_t(w)(f_{\alpha}(w) - f(w))\| : w \in \mathbb{R}\}
\leq \sup\{v(w)\|f_{\alpha}(w) - f(w)\| : w \in \mathbb{R}\}
= \|f_{\alpha} - f\|_v \to 0\]

This proves the continuity of \(\Pi_{A,g}\) in the second argument and hence \(\Pi_{A,g}\) is a (linear) dynamical system on the weighted space \(CV_b(\mathbb{R}, T)\).

References


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