Remarks on triples in enriched categories

H. Wiesler and G. Calugareanu jr

Let V be a symmetric monoidal closed category with equalizers.
The V-triples T, T', ... in the enriched category A ,
together with suitably defined morphisms form a category
V-**Trip**(A) . The V-categories A^T, $A^{T'}$, ... and the
V-functors $R : A^{T'} \to A^T$ which are compatible with the
forgetful functors form a category V-Alg(A) .

In the subsequent note it is shown that V-Trip(A) is
isomorphic to the dual of V-Alg(A) and that the morphisms of
V-Alg(A) are inverse limit preserving V-functors.

0. Introduction

In [5], Frei considers the category of the triples in a category A ,
and of the triple morphisms, Trip(A) , and the category of the categories
of algebras A^T, $A^{T'}$, ... and of the functors which are compatible with
the forgetful functors, Alg(A) . He shows that Trip(A) is isomorphic
to the dual of Alg(A) and that the morphisms of Alg(A) preserve
(inverse) limits.

Let V be a symmetric monoidal closed category, in the
Eilenberg-Kelly [3] sense. In [3] the V-categories, which are also
called enriched categories in Day-Kelly [2], are constructed. The
V-triples in V-categories and the V-categories of algebras over
V-triples are constructed in Bunge [1]. Then one can consider

---

$V-\text{Trip}(A)$, the category whose objects are the V-triples in a V-category
A , and $V-\text{Alg}(A)$, the category whose objects are the V-categories of
V-algebras over A .

In this note, using the notion of V-limit in the Kelly sense [6]
too, we extend in this enriched V-context all the results obtained by
Frei.

1. Preliminaries

We use the notions of symmetric monoidal closed category V and
V-category, in the Eilenberg-Kelly sense [3]. A V-triple $T = (T, \eta, \mu)$
in a V-category A consists of a V-functor $T : A \rightarrow A$ and two
V-natural transformations $\eta : 1_A \rightarrow T$, $\mu : TT \rightarrow T$ so that:

(i) $\mu.\mu T = \mu.T\mu$, and

(ii) $\mu.T\eta = \mu.\eta T = 1_T$.

A V-triple morphism $\tau : T \rightarrow T'$ consists of a V-natural
transformation $\tau : T \rightarrow T'$ so that:

(i) $\tau.\eta = \eta'$, and

(ii) $\tau.\mu = \mu'.\tau T'.T\tau$.

Let A_o be the underlying category of A . A V-triple in A is a
triple in A_o too. Thus, one can construct the category of algebras
A_o^T , whose objects are pairs $[A, \xi]$ where $A \in A$, $\xi \in A_o(TA, A)$,
$\xi.\eta_A = 1_A$, $\xi.\mu_A = \xi.T\xi$, and whose morphisms are $[f] : [A, \xi] \rightarrow [B, \theta]$
where $f : A \rightarrow B$ in A_o and $f.\xi = \theta.Tf$. Bunge $([1], 2.2)$ shows that
A_o^T has a V-category structure, if V has equalizers (in what follows we
suppose this condition to be fulfilled), given by $A^T : \left(A_o^T\right)^{\text{op}} \times A_o^T \rightarrow V_o$;
in this structure $U^T : A_o^T \rightarrow A_o$, given by $[A, \xi] \mapsto A$ and $[f] \mapsto f$, has
a V-functor structure and has a V-adjoint F^T given at the underlying
level by $A \mapsto [TA, \mu_A]$ and $f \mapsto [Tf]$. The functor A^T is given on

objects by $U^T_{[A,\xi][B,\theta]} : A^T([A, \xi][B, \theta]) \rightarrow A(A, B)$, the equalizer of
the following pair

$$A(A, B) \xrightarrow{T_{AB}} A(TA, TB) \underset{A(\xi, B)}{\overset{A(TA, \theta)}{\rightrightarrows}} A(TA, B)$$

In this way the V-*category* A^T is well defined.

We recall that $R : B \rightarrow A$ is a B-adjoint for $S : A \rightarrow B$ if there
exists a V-natural family of isomorphisms $n_{AB} : A(RB, A) \simeq B(B, SA)$ in
V_0 , for each $A \in A$, $B \in B$.

Let $D : K \rightarrow A_0$ be a functor and let B be a V-category. The

family of morphisms $\left\{ L \xrightarrow{\lambda_K} DK \right\}_{K \in K}$ in B_0 is a *limit in* B (or a

V-limit) in the Kelly sense [6], if the family $\{B(B, L) \rightarrow B(B, DK)\}_{K \in K}$
is a limit in V_0 , for each $B \in B$.

LEMMA 1.1. *If a V-functor $T : A \rightarrow B$ has a V-adjoint $S : B \rightarrow A$
then T preserves the limits of A .*

Proof. Let $\{L \rightarrow DK\}$ be a limit in A . Then $\{A(A, L) \rightarrow A(A, DK)\}$
is a limit in V_0 , by the above definition, for each $A \in A$, and so
$\{A(SB, L) \rightarrow A(SB, DK)\}$ are also limits in V_0 , for each $B \in B$. Using
the V-natural isomorphism n , the families $\{B(B, TL) \rightarrow B(B, TDK)\}$ are
also limits in V_0 . Hence $\{TL \rightarrow TDK\}$ is a limit in B .   □

2.   The V-isomorphism theorem

PROPOSITION 2.1. *There is a functor $G : V\text{-Trip}(A) \rightarrow [V\text{-Alg}(A)]^{op}$
given by: $G(T) = A^T$ on objects, and on $\phi : T \rightarrow T'$ by a V-functor
$G\phi : A^{T'} \rightarrow A^T$, given by $(G\phi)[A, \xi'] = [A, \xi'.\phi_A]$ and by a unique
morphism in V_0*

$$(G\phi)_{[A,\xi'][B,\theta']} : A^{T'}([A, \xi'][B, \theta']) \rightarrow A^T([A, \xi'.\phi_A][B, \theta'.\phi_B])$$

for each $[A, \xi']$, $[B, \theta']$ *in* $A^{T'}$.

Proof. Using Diagram 1 on page 379, which commutes, since $U^{T'}_{[A,\xi'],[B,\theta']}$ is an equalizer and since ϕ is V-natural, one can find

$$A(\xi'.\phi_A, B)U^{T'}_{[A,\xi'][B,\theta']} = A(TA, \theta'.\phi_B).T_{AB}.U^{T'}_{[A,\xi'][B,\theta']} .$$

$U^{T}_{[A,\xi'.\phi_A][B,\theta'.\phi_B]}$ being an equalizer it follows that there exists a unique morphism $(G\phi)_{[A,\xi'][B,\theta']}$ so that

$$U^{T'}_{[A,\xi'][B,\theta']} = U^{T}_{[A,\xi'.\phi_A][B,\theta'.\phi_B]}.(G\phi)_{[A,\xi'][B,\theta']} .$$

As in [1], the $G\phi$'s defining equality taking place also at the underlying level, it follows that $G\phi$ is a V-functor. Being compatible with the forgetful functors, $G\phi$ is a morphism in $[V\text{-Alg}(A)]^{op}$.   □

PROPOSITION 2.2. *There is a functor* $H : [V\text{-Alg}(A)]^{op} \to V\text{-Trip}(A)$ *given by* $H(A^T) = T$ *and by* $H(R) : T \to T'$, *a V-natural transformation given in* A_0 *by the family,* $H(R)^0_A = U_0 a^{-1}_{A,RF'A}(\eta'_A)$ *where* $U^T_0 : A^T_0 \to A_0$, $a^{-1}_{A,RF'A} : A_0(A, T'_0A) \approx A^T_0(FA, RF'A)$ *are the underlying corresponding notions and we denote* F^T *and* $F^{T'}$ *simply by* F , F' .

Proof. According to [5] it remains to show that $H(R)$ is V-natural. Using the formula $a^{-1}_{X,[Y,\sigma]}(g) = [\sigma.Tg]$ for $g : X \to U^T([Y, \sigma]) = Y$ and by the remark $R([X, \xi']) = [X, R(\xi')]$ one can show that $H(R)^0_A = R_0(\mu'_A).T_0\eta'_A$. The V-naturality of $H(R)$ follows now from the V-naturality of η' and μ' and from the V-functoriality of R and T $([3], I, 10.2)$.

THEOREM 2.3. *The functors* G *and* H *are inverse to each other, that is,* $V\text{-Trip}(A)$ *is isomorphic to* $[V\text{-Alg}(A)]^{op}$.

Proof. As in [5].

$$A^{T'}([A, \xi'], [B, \theta']) \xrightarrow{\;U^{T'}_{[A,\xi'][B,\theta']}\;} A(A, B) \xrightarrow{\;A(\xi', B)\;}$$

$$A(A, B) \xrightarrow{\;T'_{AB}\;} A(T'A, T'B) \xrightarrow{\;A(T'A, \theta')\;} A(T'A, B)$$

$$A(T'A, T'B) \xrightarrow{\;A(\varphi_A, T'B)\;}$$

$$A(T'A, B) \xrightarrow{\;A(\varphi_A, B)\;} A(TA, B)$$

$$A(TA, TB) \xrightarrow{\;T_{AB}\;}$$

$$A(TA, \varphi_B)$$

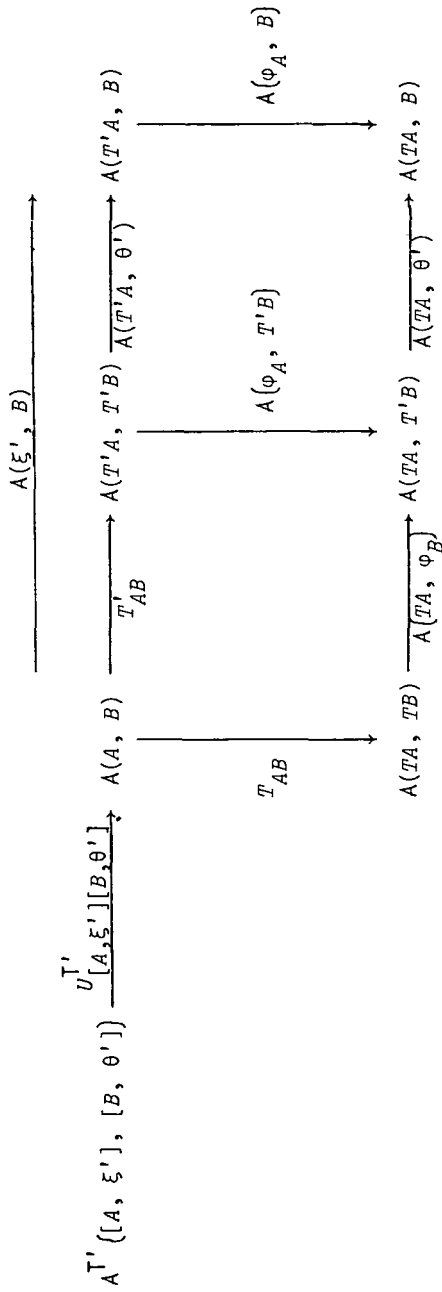$$A(TA, T'B) \xrightarrow{\;A(TA, \theta')\;} A(TA, B)$$

Diagram 1.

3. The morphisms of V-Alg(A)

PROPOSITION 3.1. *Let* T *be a* V-*triple in* A *and* $U : A^T \rightarrow A$ *be the corresponding underlying* V-*functor. Then a* V-*functor* $G : B \rightarrow A^T$ *preserves limits in* B *iff* UG *does.*

Proof.   Using Lemma 1.1, the condition is easily seen to be necessary, $F^T : A \rightarrow A^T$ being a V-adjoint for U . Conversely, let W be an object in V_o , $[A, \xi]$ an object in A_o^T and $\alpha_K : W \rightarrow A^T([A, \xi], GK)$, $K \in K$ a compatible (in K) family of morphisms in V_o . We show that there exists a unique morphism $f : W \rightarrow A^T([A, \xi], GL)$ so that

$$\alpha_K = A^T([A, \mu], G_o\lambda_K).f .$$

In Diagram 2, on page 381, all the squares commute, since $A(TA, -)$ is a functor, $G_o\lambda_K$ is an algebra morphism and since $\theta : TUGL \rightarrow UGL$ and $\theta_K : TUGDK \rightarrow UGDK$ are the T-structures of UGL and $UGDK$. Since UG preserves the limits of A , for each $A \in A$,

$A(A, UGL) \xrightarrow{A(1, U_oG_o\lambda_K)} A(A, UGDK)$ is a limit in V_o . Since α_K is a compatible family, there exists a unique morphism $g : W \rightarrow A(A, UGL)$ so that $U.\alpha_K = A(1, U_oG_o\lambda_K).g$ and hence we have another commuting square in Diagram 2.

Now, we show that g equalizes the pair $A(TA, \theta).T$ and $A(\xi, UGL)$. Indeed we have

$$A(1, UG\lambda_K).A(\xi, UGL).g = A(\xi, UGDK).A(1, UG\lambda_K).g = A(\xi, UGDK).U.\alpha_K =$$

$$A(TA, \theta_K).T.U.\alpha_K = A(TA, \theta_K).T.A(1, UG\lambda_K).g = A(1, UG\lambda_K).A(TA, \theta).T.g ,$$

and so the above statement follows from the fact that $A(1, UG\lambda_K)$ are limits in V_o . Hence there exists a unique morphism

$f : W \rightarrow A^T([A, \xi], GL)$ so that $g = U.f$. But,

$$U.\alpha_K = A(1, UG\lambda_K).g = A(1, UG\lambda_K).U.f = UA^T(1, G_o\lambda_K).f ,$$
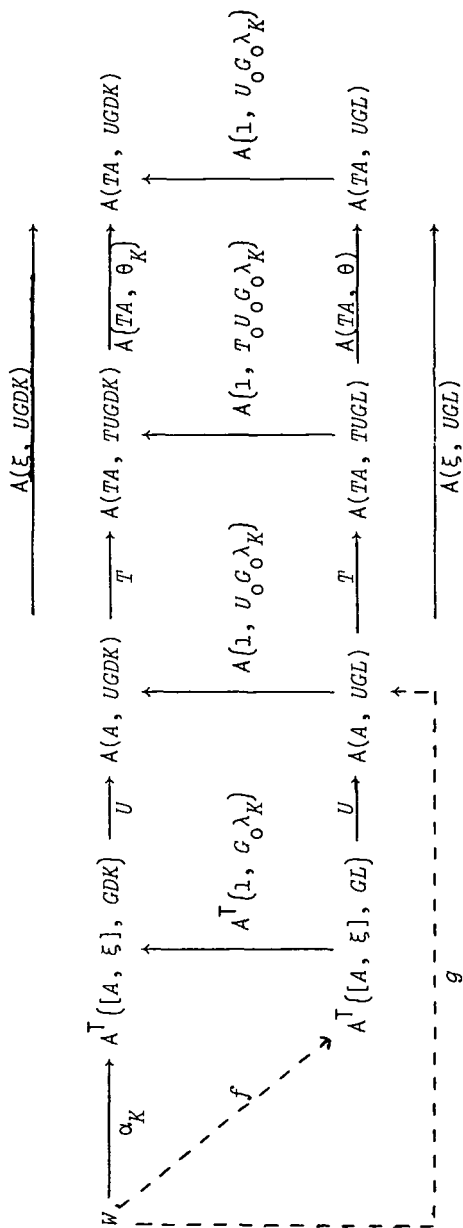
$$A(\xi,\ UGDK)$$

$$A^T([A,\ \xi],\ GDK) \xrightarrow{\ U\ } A(A,\ UGDK) \xrightarrow{\ T\ } A(TA,\ UGDK) \xrightarrow{A(TA,\ \theta_K)} A(TA,\ UGDK)$$

$$A^T(1,\ G_o\lambda_K) \qquad A(1,\ U_oG_o\lambda_K) \qquad A(1,\ T_oU_oG_o\lambda_K) \qquad A(1,\ U_oG_o\lambda_K)$$

$$A^T([A,\ \xi],\ GL) \xrightarrow{\ U\ } A(A,\ UGL) \xrightarrow{\ T\ } A(TA,\ TUGL) \quad A(TA,\ TUGDK)$$

$$A(TA,\ UGL) \xrightarrow{A(TA,\ \theta)} A(TA,\ UGL)$$

$$A(\xi,\ UGL)$$

α_K

f

g

W

Diagram 2.

and so, U being monic, the triangle in Diagram 2 commutes. In this way we already have the existence of the morphism $W \to A^T([A, \xi], GL)$ we were looking for.

The uniqueness follows from the fact that g is unique for its defining commutation and from the fact that if $f \neq f'$ then $Uf \neq Uf'$ and hence $A(1, UG\lambda_K).U.f \neq A(1, UG\lambda_K).U.f'$, the motivation of this last fact being already mentioned above.   □

The theorem and corollary which follow are immediate consequences of the last proposition.

THEOREM 3.2. *The morphisms in V-Alg(A) preserve limits.*

COROLLARY 3.3. *Let* T *be a V-triple in* A *,* $(\gamma, \varepsilon) : S \dashv T : B \to A$ *a V-adjoint pair which generates* T *, and* $L : B \to A^T$ *, the canonical V-functor given by* $LB = [TB, T\gamma_B]$ *and by* $L_{BB'}$ *, as in 2.3 in* [1]*, L preserves limits.*

References

[1] Marta C. Bunge, "Relative functor categories and categories of algebras", *J. Algebra* 11 (1969), 64-101.

[2] B.J. Day and G.M. Kelly, "Enriched functor categories", *Reports on the Midwest Category Seminar III* (Lecture notes in Mathematics, 106, 178-191. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[3] Samuel Eilenberg and G. Max Kelly, "Closed categories", *Proc. Conf. Categorical Algebra* (La Jolla, Calif., 1965), 421-562. (Springer-Verlag, Berlin, Heidelberg, New York, 1966).

[4] Samuel Eilenberg and John C. Moore, "Adjoint functors and triples", *Illinois J. Math.* 9 (1965), 381-398.

[5] A. Frei, "Some remarks on triples", *Math. Z.* 109 (1969), 269-272.

[6] G.M. Kelly, "Adjunction for enriched categories", *Reports on the*
       Midwest Category Seminar III (Lecture notes in Mathematics, 106,
       166-177. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

University of Cluj,
Cluj, Roumania.