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Divergence type of some subgroups of finitely generated Fuchsian groups

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Abstract. Let Γ be a finitely generated discrete subgroup of the isometries of the hyperbolic plane H^2 with at least one parabolic element. We prove that, if Γ_1 is a subgroup of Γ with Γ/Γ_1 abelian, the 'critical exponent' of Γ_1 is the same as that of Γ. We give necessary and sufficient conditions – in terms of the rank of Γ/Γ_1, the critical exponent of Γ, and the image of parabolic elements of Γ in Γ/Γ_1 – for the Poincaré series of Γ_1 to diverge at the critical exponent.

Introduction

The aim of this paper is to determine when a subgroup Γ_1 of a finitely generated discrete group of isometries Γ of 2-dimensional hyperbolic space H^2, with $\Gamma_1 \lhd \Gamma$, Γ/Γ_1 abelian, is *of divergence type* [**8**], i.e. whether or not the series

$$\sum_{\gamma\in\Gamma_1} \exp\{-\alpha(x, \gamma x)\}$$

diverges at $\alpha = \delta(\Gamma_1)$, for $\delta(\Gamma_1)$ the supremum of the α for which the series diverges, where $x \in H^2$ is any fixed point and $(x, \gamma x)$ denotes hyperbolic distance between x and γx. Equivalently (for $\delta(\Gamma_1) > \frac{1}{2}$), we wish to determine whether the geodesic flow $\{\phi_t\}$ on the unit tangent bundle $\mathrm{UT}(H^2/\Gamma_1)$ of H^2/Γ_1 is ergodic with respect to certain natural measures [**8**]. We restrict our discussion to the case of H^2/Γ having cusps. The case of H^2/Γ without cusps was dealt with in [**6**].

Recently, Lyons & McKean [**2**] proved that, if Γ is the fundamental group of the thrice-punctured sphere, and Γ_1 is the commutator subgroup of Γ, then hyperbolic Brownian motion on H^2/Γ_1 is not recurrent, and that Γ_1 is not of divergence type (equivalent conditions by [**8**]) (see also [**3**]). This is the sort of example we have in mind. However, we do not restrict the critical exponent of Γ_1 to be 1 (as in the cited example). The main results are summarized in theorem 1.

THEOREM 1. *Let* Γ *be a discrete finitely generated group of isometries of* H^2. *Let* $\Gamma_1 \lhd \Gamma$, $\Gamma/\Gamma_1 \simeq \mathbb{Z}^v$, *and let* $\boldsymbol{\theta}: \Gamma \xrightarrow[onto]{} \mathbb{Z}^{v_1}$, $\boldsymbol{\phi}: \Gamma \xrightarrow[onto]{} \mathbb{Z}^{v_2}$ $(v = v_1 + v_2)$ *be two homomorphisms with* $\Gamma_1 = \mathrm{Ker}\,(\boldsymbol{\theta} \oplus \boldsymbol{\phi})$, *where, if* $x_1 \cdots x_r$ *are parabolic elements of* Γ *corresponding to the r cusps of* H^2/Γ *(*$x_1 \cdots x_r$ *are unique up to taking inverses, and*

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conjugation), *then* $\boldsymbol{\phi}(x_i)=0$ *for all i, and* $\langle\boldsymbol{\theta}(x_1),\ldots,\boldsymbol{\theta}(x_r)\rangle$ *is syndetic in* $\mathbb{Z}^{v_1}$. (*Clearly any homomorphism onto* $\mathbb{Z}^v$ *can be put in the form* $\boldsymbol{\theta}\oplus\boldsymbol{\phi}$.) *Then* $\delta(\Gamma)=\delta(\Gamma_1)=\delta$, *say.*

If $\delta<1$, Γ_1 *is of divergence type if and only if* $v_1=0$ *and* $v_2\le 2$. *If* $\delta=1$, Γ_1 *is of divergence type if and only if either* $v_1\le 1$ *and* $v_2=0$, *or* $v_1=0$ *and* $v_2\le 2$.

The proof uses symbolic dynamics and the same general method, with modification, as in [**6**]. Because there is no developed theory of symbolic dynamics for cusped manifolds in higher dimensions, the methods only work in dimension 2. However, as in [**6**], the results can be extended to 'finitely determined sub-abelian' subgroups of Γ. We do not give the details here (nor the definitions) – the analogue with § 5 of [**6**] is fairly exact. But, for example, if H^2/Γ is a 1-cusped surface of genus $g\ge 1$, with $a_1\cdots a_g$, $b_1\cdots b_g$ the free generators of Γ and $\prod_{i=1}^{g}[a_i,b_i]$ representing the cusp, where $[a_i,b_i]=a_ib_ia_i^{-1}b_i^{-1}$, and if

$$\Gamma_1=\{\text{words in }\{a_i,b_i,a_i^{-1},b_i^{-1};\ i=1,\ldots,g\}:\text{sum of }a_1\text{-powers}=\text{sum of }b_1\text{-powers}=0\},$$

so that

$$\mathscr{F}=\bigcup_{i=2}^{g}\{a_i,b_i,a_i^{-1},b_i^{-1}\}\cup\bigcup_{m,n\in\mathbb{Z}\setminus\{0\}}(a_1^mb_1^na_1^{-m}b_1^{-n},\ b_1^na_1^mb_1^{-n}a_1^{-m}\}$$

is a set of free generators and inverses of generators of Γ_1, and if

$$\Gamma_2=\{\text{words in the elements of }\mathscr{F}:\text{ sum of }[a_1,b_1]\text{-powers}=0\},$$

then the generalized theorem 1 gives Γ_2 is of divergence type with $\delta(\Gamma)=\delta(\Gamma_2)=1$.

Formulation of the symbolic dynamics

In some ways, the presence of a cusp on H^2/Γ actually makes things easier. Throughout this paper, we assume without loss of generality that no $\gamma\in\Gamma$ fixes any points of H^2, so that H^2/Γ is a manifold with Γ as its fundamental group. Since we assume H^2/Γ has at least one cusp, H^2/Γ is topologically either a k-holed sphere ($k\ge 3$) or a k-holed surface of genus ≥ 1 ($k\ge 1$). So Γ must be a free group, and H^2/Γ has a geodesic triangulation – i.e. the edges of the triangulation are geodesics – with vertices only at the holes. (Note that, besides one or more cusps, some of the 'holes' may be ends of infinite volume.) The point of this is that H^2/Γ has a fundamental region F on H^2, bounded by geodesic arcs and arcs on the boundary of H^2 (possibly) with all corners on the boundary, and the images of F under Γ which are adjacent to F are $a_1F, a_2F,\ldots,a_sF, a_1^{-1}F,\ldots,a_s^{-1}F$, for some free generating set $\{a_1\cdots a_s\}$ of Γ. For instance, in the example in figure 1, F is the fundamental region of a 3-holed sphere with 2 cusps and 1 infinite volume end.

Patterson [**5**] proves that any finitely generated group Γ is of divergence type. By [**1**] the presence of a cusp means the critical exponent $\delta(\Gamma)$ satisfies $\frac{1}{2}<\delta(\Gamma)\le 1$. Equivalently to the divergence type condition, the geodesic flow $(\mathrm{UT}\,(H^2/\Gamma),\{\phi_t\},\mu_\nu)$ is ergodic, where μ_ν is the geodesic-flow invariant measure on $\mathrm{UT}\,(H^2/\Gamma)$ naturally corresponding to the (unique) Γ-invariant conformal density ν of dimension $\delta(\Gamma)$ on the limit set L_Γ of Γ [**8**].

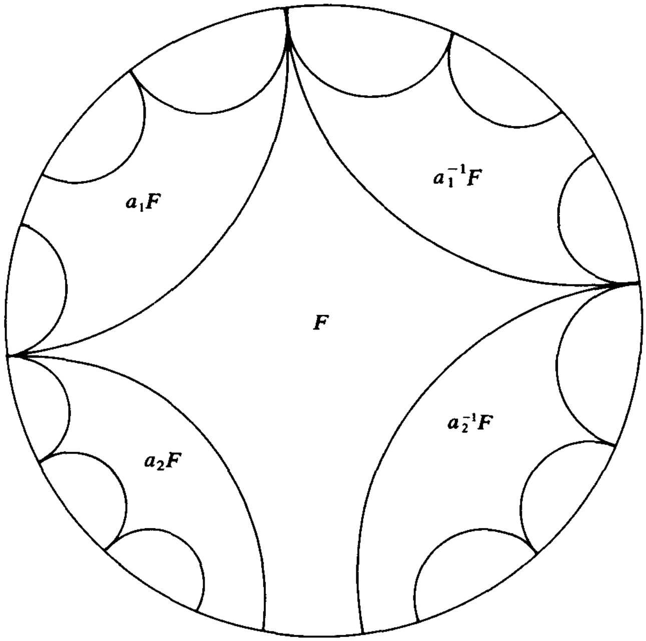

FIGURE 1

We now describe a 'subshift of finite type' (Y, σ) on an infinite set of symbols $\mathcal{K} = \mathcal{K}^{-1} \subseteq \Gamma$ such that $(\mathrm{UT}\,(H^2/\Gamma), \{\phi_t\}, \mu_\nu)$ is measure-theoretically isomorphic to the suspension under a positive (unbounded) function of a *Gibbs σ-invariant probability measure* (also denoted μ_ν) on Y. σ is defined on $Y \subseteq \mathcal{K}^{\mathbf{Z}}$ by $\sigma(\{x_i\}) = \{x_{i+1}\}$. Y is also invariant under τ, where $\tau(\{x_i\}) = \{x_{-i}^{-1}\}$, and $\tau_* \mu_\nu = \mu_\nu$.

The symbols of $\mathcal{K}$ are obtained as follows. Each cusp vertex Z_i of F is fixed by a unique (up to taking its inverse) parabolic element of Γ, which is a word of minimal length n_i in $\{a_1 \cdots a_s, a_1^{-1} \cdots a_s^{-1}\}$. Choose $N > 1$ such that $n_i | N$, $i = 1 \cdots r$ (if r is the number of cusps on F). Let $c_1 \cdots c_r$, $c_1^{-1} \cdots c_r^{-1}$ denote the parabolic elements of length N fixing the cusps of F, and $d_1 \cdots d_t$, $d_1^{-1} \cdots d_t^{-1}$ the other words of length N. Suppose a geodesic in H^2 passes successively through

$g_N^{-1} g_{N-1}^{-1} \cdots g_1^{-1} F, g_N^{-1} g_{N-1}^{-1} \cdots g_2^{-1} F, \ldots, g_N^{-1} F, F, h_1 F, h_1 h_2 F, \ldots, h_1 h_2 \cdots h_N F,$

where $g_1 \cdots g_N$, $h_1 \cdots h_N$ are words of length N in $\{a_1 \cdots a_s, a_1^{-1} \cdots a_s^{-1}\}$ (so $g_i \neq g_{i+1}^{-1}$, $h_i \neq h_{i+1}^{-1}$). Then $g_N^{-1} \neq h_1$. Moreover, the geodesic has endpoints a Euclidean distance apart bounded away from zero (in the disk model for H^2) unless $g_1 \cdots g_N = h_1 \cdots h_N = c$, where $c = c_i$ or c_i^{-1}, some $i = 1, \ldots, r$. Then

$$\mathcal{K} = \bigcup_{i=1}^{r} \bigcup_{n \in \mathbf{Z} \setminus \{0\}} \{c_i^n\} \cup \{d_1 \cdots d_t, d_1^{-1} \cdots d_t^{-1}\}$$

$$Y = \{\{x_i\} \in \mathcal{K}^{\mathbf{Z}} : x_i x_{i+1} \text{ is admissible}\}$$

where admissibility is defined by the following: ef is admissible if and only if

(i) the last element of $\{a_1 \cdots a_s, a_1^{-1} \cdots a_s^{-1}\}$ in the word e is not the inverse of the first element of the word f;

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(ii) if $e = c_k^n$, some k, n, then $f \neq c_k^m$, any m.

The realization of (UT (H^2/Γ), $\{\phi_t\}$, μ_ν) as a suspension of (Y, σ, μ_ν) is as follows. μ_ν-almost all geodesics in H^2/Γ have a unique lift in H^2 passing through a bi-infinite sequence of fundamental regions

$$\cdots f_{-1}^{-1} f_{-2}^{-1} F, f_{-1}^{-1} F, F, f_0 F, f_0 f_1 F, f_0 f_1 f_2 F, \cdots$$

where $\quad f_i \in \{a_1 \cdots a_s, a_1^{-1} \cdots a_s^{-1}\}$ and $f_{i+1} \neq f_i^{-1}$ for any i.

These symbols $\cdots f_{-1}, f_0, f_1 \cdots$ can be uniquely grouped into words of length N in $\{a_1 \cdots a_s, a_1^{-1} \cdots a_s^{-1}\}$ such that f_0 is the first element of its word. The symbols of one of the new sequences $\{g_i\}$ come from $\{c_1 \cdots c_r, c_1^{-1} \cdots c_r^{-1}\} \cup \{d_1 \cdots d_t, d_1^{-1} \cdots d_t^{-1}\}$. Now the symbols $\{g_i\}$ can be uniquely regrouped to give a bi-infinite sequence of elements of $\mathscr{K}$. The set of sequences thus obtained is residual in Y and, of course, has full μ_ν-measure in Y. See also Series [**7**].

The measure μ_ν on Y is *Gibbs* in the sense of [**6**]. The proof is the same as in [**6**], using the fact stated here that if a geodesic passes successively through $e^{-1}F, F, fF$, where, in the regrouping, e is the last symbol of a word of $\mathscr{K}$ and f the first symbol of another word of $\mathscr{K}$, then the endpoints are a Euclidean distance apart bounded away from zero. Also lemma 2 is used to show finiteness of the measure μ_ν on Y (and also something more, to be used later). Since μ_ν is Gibbs, it has a very good approximation by Markov measures and the same general method as [**6**] might work.

LEMMA 2. *There exists a constant $b_i > 0$ such that*

$$\mu_\nu([c_i^n]) = \frac{b_i}{|n|^{2\delta}}\left(1 + O\left(\frac{1}{|n|}\right)\right),$$

where ν is a Γ-invariant conformal density of dimension δ on L_Γ (i.e. $d\gamma_*\nu(\xi)/d\nu = |\gamma'(\xi)|^\delta$, $\xi \in L_\Gamma$, where $\gamma_*\nu(f) = \nu(f \circ \gamma^{-1})$) and $[c_i^n] = \{\{x_j\} \in Y : x_0 = c_i^n\}$.

Proof. Y can be identified measure-theoretically with the set of $(\xi, \eta) \in L_\Gamma \times L_\Gamma$ for which the geodesic from ξ to η passes through the interior of F, and then μ_ν identifies with the measure $d\nu(\xi)\, d\nu(\eta)/|\xi - \eta|^{2\delta}$ on this subset of $L_\Gamma \times L_\Gamma$, with ν suitably normalized [**8**]. $[c_i^n]$ then identifies with $U \times c_i^{n-1} V$ as shown, for $n \geq 1$, in figure 2. (The case $n \leq -1$ is similar. Recall that $c_i F$ is the Nth region round the cusp from F, so not next to F, since $N > 1$.)

U is bounded away from $\bigcup_{n \geq 0} c_i^n V$. Let η_0 be the fixed point of c_i. The maximum distance between η_0 and $c_i^n V$ (Euclidean distance for the disk) is $O(1/n^2)$ (since this is the size of the derivative of c_i^n over most of S^1). So

$$\left|\mu_\nu([c_i^n]) - \nu(c_i^{n-1} V) \int_U \frac{d\nu(\xi)}{|\xi - \eta_0|^{2\delta}}\right| \leq \frac{A\nu(c_i^{n-1} V)}{n^2}$$

(some A). Since ν is a conformal density,

$$\nu(c_i^{n-1} V) = \int \chi_V(c_i^{-(n-1)}\xi)\, d\nu(\xi) = \int \chi_V(\xi) |(c_i^{n-1})'(\xi)|^\delta\, d\nu(\xi).$$

Let x be a conformal map of the disk to the upper half plane, mapping c_1 to a parabolic transformation fixing 0, and V to the interval $[1, 2] \subseteq \mathbb{R}$, and let ν_1 be the

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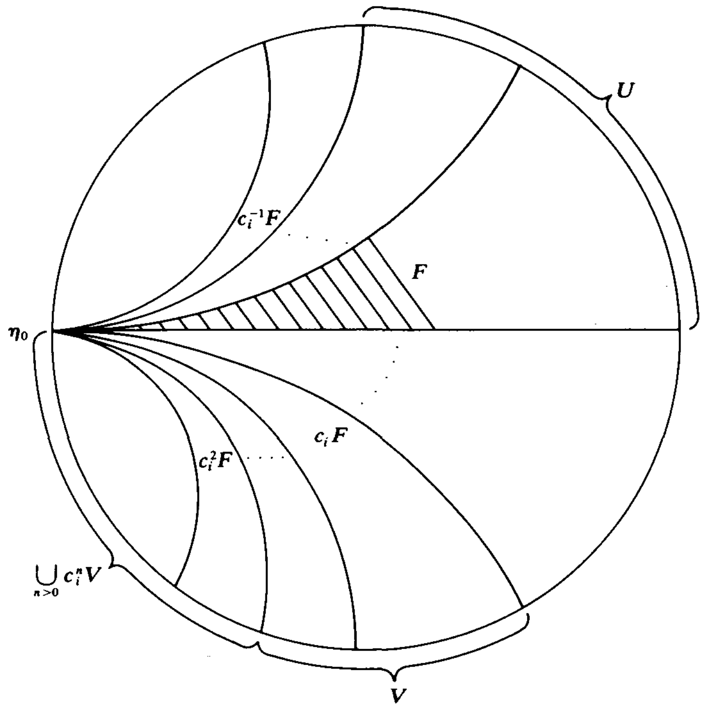

FIGURE 2

measure on $\mathbb{R}\cup\{\infty\}$ given by $\nu_1(f)=\nu(f\circ x)$. Then

$$\nu(c_i^{n-1}V)=\int_1^2 \frac{g(x)}{(1+a(n-1)x)^{2\delta}}\,d\nu_1(x)$$

for some $a>0$ and some smooth positive function g. Then

$$\nu(c_i^{n-1}V)=\frac{1}{n^{2\delta}}\left\{\int_1^2 \frac{g(x)}{(ax)^{2\delta}}\,d\nu_1(x)+O(1/n)\right\}$$

as required. □

Given the symbolic dynamics, we can now reformulate theorem 1, using the notation of theorem 1. The proof of (1) of theorem 3 is as in §§1, 2 of [**6**].

THEOREM 3. (1) *Let* $S_k=\sum\{\mu_\nu([e_0\cdots e_{k-1}]$: $e_i\in\mathcal{K}$, $[e_0\cdots e_{k-1}]\subseteq Y$ *and* $\neq\varnothing$, *and* $e_0e_1\cdots e_{k-1}\in\Gamma_1$ *i.e.* $(\boldsymbol{\theta}+\boldsymbol{\phi})(e_0e_1\cdots e_{k-1})=0\}$ *where*

$$\Gamma_1\leq\Gamma \quad \textit{and} \quad [e_0\cdots e_{k-1}]=\{\{x_i\}\in Y\colon x_i=e_i,\ 0\leq i\leq k-1\}.$$

Then $(\mathrm{UT}\,(H^2/\Gamma_1),\{\phi_t\},\mu_\nu)$ *is ergodic if and only if* $\sum_{k=1}^{\infty} S_k=\infty$, *where* μ_ν *is the unique measure on* UT (H^2/Γ_1) *such that local inverses of the natural projection onto* (UT (H^2/Γ), μ_ν) *are measure-preserving. Equivalently,* Γ_1 *is of divergence type if and only if* $\sum_{k=1}^{\infty} S_k=\infty$.

(2) *There exist constants* A, $B>0$ *such that*

$$\frac{A}{k^{v_i(2\delta-1)+\frac{1}{2}v_2}} \le S_k + S_{k+1} \le \frac{B}{k^{v_i/2(\delta-1)+\frac{1}{2}v_2}}$$

where $\delta = \delta(\Gamma)$.

Note. Since theorem 3 implies $S_k + S_{k+1} \ge c \sum_{\gamma \in A_k \le \Gamma} \exp\{-\delta(x, \gamma x)\}$ for some constant $c>0$, and $\Gamma_1 = \bigcup_{k=1}^{\infty} A_k$ a disjoint union with $A_k \subseteq \{\gamma: (x, \gamma x) \ge dk\}$, some constant d (essentially the same proof as 1.10 of [**6**]), $\delta(\Gamma_1) = \delta = \delta(\Gamma)$.

The proof of theorem 3. In modifying the proof of 4.7 in [**6**] to prove theorem 3, we have first to recall some notation, and define some new notation.

Notation. (1) Let ε_m denote the set of non-empty m-cylinders $[e_0 \cdots e_{m-1}]$ of Y, where

$$[e_0 \cdots e_{m-1}] = \{\{x_i\} \in Y: x_i = e_i,\ 0 \le i \le m-1\}$$

for $e_i \in \mathcal{K}$. ε_m can be regarded as a subset of Γ (by multiplying the symbols) so the homomorphisms $\boldsymbol{\theta}$, $\boldsymbol{\phi}$ are defined on ε_m,

$$\boldsymbol{\theta} \oplus \boldsymbol{\phi}: \varepsilon_m \to \bigoplus_{i=1}^{v_1} \langle \theta_i \rangle \oplus \bigoplus_{i=1}^{v_2} \langle \phi_i \rangle.$$

(2) W_m is a row vector, $A_m(\boldsymbol{\theta}, \boldsymbol{\phi})$ is a matrix, $V_m(\boldsymbol{\theta}, \boldsymbol{\phi})$ is a column vector, rows and columns are indexed by ε_m.

$$W_m(\mathbf{c}) = 1 \quad \text{for all } \mathbf{c} \in \varepsilon_m,$$

$$V_m(\boldsymbol{\theta}, \boldsymbol{\phi})(\mathbf{c}) = \mu_\nu(\mathbf{c}) \exp\{i(\boldsymbol{\theta} + \boldsymbol{\phi})(\mathbf{c})\},$$

$$A_m(\boldsymbol{\theta}, \boldsymbol{\phi})(\mathbf{c}, \mathbf{d}) = \exp\{i(\boldsymbol{\theta} + \boldsymbol{\phi})(c_{m-1})\} \cdot \left\{\frac{\mu_\nu(\sigma^{-1}\mathbf{c} \cap \mathbf{d})}{\mu_\nu(\mathbf{d})}\right\},$$

for $\mathbf{c} = [c_0 \cdots c_{m-1}]$.

(3) $\| \quad \|_1$ is a norm on column vectors with $\|V\|_1 = \sum_i |V_i|$ if $V = (V_i)$, and also on matrices with $\|A\|_1 = \sup_j \sum_i |a_{ij}|$ if $A = (a_{ij})$.

Note that $\|V_m(\boldsymbol{\theta}, \boldsymbol{\phi})\|_1$, $\|A_m(\boldsymbol{\theta}, \boldsymbol{\phi})\|_1 = 1$.

(4) For any $\boldsymbol{\phi}$, and $\boldsymbol{\theta} \ne \mathbf{0}$ (regarded as real variables now), and $\frac{1}{2} < \delta \le 1$, define, for a column vector-valued function $V(\boldsymbol{\theta}, \boldsymbol{\phi})$,

$$\|V\|_{1,\boldsymbol{\theta},\boldsymbol{\phi},\delta} = \frac{\|V(\boldsymbol{\theta}, \boldsymbol{\phi}) - V(\mathbf{0}, \boldsymbol{\phi})\|_1}{|\theta_1|^{2\delta-1} + \cdots + |\theta_{v_1}|^{2\delta-1}} \quad \text{if } \delta < 1,$$

$$= \frac{\|V(\boldsymbol{\theta}, \boldsymbol{\phi}) - V(\mathbf{0}, \boldsymbol{\phi})\|_1}{|\theta_1 \log \theta_1| + \cdots + |\theta_{v_1} \log \theta_{v_1}|} \quad \text{if } \delta = 1,$$

and similarly for a matrix-valued function $A(\boldsymbol{\theta}, \boldsymbol{\phi})$.

The first stage in proving (2) of theorem 3, proposition 4, is proved as 3.3 of [**6**]:

PROPOSITION 4. *For any fixed* t, u, *if* $m^{8t+3} \le k \le m^u$,

$$S_k + S_{k+1} = \frac{2}{(2\pi)^v} \int_{[-1/m^t, 1/m^t]^v} W_m(A_m(\boldsymbol{\theta}, \boldsymbol{\phi})^{k-m} V_m(\boldsymbol{\theta}, \boldsymbol{\phi})\, d\boldsymbol{\theta}\, d\boldsymbol{\phi} + O(\eta^m),$$

some $\eta < 1$.

For the next stage, the following analogue of 4.2 of [**6**] is needed:

PROPOSITION 5. *Let* $F: \mathbb{R}^{v_1}\times\mathbb{R}^{v_2}\times\mathbb{C}\times\mathbb{C}^{\varepsilon_m}\to\mathbb{C}^{\varepsilon_m}\times\mathbb{C}$ *be defined by*

$$F(\boldsymbol{\theta},\boldsymbol{\phi},\lambda,\mathbf{y})=\begin{pmatrix}(\lambda-A_m(\boldsymbol{\theta}+\boldsymbol{\phi}))(\boldsymbol{\mu}+\mathbf{y})\\ \sum_{\mathbf{c}\in\varepsilon_m} y(\mathbf{c})\end{pmatrix},$$

where $\mathbf{y}=(y(\mathbf{c}))$ *and* $\boldsymbol{\mu}=(\mu_v(\mathbf{c}))$, *so* $\boldsymbol{\mu}=V_m(\mathbf{0})$. *Then there exist* C^0 *functions* $\lambda_m(\boldsymbol{\theta},\boldsymbol{\phi})$, $\mathbf{y}(\boldsymbol{\theta},\boldsymbol{\phi})$ *defined for* $|\theta_i|\le C/m^{2/(2\delta-1)}$ *or (if* $\delta=1$*)* $C/(m^2\log m)$, *and* $|\phi_i|\le C/m^2$, *with* $F(\boldsymbol{\theta},\boldsymbol{\phi},\lambda_m(\boldsymbol{\theta},\boldsymbol{\phi}),\mathbf{y}(\boldsymbol{\theta},\boldsymbol{\phi}))=0$. *Moreover,* λ_m, $\mathbf{y}$ *are* C^∞ *in* $\boldsymbol{\phi}$, *and*

$$\left\|D^k_{\boldsymbol{\phi}}\begin{pmatrix}\lambda_m\\ \mathbf{y}\end{pmatrix}\right\|_1\le d_k m^{n_k}$$

$$\left\|D^k_{\boldsymbol{\phi}}\begin{pmatrix}\lambda_m\\ \mathbf{y}\end{pmatrix}\right\|_{1,\boldsymbol{\theta},\boldsymbol{\phi},\delta}\le d_k m^{n_k}$$

for constants d_k, n_k. *Moreover, if the sequence* $\begin{pmatrix}\lambda^r_m\\ \mathbf{y}^r\end{pmatrix}$ *is defined inductively by*

$$\begin{pmatrix}\lambda^0_m(\boldsymbol{\theta},\boldsymbol{\phi})\\ \mathbf{y}^0(\boldsymbol{\theta},\boldsymbol{\phi})\end{pmatrix}=\begin{pmatrix}1\\ \mathbf{0}\end{pmatrix},$$

$$\begin{pmatrix}\lambda^{r+1}_m(\boldsymbol{\theta},\boldsymbol{\phi})\\ \mathbf{y}^{r+1}(\boldsymbol{\theta},\boldsymbol{\phi})\end{pmatrix}=\begin{pmatrix}\lambda^r_m(\boldsymbol{\theta},\boldsymbol{\phi})\\ \mathbf{y}^r(\boldsymbol{\theta},\boldsymbol{\phi})\end{pmatrix}-(DF_{\lambda^r_m,\mathbf{y}^r})^{-1}F(\boldsymbol{\theta},\boldsymbol{\phi},\lambda^r_m,\mathbf{y}^r),$$

then

$$\left\|\begin{pmatrix}\lambda_m\\ \mathbf{y}\end{pmatrix}-\begin{pmatrix}\lambda^1_m\\ \mathbf{y}^1\end{pmatrix}\right\|_{1,\boldsymbol{\theta},\mathbf{0},\delta}\le dm^4(|\theta|^{2\delta-1}+\cdots+|\theta_{V_1}|^{2\delta-1})$$

for a constant d, *where* $|\boldsymbol{\theta}_i|^{2\delta-1}$ *is replaced by* $|\theta_i\log\theta_i|$ *if* $\delta=1$.

Similar results hold for extending the eigenvalue 1, *and eigenvector* W_m, *of* $(A_m(\mathbf{0}))^T$, *with* $\|\ \|_\infty$ *norms, by exactly dual methods.*

In order to prove proposition 5, we need the following lemma and its corollary:

LEMMA 6.

$$\sum_{n=1}^{\infty}\frac{|\exp(in\theta)-1|}{n^{2\delta}}\le A|\theta|^{2\delta-1}\quad \textit{if } \tfrac12<\delta<1$$

$$\le A|\theta\log\theta|\quad \textit{if } \delta=1.$$

Proof. For any $\varepsilon>0$, if $[|\varepsilon/\theta|]$ denotes the integral part of $|\varepsilon/\theta|$,

$$\sum_{n=1}^{\infty}\frac{|\exp(in\theta)-1|}{n^{2\delta}}\le 4|\theta|\sum_{n=1}^{[|\varepsilon/\theta|]}n^{1-2\delta}+2\sum_{[|\varepsilon/\theta|]+1}^{\infty}n^{-2\delta}$$

$$\le A_1|\theta|\cdot|\theta|^{2\delta-2}+A_2|\theta|^{2\delta-1}\quad\text{if } \delta<1$$

$$\le A_1|\theta|\,|\log\theta|+A_2|\theta|\quad\text{if } \delta=1.$$

□

COROLLARY 7.

$$\|V_m\|_{1,\boldsymbol{\theta},\boldsymbol{\phi},\delta}\le Bm$$

$$\|D^k_{\boldsymbol{\phi}}A_m\|_{1,\boldsymbol{\theta},\boldsymbol{\phi},\delta}\le A_k\quad(k\ge0)$$

for some constants B, A_k.

Proof. This follows from lemmas 2, 6, and also the following property of a Gibbs measure (see [**6**], 1.7.2) for the bound on $\|V_m\|_{1,\boldsymbol{\theta},\boldsymbol{\phi},\delta}$:

$$\mu_\nu([e_0\cdots e_{p-1}])\le D\mu_\nu([e_0\cdots e_{r-1}])\mu_\nu([e_r\cdots e_{p-1}]),$$

some constant D, any r. □

Proof of proposition 5. As in (4.2) of [**6**],

$$DF_{\lambda\mathbf{y}}=\begin{pmatrix}\boldsymbol{\mu}+\mathbf{y} & \lambda-A_m\\ 0 & 1\cdots1\end{pmatrix}.$$

As in (3.2) of [**6**],

$$\|(I-A_m(\mathbf{0},\mathbf{0}))V\|_1\ge(D^1/m)\|V\|_1,$$

some D^1, if $\sum_{\mathbf{c}\in\varepsilon_m} V(\mathbf{c})=0$, and hence (using corollary 7)

$$\|(DF_{1,\mathbf{0}})^{-1}\|_1\le Dm \quad \text{if } |\theta_i|\le c/m^{1/(2\delta-1)},\qquad |\phi_i|\le c/m \text{ for } \delta<1,$$
$$|\theta_i|\le c/(m\log m),\qquad |\phi_i|\le c/m \quad \text{if } \delta=1,$$

for some constants D, c. Hence, for $(\boldsymbol{\theta},\boldsymbol{\phi})$ in this set $\mathcal{U}$ say, $DF_{\lambda^r_m,\mathbf{y}^r}$ is invertible if $|\lambda^r_m-1|$, $\|\mathbf{y}^r\|_1\le1/(2Dm)$, and then $\|(DF_{\lambda^r_m,\mathbf{y}^r})^{-1}\|_1\le 2Dm$, so that λ^{r+1}_m, $\mathbf{y}^{r+1}$ are defined.

Fix a set $\mathcal{U}_1\subseteq\mathcal{U}$, and let

$$\varepsilon_0=\sup_{(\boldsymbol{\theta},\boldsymbol{\phi})\in\mathcal{U}_1}\|F(\boldsymbol{\theta},\boldsymbol{\phi},\lambda^0_m,\mathbf{y}^0)\|_1 \quad (\text{recall } (\lambda^0_m,\mathbf{y}^0)=(1,\mathbf{0}))$$
$$=\sup_{(\boldsymbol{\theta},\boldsymbol{\phi})\in\mathcal{U}_1}\|(I-A_m(\boldsymbol{\theta},\boldsymbol{\phi}))\boldsymbol{\mu}\|_1.$$

So from corollary 7, by suitable choice of $\mathcal{U}_1$, ε_0 can be made arbitrarily small. As in (4.2) of [**6**], we have inductively that:

$$\sup_{(\boldsymbol{\theta},\boldsymbol{\phi})\in\mathcal{U}_1}\|F(\boldsymbol{\theta},\boldsymbol{\phi},\lambda^r_m,\mathbf{y}^r)\|_1\le(2Dm)^{2^r-1}\varepsilon_0^{2^r}$$
$$\sup_{(\boldsymbol{\theta},\boldsymbol{\phi})\in\mathcal{U}_1}\left\|\begin{matrix}\lambda^r_m-\lambda^{r-1}_m\\ y^r-y^{r-1}\end{matrix}\right\|_1\le(2Dm\varepsilon_0)^{2^{r-1}}. \tag{1}$$

So by suitable choice of $\mathcal{U}_1$ (in fact, the choice indicated in the statement of the proposition) we may assume that λ^r_m, $\mathbf{y}^r$ exist for all r, $|\lambda^r_m-1|\,\|\mathbf{y}^r\|_1\le1/(2Dm)$, and the sequence $\begin{pmatrix}\lambda^r_m\\ \mathbf{y}^r\end{pmatrix}$ converges uniformly to $\begin{pmatrix}\lambda_m\\ \mathbf{y}\end{pmatrix}$ which solves $F(\boldsymbol{\theta},\boldsymbol{\phi},\lambda_m(\boldsymbol{\theta},\boldsymbol{\phi}),\mathbf{y}(\boldsymbol{\theta},\boldsymbol{\phi}))=0$.

The bound on $\left\|D^k_{\boldsymbol{\phi}}\begin{pmatrix}\lambda_m\\ \mathbf{y}\end{pmatrix}\right\|_1$ then follows from the repeated differentiation of

$$D_{\boldsymbol{\phi}}\begin{pmatrix}\lambda_m\\ \mathbf{y}\end{pmatrix}=(DF_{\lambda_m,\mathbf{y}})^{-1}((D_{\boldsymbol{\phi}}A_m(\boldsymbol{\theta},\boldsymbol{\phi}))(\boldsymbol{\mu}+\mathbf{y})), \tag{2}$$

using corollary 7 and the bound $\|(DF_{\lambda_m,\mathbf{y}})^{-1}\|_1\le 2Dm$.

It remains to compute $\|\ \ \|_{1,\boldsymbol{\theta},\boldsymbol{\phi},\delta}$-seminorms. First,

$$\|DF_{\lambda^r_m,\mathbf{y}^r}\|_{1,\boldsymbol{\theta},\boldsymbol{\phi},\delta}\le\left\|\begin{pmatrix}\lambda^r_m\\ \mathbf{y}^r\end{pmatrix}\right\|_{1,\boldsymbol{\theta},\boldsymbol{\phi},\delta}+\|A_m\|_{1,\boldsymbol{\theta},\boldsymbol{\phi},\delta}.$$

Hence, from the definition of $\begin{pmatrix}\lambda_m^{r+1}\\ \mathbf{y}^{r+1}\end{pmatrix}$,

$$\left\|\begin{pmatrix}\lambda_m^{r+1}-\lambda_m^r\\ \mathbf{y}^{r+1}-\mathbf{y}^r\end{pmatrix}\right\|_{1,\theta,\phi,\delta}$$

$$\leq 2Dm\|F(\lambda_m^r,\mathbf{y}^r)\|_{1,\theta,\phi,\delta}+\|(DF_{\lambda_m^r,\mathbf{y}^r})^{-1}\|_{1,\theta,\phi,\delta}\|F(\lambda_m^r,\mathbf{y}^r)\|_1$$

$$\leq 2Dm\|F(\lambda_m^r,\mathbf{y}^r)\|_{1,\theta,\phi,\delta}$$

$$+(2Dm)^2\left(\left\|\begin{pmatrix}\lambda_m^r\\ \mathbf{y}^r\end{pmatrix}\right\|_{1,\theta,\phi,\delta}+\|A_m\|_{1,\theta,\phi,\delta}\right)\|F(\lambda_m^r,\mathbf{y}^r)\|_1. \tag{3}$$

Since F is quadratic in λ, $\mathbf{y}$, the Taylor expansion of

$$F\left(\begin{pmatrix}\lambda_m^{r-1}\\ \mathbf{y}^{r-1}\end{pmatrix}+\begin{pmatrix}\lambda_m^r-\lambda_m^{r-1}\\ \mathbf{y}^r-\mathbf{y}^{r-1}\end{pmatrix}\right)$$

is particularly simple, and we obtain

$$\|F(\lambda_m^r,\mathbf{y}^r)\|_{1,\theta,\phi,\delta}$$

$$\leq C_1\|(DF_{\lambda_m^{r-1},\mathbf{y}^{r-1}})^{-1}F(\lambda_m^{r-1},\mathbf{y}^{r-1})\|_{1,\theta,\phi,\delta}\,\|(DF_{\lambda_m^{r-1},\mathbf{y}^{r-1}})^{-1}F(\lambda_m^{r-1},\mathbf{y}^{r-1})\|_1$$

$$\leq C_2 m\left\|\begin{pmatrix}\lambda_m^r-\lambda_m^{r-1}\\ \mathbf{y}^r-\mathbf{y}^{r-1}\end{pmatrix}\right\|_{1,\theta,\phi,\delta}\|F(\lambda_m^{r-1},\mathbf{y}^{r-1})\|_1. \tag{4}$$

Substituting from (1) and (4), (3) becomes

$$\left\|\begin{matrix}\lambda_m^{r+1}-\lambda_m^r\\ \mathbf{y}^{r+1}-\mathbf{y}^r\end{matrix}\right\|_{1,\theta,\phi,\delta}$$

$$\leq C_3 m(2Dm\varepsilon_0)^{2^{r-1}}\left(\|A_m\|_{1,\theta,\phi,\delta}+\left\|\begin{pmatrix}\lambda_m^r\\ \mathbf{y}^r\end{pmatrix}\right\|_{1,\theta,\phi,\delta}+\left\|\begin{pmatrix}\lambda_m^r-\lambda_m^{r-1}\\ \mathbf{y}^r-\mathbf{y}^{r-1}\end{pmatrix}\right\|_{1,\theta,\phi,\delta}\right). \tag{5}$$

Inductively we can prove:

$$\left\|\begin{pmatrix}\lambda_m^{r+1}-\lambda_m^r\\ \mathbf{y}^{r-1}-\mathbf{y}^r\end{pmatrix}\right\|_{1,\theta,\phi,\delta}\leq C_4 m^3(2Dm\varepsilon_0)^{2^{r-1}}\quad \text{if } r\geq 1, \tag{6}$$

provided that

$$\|A_m\|_{1,\theta,\phi,\delta}+\left\|\begin{pmatrix}\lambda_m^1\\ \mathbf{y}^1\end{pmatrix}\right\|_{1,\theta,\phi,\delta}+2\sum_{r=0}^{\infty}C_4 m^3(2Dm\varepsilon_0)^{2^r}\leq (C_4/C_3)m^2.$$

This can be arranged for C_4 large enough, and $\varepsilon_0=C/m^2$, C small enough, since

(*a*) $\|A_m\|_{1,\theta,\phi,\delta}$ is bounded, by corollary 7;

(*b*) $\left\|\begin{pmatrix}\lambda_m^1\\ \mathbf{y}^1\end{pmatrix}\right\|_{1,\theta,\phi,\delta}=\left\|\begin{pmatrix}\lambda_m^1-\lambda_m^0\\ \mathbf{y}^1-\mathbf{y}^0\end{pmatrix}\right\|_{1,\theta,\phi,\delta}=\|(DF_{1,0})^{-1}F(\cdots 1,\mathbf{0})\|_{1,\theta,\phi,\delta}$

$$\leq 2Dm\|F(\cdots 1,\mathbf{0})\|_{1,\theta,\phi,\delta}+(2Dm)^2\|DF_{1,\mathbf{0}}\|_{1,\theta,\phi,\delta}\|(I-A_m(\theta,\phi))\mu\|_1$$

$$\leq C_5 m^2,\quad \text{by corollary 7.}$$

The bound on $\|\lambda_m - \lambda_m^1\|_{1,\theta,0,\delta}$ follows from (6) since, by corollary 7, and the definition of ε_0, if $\mathscr{U}_1 = \{(\theta^1, 0): |\theta_i^1| \le |\theta_i|\}$ then

$$\varepsilon_0 \le C_6(|\theta_1|^{2\delta-1} + \cdots + |\theta_{v_1}|^{2\delta-1}) \quad \text{if } \delta < 1$$
$$\varepsilon_0 \le C_6(|\theta_1 \log \theta_1| + \cdots + |\theta_{v_1} \log \theta_{v_1}|) \quad \text{if } \delta = 1.$$

As for $\left\| D_\phi^k \binom{\lambda_m}{\mathbf{y}} \right\|_1$, the bounds on $\left\| D_\phi^k \binom{\lambda_m}{\mathbf{y}} \right\|_{1,\theta,\phi,\delta}$ follow from differentiating (2). □

Proposition 5 is used to prove the following corollaries:

COROLLARY 8 (*analogue of* (4.4) *of* [**6**]). *For any sufficiently large t, independent of m, if* $m^{8t+3} \le k \le m^n$,

$$S_k + S_{k+1} = 2(1 + O(1/m)) \frac{1}{(2\pi)^v} \int_{[-1/m^t, 1/m^t]^v} (\lambda_m(\theta, \phi))^{k-m} \, d\theta \, d\phi$$
$$+ O(\eta^m), \quad \text{some } \eta < 1.$$

Proof. Exactly as in (4.4) of [**6**], using the decomposition, for θ, ϕ near $\mathbf{0}$, $\mathbb{R}^{\varepsilon_m} = \operatorname{Im} P_m(\theta, \phi) \oplus \operatorname{Ker} P_m(\theta, \phi)$, where $A_m(\theta, \phi)$ has eigenvalue $\lambda_m(\theta, \phi)$ on $\operatorname{Im} P_m(\theta, \phi)$, and $\|(A_m(\theta, \phi))^{m+s}\|_1 < \beta < 1$ on the $A_m(\theta, \phi)$-invariant subspace $\operatorname{Ker} P_m(\theta, \phi)$, some s, β independent of m ((3.2) of [**6**] is used here). The Hölder continuity of λ_m, P_m at $\mathbf{0}$ established in proposition 5 (the dual results of proposition 5 for A_m^T are needed to prove Hölder continuity of P_m) are enough for the proof. □

Note. As in (4.1) of [**6**], $\lambda_m(\theta, \phi)$ is real, so the first ϕ-derivatives of λ_m vanish at $(\theta, \phi) = (\mathbf{0}, \mathbf{0})$.

COROLLARY 9 (*immediate from proposition* 5).

$$\lambda_m(\theta, \phi) = \lambda_m^1(\theta, \mathbf{0}) + \tfrac{1}{2}(\phi_1 \cdots \phi_{v_2}) \left(\frac{\partial^2 \lambda_m(\mathbf{0})}{\partial \phi_i \, \partial \phi_j} \right) \begin{pmatrix} \phi_1 \\ \phi_{v_2} \end{pmatrix}$$
$$+ O(m^4(|\theta_1|^{2\delta-1} + \cdots + |\theta_{v_1}|^{2\delta-1})^2) + O\Big(\sum_{i,j} m^{n_1} |\theta_i|^{2\delta-1} |\phi_j|\Big)$$
$$+ O\Big(\sum_{i,j} m^{n_2} |\theta_i|^{2\delta-1} |\phi_j|^2\Big) + O\Big(\sum_j m^{n_3} |\phi_j|^3\Big)$$

with $|\theta_i|^{2\delta-1}$ *replaced by* $|\theta_i||\log \theta_i|$ *if* $\delta = 1$.

Note. The aim is to show $\lambda_m^1(\theta, \mathbf{0}) = 1 - O(|\theta_1|^{2\delta-1} + \cdots + |\theta_{v_1}|^{2\delta-1})$. For exactly as in (4.6) of [**6**], $(\phi_1 \cdots \phi_{v_2}) \left(\frac{\partial^2 \lambda_m(\mathbf{0})}{\partial \phi_i \, \partial \phi_j} \right) \begin{pmatrix} \phi_1 \\ \phi_{v_2} \end{pmatrix}$ is boundedly negative definite of rank v_2 (we can reduce to the case of a finite symbol space by putting $\theta = \mathbf{0}$ and replacing $\{c_i^n\}_{n>0}$, $\{c_i^{-n}\}_{n<0}$ by single symbols). It is then not hard to see that $\lambda_m^1(\theta, \mathbf{0}) + \tfrac{1}{2}(\phi_1 \cdots \phi_{v_2}) \left(\frac{\partial^2 \lambda_m(\mathbf{0})}{\partial \phi_i \, \partial \phi_j} \right) \begin{pmatrix} \phi_1 \\ \phi_{v_2} \end{pmatrix}$ is the dominating part of $\lambda_m(\theta, \phi)$ (in spite of the $|\theta_i \log \theta_i|$ terms when $\delta = 1$).

Calculation of $\lambda_m^1(\theta, \mathbf{0})$. By definition,

$$\lambda_m^1(\theta, \mathbf{0}) = 1 - (DF_{1,\mathbf{0}}(\theta, \mathbf{0}))^{-1} \begin{pmatrix} (I - A_m(\theta, \mathbf{0}))\boldsymbol{\mu} \\ 0 \end{pmatrix}.$$

It can be checked that $(DF_{1,0})^{-1}$ is of the form $\begin{pmatrix} 1\cdots 1 & 0 \\ B_m(\boldsymbol{\theta}, \boldsymbol{\phi}) & \boldsymbol{\mu} \end{pmatrix}$, where, if $M = (\boldsymbol{\mu}\cdots\boldsymbol{\mu})$ (a matrix with rows and columns indexed by ε_m), then

$$B_m(\boldsymbol{\theta}, \boldsymbol{\phi})(I - A_m(\boldsymbol{\theta}, \boldsymbol{\phi})) = (I - A_m(\boldsymbol{\theta}, \boldsymbol{\phi}))B_m(\boldsymbol{\theta}, \boldsymbol{\phi}) = I - M$$

$$= \text{projection on } \left\{(y(\mathbf{c})) \colon \sum_{\mathbf{c}\in\varepsilon_m} y(\mathbf{c}) = 0\right\} \text{ along sp } (\boldsymbol{\mu}).$$

($B_m(\boldsymbol{\theta}, \boldsymbol{\phi})$ does exist, since by § 3 of [6] $\|A_m(\mathbf{0}, \mathbf{0})\|_1 < 1$ on Im $(I - M)$, hence also for nearby $(\boldsymbol{\theta}, \boldsymbol{\phi})$.) So

$$\lambda^1_m(\boldsymbol{\theta}, \mathbf{0}) = 1 - (1\cdots 1)(I - A_m(\boldsymbol{\theta}, \mathbf{0}))\boldsymbol{\mu}$$

$$= 2\sum_{i=1}^{r}\sum_{n=1}^{\infty} \mu_\nu([c_i^n]) \cos n\boldsymbol{\theta}(c_i)$$

(using the fact that $\mu_\nu([c_i^n]) = \mu_\nu([c_i^{-n}])$).

Hence, from lemma 2,

$$\lambda^1_m(\boldsymbol{\theta}, \mathbf{0}) = 2\sum_{i=1}^{r} b_i \sum_{n=1}^{\infty} \frac{1}{n^{2\delta}} \cos n\boldsymbol{\theta}(c_i) + O(|\boldsymbol{\theta}(c_1)| + \cdots + |\boldsymbol{\theta}(c_r)|),$$

since any cosine series with nth term $O(1/n^{2\delta+1})(\delta > \frac{1}{2})$ is C^1 with first derivative at 0 vanishing.

LEMMA 10. *If*

$$f(\theta) = \sum_{n=1}^{\infty} \frac{\cos n\theta}{n^{2\delta}}, \qquad f(\theta) = \sum_{n=1}^{\infty} \frac{1}{n^{2\delta}}(1 - C|\theta|^{2\delta-1} + O(|\theta|^{2\delta-1}))$$

for some constant $C > 0$ $(\delta > \frac{1}{2})$.

Proof. This is standard complex analysis. Since, clearly, $f(\theta) = f(-\theta)$, we need only consider the expansion for $\theta > 0$. If

$$0 \le \theta \le 2\pi, \qquad f(\theta) = \lim_{N\to\infty} \frac{1}{2i}\int_{\gamma_N} \frac{\cos(\pi - \theta)z}{z^{2\delta} \sin \pi z}\, dz$$

where γ_N is the contour shown in figure 3. In the limit as $N \to \infty$, the integral vanishes except on the imaginary axis and half-circle. If $\delta = 1$, the imaginary axis integral also cancels out, and the integral around the half-circle, which is a C^∞ function, has derivative $-\frac{1}{2}\pi$ at 0. If $\frac{1}{2} < \delta < 1$,

$$f(\theta) = g(\theta) - \sin \pi\delta \int_0^\infty \frac{1 - \exp(-y\theta)}{y^{2\delta}}\, dy, \quad \text{where } g \text{ is } C^\infty.$$

But

$$\int_0^\infty \frac{1 - \exp(-y\theta)}{y^{2\delta}}\, dy = \theta^{2\delta-1}\int_0^\infty \frac{1 - \exp(-y)}{y^{2\delta}}\, dy \quad \text{by change of variable.} \qquad \square$$

Completion of the proof of theorem 3. It has now been proved that

$$\lambda_m(\boldsymbol{\theta}, \boldsymbol{\phi}) = 1 - \sum_{i=1}^{\infty} a_i |\theta(c_i)|^{2\delta-1} + \tfrac{1}{2}(\phi_1 \cdots \phi_{v_2})\left(\frac{\partial^2 \lambda_m(\mathbf{0})}{\partial\phi_i\, \partial\phi_j}\right)\begin{pmatrix}\phi_1 \\ \phi_{v_2}\end{pmatrix} + \text{higher order terms},$$

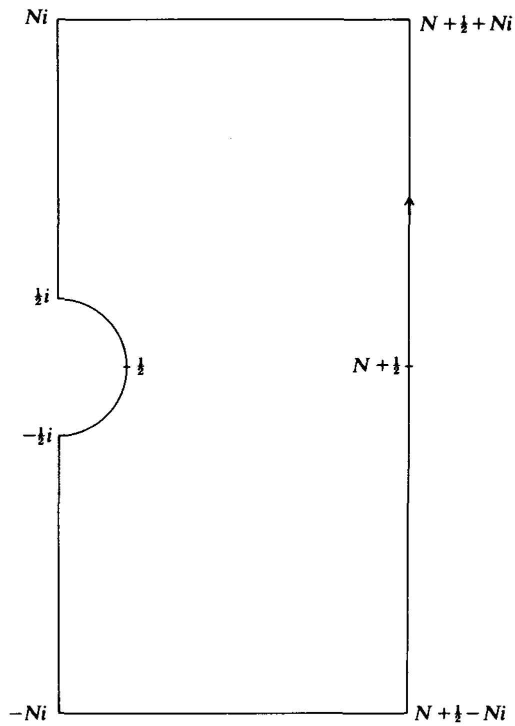

FIGURE 3

for constants $a_i > 0$, where $-\frac{1}{2}(\phi_1 \cdots \phi_{v_2})\left(\frac{\partial^2 \lambda_m(\mathbf{0})}{\partial\phi_i\,\partial\phi_j}\right)\begin{pmatrix}\phi_1\\ \phi_{v_2}\end{pmatrix}$ converges geometrically fast to a function $G(\phi_1 \cdots \phi_{v_2})$, where G is boundedly positive definite (from [**6**], (4.6)). So from corollary 8,

$$S_k + S_{k+1} = \frac{2}{(2\pi)^v}\left(1 + O\left(\frac{1}{m}\right)\right)\int_{[-1/m',1/m']^v}\left(\exp\left\{-(k-m)\left(\sum_{i=1}^{r} a_i|\theta(c_i)|^{2\delta-1}\right)\right.\right.$$
$$\left.\left.+ G(\phi_1 \cdots \phi_{v_2}) + \text{higher order terms}\right\}\right)\, d\boldsymbol{\theta}\, d\boldsymbol{\phi} + O(\eta^m), \quad \text{some } \eta < 1.$$

Change of variable then gives

$$S_k + S_{k+1} \sim \frac{2}{(2\pi)^v} k^{-(v_1/(2\delta-1)+\frac{1}{2}v_2)}\int_{\mathbb{R}^v} \exp\left\{-\left(\sum_{i=1}^{r} a_i\,|\theta(c_i)|^{2\delta-1}\right.\right.$$
$$\left.\left.+ G(\phi_1 \cdots \phi_{v_2})\right)\right\}\, d\boldsymbol{\theta}\, d\boldsymbol{\phi}$$

and theorem 3 is proved.

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