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COMPACTIFICATION OF HEREDITARILY LOCALLY CONNECTED SPACES

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All spaces considered in this paper are completely regular and T_1. A *continuum* is a compact, connected, Hausdorff space. A continuum is *hereditarily locally connected* if each of its subcontinua is locally connected. The reader may consult Whyburn [**5**] or Kuratowski [**2**] for a discussion on hereditarily locally connected metric continua. Nishiura and Tymchatyn [**3**] recently obtained some metric characterizations of connected subsets of hereditarily locally connected metric continua. Simone [**4**] extended to arbitrary hereditarily locally connected continua some well-known characterizations of hereditarily locally connected metric continua. In the first section of this paper some other characterizations of hereditarily locally connected metric continua are extended to the non-metric case. In particular, we extend Wilder's theorem to say that a continuum is hereditarily locally connected if and only if every connected subset is locally connected. In the second section of this paper there are given some uniform and some topological characterizations of connected spaces which admit a hereditarily locally connected compactification.

1. Characterizations of hereditarily locally connected continua. Let X be a uniform space. A family $\mathscr{A}$ of subsets of X is said to be *null* if for each uniform open cover $\mathfrak{A}$ of X there exist at most finitely many $A \in \mathscr{A}$ such that $A \not\subset U$ for any $U \in \mathfrak{A}$. If X is a compact space this is equivalent to the condition that for each pair P and Q of open subsets of X with disjoint closures at most finitely many members of $\mathscr{A}$ meet both P and Q.

We shall need the following result of Simone [**4**].

THEOREM 1 (Simone [**4**]). *A continuum X is hereditarily locally connected if and only if the components of every closed subset of X form a null family.*

We let N denote the set of natural numbers. We let $\mathrm{Cl}_Y(A)$ denote the closure of a set A in a space Y. By a neighbourhood of a point we always mean open neighbourhood.

THEOREM 2. *A continuum X is hereditarily locally connected if and only if the components of every open set in X form a null family.*

Proof. Let X be hereditarily locally connected. The components of each open set in X are open. Suppose U is an open subset of X such that the com-

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ponents of U do not form a null family. Then there exists a sequence $C_i)_{i \in N}$ of components of U which is not a null family. For each i there exists a continuum $D_i \subset C_i$ such that the family $D_i)_{i \in N}$ is not a null family. Let

$$D = \mathrm{Cl}_X(D_0 \cup D_1 \cup \ldots).$$

For each $i \in N$, C_i is a neighbourhood of D_i in X such that $C_i \cap D = D_i$. The sets $D_i)_{i \in N}$ are components of D, contrary to Theorem 1. Hence, the components of every open subset of X form a null family.

Suppose now that the family of components of every open set in X is a null family. Then X is locally connected. Just suppose X is not hereditarily locally connected. By Theorem 1 there exists a closed subset K of X such that the components of K do not form a null family. Let U and V be open sets in X with disjoint closures such that infinitely many components of K meet both U and V. Let C_1 and C_2 be two components of K each of which meets both U and V. Since K is compact the components of K are quasi-components so $K = U_1 \cup V_1$ where U_1 and V_1 are disjoint open sets in K such that $C_1 \subset U_1$ and $C_2 \subset V_1$. Without loss of generality, V_1 contains infinitely many components of K each of which meets both U and V. Since X is normal and K is closed in X, there exist disjoint open sets P_1 and Q_1 in X such that $U_1 \subset P_1$ and $V_1 \subset Q_1$. Let R_1 be the component of P_1 which contains C_1. Let C_3 be a component of K distinct from C_2 which meets both U and V and such that $C_3 \subset V_1$. Then $V_1 = U_2 \cup V_2$, where U_2 and V_2 are disjoint open sets in K such that $C_2 \in U_2$ and $C_3 \in V_2$. Without loss of generality, V_2 contains infinitely many components of K each of which meets both U and V. Since X is normal and V_1 is closed, there exist disjoint open sets P_2 and Q_2 in X such that $U_2 \subset P_2$, $V_2 \subset Q_2$ and $P_2 \cup Q_2 \subset Q_1$. Let R_2 be the component of P_2 which contains C_2. Inductively we can define a non-null sequence $R_i)$ of disjoint connected open sets in X. This is a contradiction. Thus X is hereditarily locally connected.

THEOREM 3. *A continuum X is hereditarily locally connected if and only if the quasi-components of each subset of X form a null family.*

Proof. If the quasi-components of every subset of X form a null family then the components of every closed subset of X form a null family. By Theorem 1 X is hereditarily locally connected.

Suppose now that X is locally connected and C is a subset of X such that the quasi-components of C do not form a null family. Let U and V be open sets in X with disjoint closures such that infinitely many quasi-components of C meet both U and V. Let Q_1 and Q_2 be two quasi-components of C each of which meets both U and V. Let V_1 and U_1 be disjoint open sets in C such that $C = U_1 \cup V_1$, $Q_1 \subset U_1$ and $Q_2 \subset V_1$. We may suppose, without loss of generality, that V_1 contains infinitely many quasi-components of C each of which meets both U and V. Let W_1 be an open set in X such that $W_1 \cap C = U_1$. The component of W_1 which contains Q_1 is an open connected set in X which

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meets both U and V. Let S_1 be a connected open set in W_1 which meets both U and V and such that $\mathrm{Cl}_X(S_1) \subset W_1$.

Let n be a positive integer. Suppose $Q_1, \ldots, Q_{n+1}$ are quasi-components of C each of which meets both U and V. Suppose U_i, V_i, S_i, W_i have been defined for each $i = 1, \ldots, n$ such that $V_{i-1} = U_i \cup V_i$ where U_i and V_i are disjoint open sets in C, $Q_i \subset U_i$, $Q_{i+1} \in V_i$, W_i is an open set in X such that $W_i \subset X \backslash \mathrm{Cl}_X(S_1 \cup \ldots \cup S_{i-1})$, $W_i \cap C = U_i$, S_i is a connected open set in X which meets both U and V, $\mathrm{Cl}_X(S_i) \subset W_i$ and V_i contains infinitely many quasi-components of C each of which meets both U and V.

Let $Q_{n+2} \neq Q_{n+1}$ be a quasi-component of C in V_n such that Q_{n+2} meets both U and V. There exist disjoint open sets U_{n+1} and V_{n+1} in C such that $V_n = U_{n+1} \cup V_{n+1}$, $Q_{n+1} \subset U_{n+1}$ and $Q_{n+2} \subset V_{n+1}$. We may suppose, without loss of generality, that V_{n+1} contains infinitely many quasi-components of C each of which meets both U and V. Let $W_{n+1} \subset X \backslash \mathrm{Cl}_X(S_1 \cup \ldots \cup S_n)$ be an open set in X such that $W_{n+1} \cap C = U_{n+1}$. The component of W_{n+1} which contains Q_{n+1} meets both U and V. Let S_{n+1} be a connected open set in X such that S_{n+1} meets both U and V and $\mathrm{Cl}_X(S_{n+1}) \subset W_{n+1}$. By induction we define a non-null sequence $S_i)$ of connected pairwise disjoint open sets in X. By Theorem 2, X is not hereditarily locally connected. We have proved that if X is hereditarily locally connected, then the quasi-components of every subset of X form a null family.

THEOREM 4. *A continuum X is hereditarily locally connected if and only if the components of every subset of X form a null family.*

Proof. If the components of every subset of X form a null family then X is hereditarily locally connected by Theorem 1.

Suppose X is hereditarily locally connected. Just suppose C is a subset of X which contains a non-null family $C_i)_{i \in N}$ of components. Let U and V be open sets in X with disjoint closures such that each of infinitely many $C_i)_{i \in N}$ meet both U and V. Without loss of generality, we may suppose each C_i meets both U and V. Let $D = C_0 \cup C_1 \cup C_2 \cup \ldots$. Then D contains a non-null family of quasi-components contrary to Theorem 3. This completes the proof of Theorem 4.

The following theorem was proved by Wilder (see [**5**]) for the case of metric continua.

THEOREM 5. *A continuum X is hereditarily locally connected if and only if every connected subset of X is locally connected.*

Proof. Suppose X is hereditarily locally connected. Let C be a connected subset of X. Let $x \in C$, let U be a neighbourhood of x in X and let Q be the quasi-component of $U \cap C$ which contains x. Just suppose there is a net $x_\alpha)$ in $C \backslash Q$ which converges to x. Let V be a neighbourhood of x in X such that $\mathrm{Cl}(V) \subset U$. If W is a closed and open neighbourhood of x_α in $U \cap C$ then

$\mathrm{Cl}(W)$ meets the boundary of U since C is connected. By the proof of Theorem 3 there is a sequence V_i) of disjoint open connected sets in X each of which meets both $X \backslash U$ and V. This contradicts Theorem 2. Thus, x is a C interior point of Q. It now follows that the quasi-components of open sets in C are open and hence C is locally connected.

It is well-known (see [**5**]) that a metric continuum is hereditarily locally connected if and only if the quasi-components of every subset of X are connected. Simone proved in [**4**] that if X is a continuum in which the quasi-components of each subset are connected then X is hereditarily locally connected. The following question remains open:

Question. If X is a hereditarily locally connected continuum are the quasi-components of each subset of X connected?

2. Connected subsets of hereditarily locally connected continua. A space Y is called a *perfect extension* of a space X provided for each closed C in X which separates A and B in X $\mathrm{Cl}_Y(C)$ separates A and B in Y.

LEMMA 6. *Let X be a uniform space that has a basis of uniform coverings of connected sets. Let Y be the completion of X. Then Y is a perfect extension of X.*

Proof. Let C be a closed set in X which separates A and B in X. Then, $X \backslash C = M \cup N$ where M and N are disjoint open sets in X such that $A \subset M$ and $B \subset N$.

If $\mathfrak{A}$ is an uniform cover of X let $\overline{\mathfrak{A}} = \{\mathrm{Cl}_Y(U) | U \in \mathfrak{A}\}$. Then $\overline{\mathfrak{A}}$ is a uniform cover of Y by [**1**, Theorem II.9].

Let $p \in \mathrm{Cl}_Y(M) \backslash \mathrm{Cl}_Y(C)$. Let $\mathfrak{A}$ be a uniform cover of X by connected sets such that $\mathrm{St}(p, \overline{\mathfrak{A}})$ is disjoint from $\mathrm{Cl}_Y(C)$. Let $U \in \mathfrak{A}$ such that $\mathrm{Cl}_Y(U)$ contains a neighbourhood of p in Y. Then $U \cap M \neq \emptyset$. Since U is connected and C separates M and N, $U \cap N = \emptyset$. Thus, $p \notin \mathrm{Cl}_Y(N)$. It follows that $\mathrm{Cl}_Y(M) \cap \mathrm{Cl}_Y(N) \subset \mathrm{Cl}_Y(C)$ and $\mathrm{Cl}_Y(C)$ separates A and B in Y.

A uniform space X is said to have *property S* if each uniform open cover of X has a finite refinement whose members are connected sets.

Let Y be a compactification of a space X. Then Y is said to be a *perfect compactification* of X if Y is a perfect extension of X. We say Y has *ponctiforme remainder* if $Y \backslash X$ contains no non-degenerate continuum.

We shall call a space *hereditarily locally connected* if and only if all of its subsets are locally connected. By Theorem 5 this definition agrees with the usual definition of hereditary local connectedness on continua.

Let D denote the set of dyadic rationals in $[0, 1]$.

THEOREM 7. *For a connected, locally connected Tychonoff space X the following conditions are equivalent:*

i) *X has a hereditarily locally connected compactification.*

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ii) *X is herediiarily locally connected and has a perfect compactification with ponctiforme remainder.*

iii) *There exists a uniformity for X such that every connected open subset of X has property S.*

iv) *There exists a uniformity for X such that each family of pairwise disjoint, open, connected subsets of X is null.*

v) *For each* $x \in X$ *and each closed set A in* $X\backslash\{x\}$ *there exists a family* $U_d)_{d\in D}$ *of neighbourhoods of x such that for each* $d < e$ *in D* $\mathrm{Cl}(U_d) \subset U_e \subset X\backslash A$ *and there does not exist a family* $V_i)_{i\in N}$ *of open, connected, pairwise disjoint sets each of which meets both* U_d *and* $X\backslash U_e$.

vi) *There exists a family* $f_\alpha)_{\alpha\in A}$ *of continuous functions of X into the unit interval* $[0, 1]$ *which separate points and closed sets of X and such that for each* $\alpha \in A$ *and* $x < y$ *in* $[0, 1]$ *there does not exist a family* $V_i)_{i\in N}$ *of pairwise disjoint, open, connected sets each of which meets both* $f_\alpha^{-1}(x)$ *and* $f_\alpha^{-1}(y)$.

Proof. i) $\Rightarrow$ iv). This follows from Theorem 4.

iv) $\Rightarrow$ iii). Let μ be a uniformity on X such that every family of pairwise disjoint, connected, open sets in X is a null family. We prove first that X with the uniformity μ has property S.

Let $\mathscr{U} \in \mu$ be a uniform cover of X by open sets. Suppose that no finite subfamily of $\mathscr{U}$ covers X. Let $\mathscr{V} \in \mu$ such that $\mathscr{V}^{**} < \mathscr{U}$. Let $V_0 \in \mathscr{V}$ and let $U_0 \in \mathscr{U}$ such that $\mathrm{St}(V_0, \mathscr{V}^*) \subset U_0$. Let $x_1 \in X\backslash U_0$. Let $V_1 \in \mathscr{V}$ such that $x_1 \in V_1$ and let $U_1 \in \mathscr{U}$ such that $\mathrm{St}(V_1, \mathscr{V}^*) \subset U_1$. Then $\mathrm{St}(V_0, \mathscr{V}) \cap \mathrm{St}(V_1, \mathscr{V}) = \emptyset$ and $X\backslash(U_0 \cup U_1) \neq \emptyset$. Inductively, we can construct a pairwise disjoint family $\mathrm{St}(V_i, \mathscr{V})_{i\in N}$ of open non-empty sets. For each $i \in N$ at least one component C_i of $\mathrm{St}(V_i, \mathscr{V})$ meets V_i. Let $\mathscr{W} \in \mu$ such that $\mathscr{W}^* < \mathscr{V}$. Then $C_i \not\subset W$ for any $W \in \mathscr{W}$. Thus $C_i)$ is a non-null family of pairwise disjoint, open, connected subsets of X. This is a contradiction. Hence each uniform open cover of X has a finite subcover.

Let C be a connected open set in X. Let $\mathscr{U}, \mathscr{V} \in \mu$ be uniform covers of X by open sets such that $\mathscr{V}^{**} < \mathscr{U}$. Let $V \in \mathscr{V}$ and $U \in \mathscr{U}$ such that $\mathrm{St}(V, \mathscr{V}^*) \subset U$. Each component of $U \cap C$ which meets V is open and is not contained in any member of $\mathscr{V}$. Since every family of pairwise disjoint open connected sets in X is null at most finitely many components of $U \cap C$ meet V. Since X is covered by finitely many members of $\mathscr{V}$ it follows that C has property S.

iii) $\Rightarrow$ ii). Let μ be a uniformity for X such that every open connected subset of X has property S. For $\mathscr{U} \in \mu$ let $\mathscr{U}' = \{C|C$ is a component of $\mathrm{St}(U, \mathscr{U})$, $U \in \mathscr{U}$ and $U \cap C \neq \emptyset\}$. Let ν be the uniformity generated by $\{\mathscr{U}'|\mathscr{U} \in \mu\}$. Then ν generates the topology of X. Every member of ν has a finite uniform refinement consisting of connected sets and every open connected subset of X has property S.

Let Y be the completion of the uniform space X with uniformity ν. Then Y is a perfect extension of X by Lemma 6. By Isbell [**1**, Theorem II.29], Y is

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compact. Since X has property S, Y has property S by Isbell [**1**, Theorem II.9]. Hence Y is locally connected.

Just suppose $Y\backslash X$ contains a non-degenerate continuum K. Let $x, y \in K$ so that $x \neq y$. Let U and V be neighbourhoods in Y of x and y respectively such that U and V have disjoint closures and U is connected. Let W_0 be a connected neighbourhood of K in Y. Since Y is a perfect compactification of X, $W_0 \cap X$ is connected. So $W_0\backslash K$ is continuumwise connected. Let K_0 be a continuum in $W_0\backslash K$ which meets both U and V. Let O_0 be a connected neighbourhood of K_0 in Y such that $\mathrm{Cl}_Y(O_0) \subset W_0\backslash K$. Then $O_0 \cap X$ is a connected open set in X which meets both U and V. Let W_1 be a connected neighbourhood of K in Y such that $W_1 \subset W_0\backslash \mathrm{Cl}_Y(O_0)$. By a similar argument one can find a connected open set O_1 in Y, with $\mathrm{Cl}_Y(O_1) \subset W_1\backslash K$ such that O_1 meets both U and V. By induction there exists a sequence $O_i)_{i\in N}$ of pairwise disjoint open connected sets in Y such that each O_i meets both U and V. Hence $(U \cup O_0 \cup O_1 \cup \ldots) \cap X$ is a connected open set in X which does not have property S. This is a contradiction. Hence $Y\backslash X$ is ponctiforme.

If W is an open set in Y the components of W are open since Y is locally connected. Let $\{C_\alpha\}_{\alpha\in A}$ denote the family of components of W. Just suppose there exist open sets U and V in Y with disjoint closures such that infinitely many of C_α meet both U and V. Since Y is a locally connected continuum we may suppose U is connected. Then

$$O = U \cup \{C_\alpha|_{\alpha\in A},\ U \cap C_\alpha \neq \emptyset \text{ and } V \cap C_\alpha \neq \emptyset\}$$

is a connected open set in Y. Since Y is a perfect compactification of X, $O \cap X$ is a connected open set in X. However, $O \cap X$ does not have property S. This is a contradiction. Therefore, the components of every open subset of Y form a null family. By Theorem 2, Y is hereditarily locally connected. By Theorem 5, X is hereditarily locally connected.

ii) $\Rightarrow$ i). Suppose X is hereditarily locally connected and has a perfect compactification Y with ponctiforme remainder. It is clear that Y is locally connected at each point of X. Since a continuum cannot fail to be locally connected only at points of a ponctiforme set, Y is locally connected.

Just suppose Y is not hereditarily locally connected. By Theorem 2, there exists an open set U in Y and a non-null family $U_i)_{i\in N}$ of components of U. For each $i \in N$, let $V_i \subset \mathrm{Cl}_Y(V_i) \subset U_i$ be a connected open set such that $V_i)_{i\in N}$ is not a null family. Then $\mathrm{Cl}_Y(V_i) \subset \mathrm{Cl}_Y(\bigcup_{j\neq i} V_j) = \emptyset$ for each $i \in N$. Let x and y be two points of X which lie in the same component of lim sup V_i. Such points exist since $Y\backslash X$ is ponctiforme. Let U and V be connected neighbourhoods in Y of x and y respectively such that $\mathrm{Cl}_Y(U) \cap \mathrm{Cl}_Y(V) = \emptyset$. Without loss of generality, one may suppose U meets V_i for each i. Then $W = U \cup V_0 \cup V_1 \cup \ldots$ is a connected open set in Y. Since Y is a perfect compactification of X, $W \cap X$ is connected. Hence $(W \cap X) \cup \{y\}$ is a connected set in X which is not locally connected at y. This is a contradiction. Hence Y is hereditarily locally connected.

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i) $\Rightarrow$ v). This follows from Theorem 2.

v) $\Rightarrow$ vi). Construct Urysohn functions using the sets $U_d)_{d \in D}$.

vi) $\Rightarrow$ iv). Let $\mathfrak{F}$ be a family of continuous functions of X into $[0, 1]$ which satisfy the conditions of vi). Let μ be the coarsest uniformity on X which includes all of the inverse images of uniform coverings of $[0, 1]$ under these mappings. Then μ satisfies the conditions of iv).

A space X is said to be *rim-compact* if it has a basis of open sets with compact boundaries.

The following result was obtained in [**3**] for the case of separable metric spaces.

COROLLARY 8. *A rim-compact, hereditarily locally connected, completely regular, T_1 space has a hereditarily locally connected compactification.*

Proof. The Freudenthal compactification Y of a rim-compact space X is perfect and has zero-dimensional remainder (see [**1**, Theorem VI.36]).

THEOREM 9. *For a separable metric connected space the following conditions are equivalent:*

i) *X has a hereditarily locally connected compactification.*

ii) *X has a hereditarily locally connected metric compactification.*

Proof. Suppose X has a hereditarily locally connected compactification Y. Let $\mathcal{G}$ be the set of all continuous functions of Y onto $[0, 1]$. Then $\mathfrak{F} = \{g|X | g \in \mathcal{G}\}$ satisfies vi) of Theorem 7.

Let U be an open set in X and let $x \in U$. There exists $f \in \mathfrak{F}$ such that f separates x and $X \backslash U$. Since f is continuous it separates a neighbourhood of x and $X \backslash U$. Since U is Lindelöf there exists a countable subfamily $\mathfrak{F}_U$ of $\mathfrak{F}$ such that for each $y \in U$ there is an $f_y \in \mathfrak{F}_U$ such that f_y separates y and $X \backslash U$. Since X has a countable base there is a countable subfamily $\mathfrak{F}'$ of $\mathfrak{F}$ which separates points and closed sets of X. This family defines a metric hereditarily locally connected compactification of X.

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