EXTENSIONS OF Mccoy’S THEOREM

CHAN YONG HONG
Department of Mathematics and Research Institute for Basic Sciences,
Kyung Hee University, Seoul 131-701, Korea
e-mail: hcy@khu.ac.kr

NAM KYUN KIM∗
College of Liberal Arts and Sciences, Hanbat National University,
Daejeon 305-719, Korea
e-mail: nkkim@hanbat.ac.kr

and YANG LEE
Department of Mathematics Education, Pusan National University,
Pusan 609-735, Korea
e-mail: ylee@pusan.ac.kr

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Abstract. McCoy proved that for a right ideal A of $S = R[x_1, \ldots, x_k]$ over a ring $R$, if $r_S(A) \neq 0$ then $r_R(A) \neq 0$. We extend the result to the Ore extensions, the skew monoid rings and the skew power series rings over non-commutative rings and so on.


Over a commutative ring $R$, McCoy [4, Theorem 2] obtained the following in 1942: $f(x)$ is a zero divisor in $R[x]$ if and only if $f(x)c = 0$ for some non-zero $c \in R$, where $R[x]$ is the polynomial ring with indeterminate $x$ over $R$. But Weiner [9] showed that this theorem fails in non-commutative rings.

Based on these results, Nielsen [6] called a ring $R$ right McCoy when the equation $f(x)g(x) = 0$ implies $f(x)c = 0$ for some non-zero $c \in R$, where $f(x)$, $g(x)$ are non-zero polynomials in $R[x]$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy then the ring is called a McCoy ring. Nielsen [6, Theorem 2] proved that if a ring $R$ is reversible (i.e. for $a, b \in R$, $ab = 0$ implies $ba = 0$) then $R$ is McCoy.

As stated above, McCoy’s theorem fails in non-commutative rings. However McCoy [5] proved the following result.

THEOREM †. Let $R$ be a ring and $A$ a right ideal of $S = R[x_1, \ldots, x_k]$. If $r_S(A) \neq 0$ then $r_R(A) \neq 0$.

In 2002, Hirano [3, Theorem 2.2] proved independently that if for $f(x) \in R[x]$, $r_R(f(x)R[x]) \neq 0$ then $r_R(f(x)R[x]) \neq 0$.

On the other hand, McCoy’s theorem fails in the formal power series ring $R[[x]]$ over a commutative ring $R$ by [1, Example 3] in general. However, Gilmer [2] provided several conditions that are sufficient in order that the analogue of McCoy’s theorem should be valid in a commutative $R[[x]]$. Such conditions include the reducedness

∗ Corresponding author
and the von Neumann regularity of the total quotient ring, etc. Moreover, Fields [1, Theorem 5] proved that if $R$ is a commutative Noetherian ring in which $Q_1 \cap Q_2 \cap \cdots \cap Q_n = 0$ is a shortest primary representation of $0$, then $f(x)g(x) = 0$ implies $f(x)c = 0$ for some non-zero $c \in R$.

We extend, in this paper, Theorem † to the Ore extensions of several types, the skew monoid rings and the skew power series rings over non-commutative rings, and so on.

Throughout this paper, $R$ denotes associative ring with identity. We denote the right annihilator of $A$ in $R$ by $r_R(A)$, where $A$ is a subset of an extension of $R$. We assume that $\sigma$ is an automorphism of $R$ and $\delta$ is a $\sigma$-derivation of $R$. Recall that the Ore extension $R[x; \sigma, \delta]$ of a ring $R$ is the ring obtained by giving the polynomial ring over $R$ with the new multiplication $xr = \sigma(r)x + \delta(r)$ for any $r \in R$.

**Theorem 1.** Let $R$ be a ring and and $A$ a right ideal of $S = R[x; \sigma, \delta]$. If $r_S(A) \neq 0$ then $r_R(A) \neq 0$.

**Proof.** Let $g(x) = b_0 + b_1x + \cdots + b_nx^n$ be a non-zero element in $r_S(A)$ with minimal degree. Then $Ag(x) = 0$ and so $f(x)Sg(x) = 0$ for any $f(x) = a_0 + a_1x + \cdots + a_mx^m \in A$. Note that for any $r \in R$,

$$rx^i = x^i\sigma^{-i}(r)\left(\sum_{s+t=i-1} \sigma^{s}\delta^{t}(\sigma^{-i}(r))\right)x^{i-1} - \cdots - \left(\sum_{s+t=i-1} \delta^{s}\delta^{t}(\sigma^{-i}(r))\right)x - \delta^{i}(\sigma^{-i}(r)).$$

Then we can rewrite $f(x) = c_0 + xc_1 + \cdots + x^mc_m$. Thus we have the following:

$$(c_0 + xc_1 + \cdots + x^mc_m)R(b_0 + b_1x + \cdots + b_nx^n) = 0. \quad (*)$$

We will show that $f(x)b_j = 0$ for any $0 \leq j \leq n$. If $n = 0$, then we are done. Suppose that $n \geq 1$. From equation $(*)$, we have $c_mb_n = 0$. Then $f(x)R(c_mg(x)) \subseteq f(x)Rg(x) = 0$ and so equation $(*)$ becomes

$$(c_0 + xc_1 + \cdots + x^mc_m)c_mR(b_0 + b_1x + \cdots + b_nx^n) = 0.$$ 

By the choice of $g(x)$, we have $c_mb_0 + c_mb_1x + \cdots + c_mb_{n-1}x^{n-1} = 0$ and so $c_m\delta_j = 0$ for any $0 \leq j \leq n$. Assume that $c_i\delta_j = 0$, where $i = t + 1, \ldots, m$ and $0 \leq j \leq n$ and that for each $0 \leq i \leq t$, $c_ib_j \neq 0$ for some $j$. Then equation $(*)$ becomes

$$0 = f(x)Rg(x) = (c_0 + xc_1 + \cdots + x^tc_t)R(b_0 + b_1x + \cdots + b_nx^n).$$

Thus we also have $c_i\delta_j = 0$. Then $f(x)R(c_ig(x)) \subseteq f(x)Rg(x) = 0$ and so $f(x)R(c_ib_0 + c_ib_1x + \cdots + c_ib_{n-1}x^{n-1}) = 0$. By the choice of $g(x)$, we have $c_ib_0 + c_ib_1x + \cdots + c_ib_{n-1}x^{n-1} = 0$ and so $c_i\delta_j = 0$ for any $0 \leq j \leq n$, which is a contradiction. Consequently $n$ must be zero. Hence $f(x)b_0 = 0$ and therefore $Ab_0 = 0$ with $b_0 \neq 0$. \hfill $\square$

**Corollary 2.** For a ring $R$, let $T$ be $R[x; \sigma]$. $R[x, x^{-1}; \sigma]$ or $R[x; \delta]$ and $A$ a right ideal of $T$. If $r_T(A) \neq 0$ then $r_R(A) \neq 0$. 

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Recall that a monoid \( G \) is called a unique product monoid (simply, u.p.-monoid) if any two non-empty finite subsets \( A, B \subseteq G \) there exists \( c \in G \) uniquely presented in the form \( ab \) where \( a \in A \) and \( b \in B \). The class of u.p.-monoids is quite large and important (see [7] and [8] for details). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups.

Let \( R \) be a ring and \( G \) a u.p.-monoid. Assume that \( G \) acts on \( R \) by means of a homomorphism into the automorphism group of \( R \). We denote by \( \sigma_g(r) \) the image of \( r \in R \) under \( g \in G \). The skew monoid ring \( R * G \) is a ring which as a left \( R \)-module is free with basis \( G \) and multiplication defined by the rule \( gr = \sigma_g(r)g \).

**Theorem 3.** Let \( R \) be a ring, \( G \) a u.p.-monoid and \( A \) a right ideal of \( R * G \). If \( r_{R*G}(A) \neq 0 \) then \( r_G(A) \neq 0 \).

**Proof.** Let \( \beta = b_0h_0 + b_1h_1 + \cdots + b_nh_n \) be a non-zero element in \( r_{R*G}(A) \) with minimal non-zero terms, where \( b_j \in R \) and \( h_j \in G \). Then \( A\beta = 0 \) and so \( \alpha(R * G)\beta = 0 \) for any \( \alpha = a_0g_0 + a_1g_1 + \cdots + a_mg_m \in A \) with \( a_i \in R \) and \( g_i \in G \). Thus we have the following:

\[
(a_0g_0 + a_1g_1 + \cdots + a_mg_m)R(b_0h_0 + b_1h_1 + \cdots + b_nh_n) = 0.
\]

We will show that \( a_iR\sigma_{g_i}(b_j) = 0 \) for any \( 0 \leq i \leq m \) and \( 0 \leq j \leq n \). If \( n = 0 \), then

\[
0 = (a_0g_0 + a_1g_1 + \cdots + a_mg_m)r(b_0h_0) = a_0g_0(r)(b_0h_0) = a_0g_0(b_0h_0) = 0.
\]

By equation (**) we have \( a_mR\sigma_{g_m}(b_n) = 0 \). Since \( \sigma_{g_m}(\cdot) \) is an automorphism of \( R \), \( \sigma_{g_m}^{-1}(a_m)Rb_n = 0 \). Now for any \( s \in R \), \( \alpha R(\sigma_{g_m}^{-1}(a_m)s\beta) \subseteq \alpha R\beta = 0 \) and so \( \alpha R(\sigma_{g_m}^{-1}(a_m)s\beta) = 0 \), where

\[
\sigma_{g_m}^{-1}(a_m)s\beta = \sigma_{g_m}^{-1}(a_m)s(b_0h_0 + \sigma_{g_m}^{-1}(a_m)s(b_1h_1 + \cdots + \sigma_{g_m}^{-1}(a_m)s(b_nh_n - 1).\]

By the choice of \( \beta \), \( \sigma_{g_m}^{-1}(a_m)s\beta = 0 \), and hence \( a_mR\sigma_{g_m}(b_j) = 0 \) for any \( 0 \leq j \leq n \). After reordering if necessary, we may assume that \( p = m = q = n \). Then from equation (**) we have \( a_R\beta = (a_0g_0 + a_1g_1 + \cdots + a_1g_1)R(b_0h_0 + b_1h_1 + \cdots + b_nh_n) = 0 \). Since \( G \) is an u.p.-monoid, there exist \( p, q \) with \( 0 \leq p \leq t \) and \( 0 \leq q \leq n \) such that \( g_ph_q \) is uniquely presented by considering two subsets \( A = \{g_0, g_1, \ldots, g_t\} \) and \( B = \{h_0, h_1, \ldots, h_n\} \) of \( G \). After reordering if necessary, we may assume that \( p = t \) and \( q = n \). Then \( a_R\sigma_{g_i}(b_j) = 0 \) and so \( \sigma_{g_i}^{-1}(a_i)Rb_n = 0 \). Hence

\[
0 = \alpha R(\sigma_{g_i}^{-1}(a_i)s\beta) = \alpha R(\sigma_{g_i}^{-1}(a_i)s(b_0h_0 + \sigma_{g_i}^{-1}(a_i)s(b_1h_1 + \cdots + \sigma_{g_i}^{-1}(a_i)s(b_nh_n - 1).\]

By choice of \( \beta \), we have \( \sigma_{g_i}^{-1}(a_i)s(b_0h_0 + \sigma_{g_i}^{-1}(a_i)s(b_1h_1 + \cdots + \sigma_{g_i}^{-1}(a_i)s(b_nh_n - 1 = 0 \) and hence \( a_R\sigma_{g_i}(b_j) = 0 \) for any \( 0 \leq j \leq n \), which is a contradiction. Consequently \( n \) must be zero. Hence we have \( ab_0 = 0 \), and therefore \( Ab_0 = 0 \) with \( b_0 \neq 0 \). \( \Box \)

By [1, Example 3], McCoy’s theorem fails in the formal power series ring \( R[[x]] \) over a commutative ring \( R \). However, Gilmer [2] proved that a commutative ring satisfies
McCoy’s theorem for the formal power series ring case, when it is reduced (i.e. a ring with no non-zero nilpotent elements).

We here show that Theorem † holds for the skew power series rings and the skew Laurent power series rings over semi-prime rings, noting that Theorem † does not hold for the formal power series ring case in general.

**Lemma 4.** Let $R$ be a semi-prime ring. Then for $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x; \sigma]]$, $f(x)R[[x; \sigma]]g(x) = 0$ if and only if $a_i R \sigma^{i+1}(b_j) = 0$ for all $t, i, j \geq 0$.

**Proof.** It is enough to show the necessity. Suppose that $f(x)R[[x; \sigma]]g(x) = 0$, equivalently, $f(x)x^r g(x) = 0$ for any $r \in R$ and integer $t \geq 0$. So we have the following:

\[
\begin{align*}
a_0 \sigma^t(r b_0) &= 0, \\
a_0 \sigma^t(r b_1) + a_1 \sigma^{t+1}(r b_0) &= 0, \\
&\quad \ldots \\
a_0 \sigma^t(r b_n) + a_1 \sigma^{t+1}(r b_{n-1}) + \cdots + a_n \sigma^{t+n}(r b_0) &= 0. \tag{n}
\end{align*}
\]

From equation (0), $a_0 \sigma^t(r b_0) = 0$. In equation (1), we replace $r$ by $r b_0 s$ for any $s \in R$. Then $0 = a_0 \sigma^t(r b_0 s b_1) + a_1 \sigma^{t+1}(r b_0 s b_0) = a_1 \sigma^{t+1}(r b_0 s b_0)$. Thus $a_1 R \sigma^{t+1}(b_0) R \sigma^{t+1}(b_0) = 0$. Since $R$ is semi-prime, $a_1 R \sigma^{t+1}(b_0) = 0$ and so $a_1 \sigma^{t+1}(r b_0) = 0$ for all $r \in R$. From equation (1), $a_0 \sigma^t(r b_1) = 0$ for all $r \in R$. Now suppose that $a_i \sigma^{t+i}(r b_j) = 0$ for all $t \geq 0$ and $0 \leq i + j \leq n - 1$. In equation (n), we first replace $r$ by $r b_0 s$. Then $a_n \sigma^{t+n}(r b_0 s b_0) = 0$ and so $a_n \sigma^{t+n}(r b_0) = 0$ by the same method as above. So we have

\[
a_0 \sigma^t(r b_n) + a_1 \sigma^{t+1}(r b_{n-1}) + \cdots + a_{n-1} \sigma^{t+n-1}(r b_1) = 0. \tag{n'}
\]

Next, we replace $r$ by $r b_1 s$ for any $s \in R$ in equation (n'). Then $a_{n-1} \sigma^{t+n-1+i}(r b_1) = 0$ using $R$ is semi-prime. Continuing this process, we have $a_i \sigma^{t+i}(r b_j) = 0$ for all $t \geq 0$ and $0 \leq i + j \leq n$. By induction, we have $a_i \sigma^{t+i}(r b_j) = 0$ and therefore $a_i R \sigma^{t+i}(b_j) = 0$ for all $k, i, j \geq 0$. □

We also have the same result as Lemma 4 for the skew Laurent power series ring $R[[x, x^{-1}; \sigma]]$, using a slightly modified method. Now we have the following.

**Theorem 5.** Let $R$ be a semi-prime ring and $A$ a right ideal of $T = R[[x; \sigma]]$ or $T = R[[x, x^{-1}; \sigma]]$. If $r_T(A) \neq 0$ then $r_R(A) \neq 0$.

**Proof.** It is enough to show the skew power series ring case. Let $0 \neq g(x) = \sum_{j=0}^{\infty} b_j x^j \in r_T(A)$. Then $Ag(x) = 0$ and so $f(x)Tg(x) = 0$ for any $f(x) = \sum_{i=0}^{\infty} a_i x^i \in A$. By Lemma 4, we have $a_i R \sigma^{i+t}(b_j) = 0$ for any integers $t, i, j \geq 0$. Then $f(x)b_j = 0$ and therefore $Ac = 0$, where $c = b_j$ for any non-zero $b_j$. □

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