The Riemann surfaces of a function and its fractional integral.

By William Fabian.

1. Introduction. For a many-valued function $f(z)$ of the complex variable $z$, a Riemann surface can be constructed such that, at any point $z$ on the surface, the function has only one value; a function normally multiform, is therefore uniform on a certain Riemann surface.

The operator $D^{-\lambda}$ represents a $\lambda$th integral of a function and is defined by

$$D^{-\lambda}(l_a) f(z) = \frac{1}{\Gamma(\lambda+\gamma)} \left( \frac{d}{dz} \right)^{\gamma} \int_a^z (z-t)^{\lambda+\gamma-1} f(t) \, dt,$$

where $l$ is a simple curve in the plane of the complex variable, along which the integration is carried out. $\lambda$ may be real or complex, and $\gamma$ is the least integer greater than or equal to zero such that $R(\lambda) + \gamma > 0$, $R(\lambda)$ being the real part of $\lambda$.

In this note we are concerned with relations between the Riemann surfaces of a function and its fractional integral.

2. Transformation of Riemann surfaces.

Theorem 1. Let $f(z)$ be analytic within a circle with centre at $a$, and which contains $l$ in its interior. Then $a$ is a branch-point of $D^{-\lambda}(l_a)f(z)$ for non-integral values of $\lambda$.

If $\lambda$ is a rational fraction $\frac{r}{s}$ expressed in its lowest terms, then $a$ is the vertex of a cycle of $s$ roots.

If $\lambda$ is irrational or complex, then $a$ is the vertex of an infinite number of roots.

Proof. The Taylor series for $f(z)$ at $a$ within the given circle is

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n.$$

Then applying the operator $D^{-\lambda}$ to each term of this series, we easily find that, within the given circle,

$$D^{-\lambda}(l_a)f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\Gamma(\lambda+n+1)} (z-a)^{\lambda+n}.$$
Theorem 2. Let \( f(z) \) be analytic in a bounded region \( E \), except for an isolated singularity within \( E \) at a point \( p \) different from \( a \), at which \( f(z) \) can be expanded in a Laurent series.

Then, for non-integral values of \( \lambda \), \( p \) is a branch-point of \( D^{-\lambda}(l_a)f(z) \), with cycles of an infinite number of roots.

Proof. In the \( t \)-plane, where \( l \) joins the points \( t = a \) and \( t = z \), let \( C \) be a closed contour through the point \( t = z \), which lies wholly in \( E \), encloses \( p \), and excludes \( l \). Denote by \( S_m \) the curve traced out by a point \( t \) which passes along \( l \) from \( a \) to \( z \) and then describes \( C \) \( m \) times. Then

\[
D^{-\lambda}(S_m)f(z) = D^\gamma D^{-\lambda-\gamma}(S_m)f(z)
= D^\gamma[D^{-\lambda-\gamma}(l_a)f(z) + mD^{-\lambda-\gamma}(C)f(z)]
= D^{-\lambda}(l_a)f(z) + mD^{-\lambda-\gamma}(C)f(z)
= D^{-\lambda}(l_a)f(z) + mD^{-\lambda-\gamma}(C)f^{(\gamma)}(z),
\]

on integrating \( D^{-\lambda-\gamma}(C)f(z) \) by parts \( \gamma \) times.

By a previous theorem \(^1\)

\[
D^{-\lambda-\gamma}(C)f^{(\gamma)}(z) = 2\pi i \sum_{\sigma = 1}^{\infty} (-1)^{\sigma-1} A_{\sigma} \frac{(z - p)^{1-\sigma}}{\Gamma(\lambda - \sigma + 1)(\sigma - 1)!},
\]

where \( \sum_{\sigma = -\infty}^{\infty} A_{\sigma} (z - p)^{-\sigma} \) is the Laurent series for \( f(z) \) at \( p \).

The conclusion now follows from (1).

Theorem 3. Let \( f(z) \) be analytic in a bounded region \( E \) on the Riemann surface associated with \( f(z) \), except for a branch-point within \( E \) at a point \( p \) different from \( a \), at which \( f(z) \) can be expanded in a Puiseux series. Let the number of roots of \( f(z) \) in the cycle \(^2\) at \( p \) be \( r \).

If the Puiseux series for \( f(z) \) at \( p \) does not contain negative integral powers of \( (z - p) \), the number of roots of \( D^{-\lambda}(l_a)f(z) \), where \( \lambda \) is non-integral, in the corresponding cycle at \( p \) does not exceed \( r \). If the series contains negative integral powers of \( (z - p) \), the number of roots of \( D^{-\lambda}(l_a)f(z) \), where \( \lambda \) is non-integral, in the corresponding cycle at \( p \) is infinite.

\(^1\) Fabian : Phil. Mag., 21, 277 (1936).

\(^2\) If \( f(z) \) has \( M \) cycles at \( p \), \( f(z) \) is to be regarded as having \( M \) branch-points at \( p \), and the theorem applies to each of these branch-points separately.

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Proof. On the Riemann surface associated with \( f(t) \), let \( C \) be a closed contour through the point \( t = z \), which lies wholly in \( E \), encloses \( p \) and excludes \( l \), where \( l \) joins \( a \) and \( z \). Denote by \( S_m \) the curve traced out by a point \( t \) which passes along \( l \) from \( a \) to \( z \) and then describes \( C \) \( m \) times.

As in the proof of Theorem 2, we have

\[
D^{-\lambda} (S_m) f(z) = D^{-\lambda} (l_a) f(z) + m D^{-\lambda - \gamma} (C) f^{(\nu)}(z).
\]

(1)

By a previous theorem,\(^1\) from which the value of \( D^{-\lambda - \gamma} (C) f^{(\nu)}(z) \) can be immediately deduced, it follows that \( D^{-\lambda - \gamma} (C) f^{(\nu)}(z) \), for non-integral values of \( \lambda \), is or is not zero, according as the Puiseux series for \( f(z) \) at \( p \) does not or does contain negative integral powers of \( (z - p) \). The result then follows from (1).

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\(^1\) Fabian: Phil. Mag., 21, 276 (1936).

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