Canad. Math. Bull. Vol. **26** (3), 1983

ORIENTED MATROIDS AND GEOMETRIC SORTING

BY
RAUL CORDOVIL

ABSTRACT. Recently Goodman and Pollack [3] have proved a theorem which can be applied to encode in a compact form the order properties of an arbitrary configuration of points in $\mathbb{R}^d$. In this note we generalise this theorem to oriented matroids.

1. Introduction. We suppose that the reader has some familiarity with the fundamentals of the oriented matroid theory (see [1, 2, 4, 5, 6, 7, 8]). We will be using Las Vergnas' notation with only a small difference. We consider a matroid $M(E)$ *oriented*, and denote by $\mathcal{O}$ its *orientation*, if we know the families of their oriented circuits and oriented cocircuits. We remark, however, that the structure $\mathcal{O}$ is uniquely determined if we know only the family of oriented circuits [resp. oriented cocircuits] of $M(E)$ (see [1, 2]).

For a perfect understanding of our results we will need some terminology. We remark nevertheless that the paragraph 2 is self-contained: Theorem 2.6 (the main theorem of this note) is any easy consequence of our definition of orientations of the bases of a matroid (see Definition 2.1).

An element of the cartesian product X^{n+1} of a set X will be called an *n-simplex* (or *simplex of dimension n*). If $\sigma_1, \sigma_2, \ldots, \sigma_m$ are simplexes respectively of dimensions $d_1, d_2, \ldots, d_m$ then we denote by $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ the $(d_1+d_2+\cdots+d_m+m-1)$-simplex obtained by their concatenation.

DEFINITION 1.1 [8, 9]. A *n-ordered set* (for $n \geq 0$) is a pair (X, Φ), where X is a set and $\Phi, \Phi: X^{n+1} \to \{-1, 0, 1\}$, is a function not identically zero satisfying the conditions 1.2 and 1.3 below:

1.2. Φ is *alternating*, i.e. if the n-simplex σ is obtained from the n-simplex τ by interchanging two entries, then $\Phi(\sigma) = -\Phi(\tau)$;

1.3. If $\sigma = (a_0, a_1, \ldots, a_n)$ and $\tau = (b_0, b_1, \ldots, b_n)$ are two n-simplexes and $\Phi(\sigma) \cdot \Phi(\tau) = -1$, then for some i, $0 \leq i \leq n$,

$$\Phi(b_i, a_1, \ldots, a_n) \cdot \Phi(b_0, \ldots, b_{i-1}, a_0, b_{i+1}, \ldots, b_n) = -1.$$

If (X, Φ) is a n-ordered set we denote by $(X, -\Phi)$ the n-ordered set such that for every n-simplex σ, $-\Phi(\sigma) = -(\Phi(\sigma))$.

Received by the editors April 30, 1982 and, in revised form, September 10, 1982
1980 Mathematics Subject Classification: Primary 05B35, 51E30.

The notion of n-ordered set was introduced by L. G. Novoa [9] as a natural generalization of that of a totally ordered set. J. Lawrence [8] has proved that *a n-ordered set (E, Φ), where E is a finite set, has a canonical structure of oriented matroid $(M(E), \mathcal{O})$ and conversely, for every oriented matroid $(M(E), \mathcal{O})$ there exists exactly two n-ordered sets (E, Φ) and $(E, -\Phi)$ having $(M(E), \mathcal{O})$ as associate structure.* For convenience of the reader we summarise these results.

PROPOSITION 1.4 [8]. *Let (E, Φ) be a n-ordered set, where E is a finite set. Then $\mathcal{B} = \{B : B \in E^{n+1}, \Phi(B) \neq 0\}$ is the set of ordered bases of a matroid $M(E)$ of rank $n+1$.*

Proof. The set $\mathcal{B}$ is nonempty, because Φ is not identically zero. If $n = 0, 1$ the proposition is trivialy true. Suppose $n \geq 2$ and let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$. By reordering, if necessary, the elements of B_1 and B_2 we can suppose that $B_1 = \{x, a_1, \ldots, a_n\}$ and $\Phi(B_1) \cdot \Phi(B_2) = -1$. But in this case, by the condition 1.3, there exists b_i, $b_i \in B_2 - B_1$, such that $(b_i, a_1, \ldots, a_n) \in \mathcal{B}$. □

Theorem 1.5 below is equivalent to two of Lawrence's theorems (see [8], Theorems 3 and 6).

THEOREM 1.5 [8]. *Let (E, Φ) be a n-ordered set, where E is a finite set. Then there exists an orientation $\mathcal{O}$ of the matroid $M(E)$, determined by (E, Φ), such that:*

1.6. *For every cocircuit X of M and for every ordered base B of the hyperplane $E - \mathbf{X}$ the elements $x, y \in \mathbf{X}$ have the same sign in X if and only if $\Phi(x, B) = \Phi(y, B)$.*

Conversely, if $\mathcal{O}$ is an orientation of a matroid $M(E)$ of rank $n+1$, then there exist exactly two n-ordered sets (E, Φ) and $(E, -\Phi)$ such that the function Φ [resp. $-\Phi$] satisfy the condition 1.6. □

We remark that the converse part of Theorem 1.5 is an easy consequence of Las Vergnas' characterisation of oriented matroids in terms of orientations of their bases [6].

EXAMPLE 1.7. Let V be a n-dimensional vector space over an ordered field with a distinguished (ordered) base B. Let $\Phi : V^n \to \{1, -1, 0\}$ be the function such that $\Phi(B') = 1$ [resp. $\Phi(B') = -1$] when the transition matrix P from B to B' has positive determinant $|P| > 0$ [resp. negative determinant $|P| < 0$]. It is clear that the pair (V, Φ) is a $(n-1)$-ordered set. Likewise the pair (E, Φ) where E is a finite set spanning set of V and Φ' is the restriction of Φ to E^n, is a $(n-1)$-ordered set. Let $(M(E), \mathcal{O})$ be the oriented matroid determined by (E, Φ'). Then, for every base B'' of the $(n-1)$-dimensional subspace of V spanned by $E - \mathbf{X}$, the elements x, y, $x, y \in \mathbf{X}$, have the same sign in X if and only if $\Phi(x, B'') = \Phi(y, B'')$. (I.e. $x, y \in \mathbf{X}$ have the same sign in the cocircuit X if and only if they are on the same side of the vector subspace of V generated by

$E-\mathbf{X}$.) The matroid M is called the *oriented matroid on E determined by linear dependence in* V [1].

2. **The basic theorem of geometric sorting**

DEFINITION 2.1. Let E be a finite set and let $\mathcal{B}$ be a set of ordered subsets of size n of E (i.e., $\mathcal{B} \subseteq E^n$). By definition we say that the triple $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$, where $\mathcal{B}_1 \cup \mathcal{B}_2$ is a partition of $\mathcal{B}$, is an *orientation of the bases* (*of a matroid*) if the conditions 2.2 and 2.3 below are satisfied. (If $B, B' \in \mathcal{B}$ we write $B \sim B'$ [resp. $B \not\sim B'$] when $B, B' \in \mathcal{B}_i$ [resp. $B \in \mathcal{B}_i$, $B' \in \mathcal{B} - \mathcal{B}_i$], $i = 1$ or 2.):

2.2. *If B' is obtained from B by interchanging two entries, then $B \not\sim B'$*;

2.3. *If* $(a_1, a_2, \ldots, a_n) \not\sim (b_1, b_2, \ldots, b_n)$ *there exists i, $1 \le i \le n$, such that* $(b_i, a_2, \ldots, a_n) \not\sim (b_1, \ldots, b_{i-1}, a_1, b_{i+1}, \ldots, b_n)$.

REMARK 2.4. If $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$ is an *orientation of the bases* (of a matroid) then $\mathcal{B}$ is the set of (ordered) bases of a matroid M. (The proof is similar to the proof of Proposition 1.4.) By Theorem 1.5 it is also clear that an orientation $\mathcal{O}$ of the matroid M is also determined. Example 1.7 establishes that this notion of orientation of bases reduces to the usual one when $(M, \mathcal{O})$ is an oriented matroid determined by linear dependence in a vector space over an ordered field. (In this last case it is possible to prove shortly the Conditions 2.2 and 2.3. See [8] for more details.)

Proposition 2.5 below is a technical result necessary to prove our main result (Theorem 2.6). If X is a set and $\mathcal{B} \subseteq X^n$, $a \in X$ then we note by $\mathcal{B}/a$ [resp. $\mathcal{B} \setminus a$] the set $\{B - \{a\} : B = (a = b_1, \ldots, b_n) \in \mathcal{B}\}$ [resp. $\{B : B \in \mathcal{B}$ and $a \notin B\}$].

PROPOSITION 2.5. *Let $(M(E), \mathcal{O})$ be an oriented matroid and let $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$ be the orientation of the bases of M determined by $\mathcal{O}$. Then $(\mathcal{B}/a, \mathcal{B}_1/a, \mathcal{B}_2/a)$ [resp. $(\mathcal{B} \setminus a, \mathcal{B}_1 \setminus a, \mathcal{B}_2 \setminus a)$] is the orientation of the bases of the matroid M/a [resp. $M \setminus a$] determined by the orientation $\mathcal{O}/a$ [resp. $\mathcal{O} \setminus a$].*

Proof. The proof of the statement concerning the matroid $M \setminus a$ is routine. We prove that concerning M/a. It is clear that $(\mathcal{B}/a, \mathcal{B}_1/a, \mathcal{B}_2/a)$ verifies Condition 2.2. We prove 2.3. If $n = 1, 2$ then 2.3 is trivially true. Suppose $n \ge 3$ and let $(s_1, \ldots, s_{n-1}) \in (\mathcal{B}_1/a)$, $(t_1, \ldots, t_{n-1}) \in (\mathcal{B}_2/a)$. Then $(a, s_1, \ldots, s_{n-1}) \in \mathcal{B}_1$, $(a, t_1, \ldots, t_{n-1}) \in \mathcal{B}_2$ and by the Condition 2.2 $(s_1, a, s_2, \ldots, s_{n-1}) \in \mathcal{B}_2$ and $(t_1, a, t_2, \ldots, t_{n-1}) \in \mathcal{B}_1$. But in this case, by the Conditions 2.2 and 2.3 related to $(\mathcal{B}, \mathcal{B}_1' = \mathcal{B}_2, \mathcal{B}_2' = \mathcal{B}_1)$ there exists t_i, $1 \le i \le n$, such that $(t_i, a, s_2, \ldots, s_{n-1}) \in \mathcal{B}_2$ and $(t_1, a, t_2, \ldots, t_{i-1}, s_1, t_{i+1}, \ldots, t_{n-1}) \in \mathcal{B}_1$. Then $(t_i, s_2, \ldots, s_{n-1}) \in \mathcal{B}_1/a$ and $(t_1, \ldots, t_{i-1}, s_1, t_{i+1}, \ldots, t_{n-1}) \in \mathcal{B}_2/a$.

We leave to the reader the proof that $(\mathcal{B}/a, \mathcal{B}_1/a, \mathcal{B}_2/a)$ is the orientation of the bases of the matroid M/a determined by $\mathcal{O}/a$. $\square$

We remark that in the proof of Theorem 2.6 below we use only the fact that if $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$ is an orientation of bases then $(\mathcal{B}/a, \mathcal{B}_1/a, \mathcal{B}_2/a)$ and $(\mathcal{B}\setminus a, \mathcal{B}_1\setminus a, \mathcal{B}_2\setminus a)$ are also orientation of bases. Theorem 2.6 generalises the "basic theorem of geometric sorting" of Goodman and Pollack (see [3] Theorem 1.8).

THEOREM 2.6. *Let $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$ be an orientation of bases. Suppose $\mathcal{B} \subseteq E^n$ and let $\mathcal{T} = \bigcup_{x\in E} \mathcal{B}/x$. Then $(\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2)$ is uniquely determined if we know $\mathcal{B}$, and for every I, $I \in \mathcal{T}$, we know the number of elements y of E such that $I \in \mathcal{B}_1/y$.*

Proof. The proof is by induction on $|E| = m$. If $m = 1$ the theorem is trivially true. Suppose $m \geq 2$ and let a be an element of E. By hypothesis, for every $I \in E^{n-2}$, we know the number of elements x fo E such that $(a, I) \in \mathcal{B}_1/x$ and also the number of elements y of E such that $(y, a, I) \in \mathcal{B}$. But in this case we know also the number of elements z of E such that $(a, I) \in \mathcal{B}_2/z$. By Condition 2.2 this is equivalent to the knowledge of the number of elements z of E such that $(z, I) \in \mathcal{B}_1/a$. By Proposition 2.5 $(\mathcal{B}/a, \mathcal{B}_1/a, \mathcal{B}_2/a)$ is an orientation of bases. Then, by induction hypothesis relative to $(\mathcal{B}/a, \mathcal{B}_1/a, \mathcal{B}_2/a)$, $\mathcal{B}_1/a$ is uniquely determined. i.e. we know the ordered bases B, $B \in \mathcal{B}_1$, such that their first element is a. As, by Proposition 2.5, $(\mathcal{B}\setminus a,\ \mathcal{B}_1\setminus a,\ \mathcal{B}_2\setminus a)$ is also an orientation of bases, Theorem 2.6 follows by induction hypothesis relative to $(\mathcal{B}\setminus a, \mathcal{B}_1\setminus a, \mathcal{B}_2\setminus a)$. □

RERERENCES

1. R. Bland and M. Las Vergnas, *Orientability of matroids*, J. Combinatorial Theory B **24** (1978), 94–123.

2. J. Folkman and J. Lawrence, *Oriented matroids*, J. Combinatorial Theory B **25** (1978), 199–236.

3. J. E. Goodman and R. Pollack, *Multidimensional sorting*, preprint.

4. M. Las Vergnas, *Matroïdes orientables*, *C.R. Acad. Sci. Paris Sér. A*, **280** (1975), 61–64.

5. M. Las Vergnas, *Convexity in oriented matroids*, J. Combinatorial Theory *B* **29** (1980), 231–243.

6. M. Las Vergnas, *Bases in oriented matroids*, J. Combinatorial Theory B **25** (1978), 283–289.

7. M. Las Vergnas, *Extensions ponctuelles d'une géométrie orientée*, dans "Problèmes combinatoires et théorie des graphes" Actes du Colloque International C.N.R.S. no. 260, 263–268, Orsay 1976, Paris 1978.

8. J. Lawrence, *Oriented matroids and multiply ordered sets*, to appear in Linear Algebra and its Applications.

9. L. G. Novoa, *On n-ordered sets and order completeness*, Pac. J. of Math. **15** (1965), 1337–1345.

CENTRO DE FÍSICA DA MATÉRIA CONDENSADA (INIC)
AV. PROF. GAMA PINTO 2,
1699 LISBOA CODEX, PORTUGAL

CENTRO DE MATEMÁTICA DA UNIVERSIDADE DE COIMBRA (INIC)
UNIVERSIDADE DE COIMBRA,
3000 COIMBRA, PORTUGAL