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EXTENSION OF THE BIVARIATE CHARACTERIZATION FOR STOCHASTIC ORDERS

RHONDA RIGHTER,* *Santa Clara University*
J. GEORGE SHANTHIKUMAR,** *University of California, Berkeley*

Abstract

The bivariate characterization of stochastic ordering relations given by Shanthikumar and Yao (1991) is based on collections of bivariate functions $g(x, y)$, where $g(x, y)$ and $g(y, x)$ satisfy certain properties. We give an alternate characterization based on collections of pairs of bivariate functions, $g_1(x, y)$ and $g_2(x, y)$, satisfying certain properties. This characterization allows us to extend results for single machine scheduling of jobs that are identical except for their processing times, to jobs that may have different costs associated with them.

LIKELIHOOD RATIO ORDERING; HAZARD RATE ORDERING; STOCHASTIC SCHEDULING

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1. Preliminaries

For convenience we list the following results for the bivariate characterization of likelihood ratio, hazard rate, and stochastically ordered random variables (Shanthikumar and Yao (1991)). Throughout we assume X and Y are independent random variables. For any bivariate function, $g(x, y)$, define $\Delta g(x, y) := g(x, y) - g(y, x)$. Also define

$$\mathcal{G}_{\mathrm{lr}} := \{g(x, y): g(x, y) \geqq g(y, x)\ \forall x \geqq y\},$$
$$\mathcal{G}_{\mathrm{hr}} := \{g(x, y): \Delta g(x, y) \text{ is increasing in } x\ \forall x \geqq y\},$$
$$\mathcal{G}_{\mathrm{st}} := \{g(x, y): \Delta g(x, y) \text{ is increasing in } x\ \forall x\}.$$

Lemma 1. $X \geqq_a Y \Leftrightarrow E[g(X, Y)] \geqq E[g(Y, X)]\ \forall g \in \mathcal{G}_a$, for a = lr, hr, st respectively.

2. Main result

Let $g_1(x, y)$ and $g_2(x, y)$ be two bivariate functions, and let $\Delta g_{12}(x, y) = g_1(x, y) - g_2(x, y)$. We consider the following set of conditions on g_1 and g_2:

(a) $\Delta g_{12}(x, y) \geqq -\Delta g_{12}(y, x)$, i.e. $g_1(x, y) - g_2(x, y) \geqq g_2(y, x) - g_1(y, x)$, for all $x \geqq y$.
(b) $\Delta g_{12}(x, y) \geqq 0$, i.e. $g_1(x, y) \geqq g_2(x, y)$ for all $x \geqq y$.
(c) $g_1(x, y) \geqq g_2(y, x)$ for all $x \geqq y$.
(d) $g_1(y, x) \geqq g_2(x, y)$ for all $x \geqq y$.
(e) $g_1(x, y)$ increasing in x for all $x \geqq y$.
(f) $g_1(x, y)$ decreasing in y for all $y \leqq x$.
(g) $\Delta g_{12}(x, y)$ increasing in x for all $x \geqq y$.
(h) $\Delta g_{12}(x, y)$ decreasing in y for all $y \leqq x$.

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* Postal address: Department of Decision and Information Sciences, Santa Clara University, Santa Clara, CA 95053, USA.

** Postal address: Walter A. Haas School of Business, University of California, Berkeley, CA 94720, USA.

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Theorem 1.

(i) $X \geqq_{\text{lr}} Y \Leftrightarrow \boldsymbol{E}[g_1(X, Y)] \geqq \boldsymbol{E}[g_2(X, Y)]$ for all g_1, g_2 satisfying conditions (a) and (b).

(ii) $X \geqq_{\text{lr}} Y \Leftrightarrow g_1(X, Y) \geqq_{\text{icx}} g_2(X, Y)$ for all g_1, g_2 satisfying conditions (a), (b), and (c).

(iii) $X \geqq_{\text{lr}} Y \Leftrightarrow g_1(X, Y) \geqq_{\text{st}} g_2(X, Y)$ for all g_1, g_2 satisfying conditions (b), (c), and (d) (and therefore (a)).

(iv) $X \geqq_{\text{hr}} Y \Leftrightarrow \boldsymbol{E}[g_1(X, Y)] \geqq \boldsymbol{E}[g_2(X, Y)]$ for all g_1, g_2 satisfying conditions (a) and (g).

(v) $X \geqq_{\text{hr}} Y \Leftrightarrow g_1(X, Y) \geqq_{\text{icx}} g_2(X, Y)$ for all g_1, g_2 satisfying conditions (a), (b), (c), (e), and (g).

(vi) $X \geqq_{\text{st}} Y \Leftrightarrow \boldsymbol{E}[g_1(X, Y)] \geqq \boldsymbol{E}[g_2(X, Y)]$ for all g_1, g_2 satisfying conditions (a), (g), and (h).

(vii) $X \geqq_{\text{st}} Y \Leftrightarrow g_1(X, Y) \geqq_{\text{icx}} g_2(X, Y)$ for all g_1, g_2 satisfying conditions (a), (b), (c), (e), (f), (g), and (h).

Proof.

$\Rightarrow$: For simplicity, let us assume X and Y are continuous random variables, with densities f_X and f_Y respectively. Then,

$$\begin{aligned}
\boldsymbol{E}[\Delta g_{12}(X, Y)] &= \int_y \int_x [g_1(x, y) - g_2(x, y)] f_X(x) f_Y(y)\, dx\, dy \\
&= \int_y \int_{x \geqq y} \{[g_1(x, y) - g_2(x, y)] f_X(x) f_Y(y) \\
&\quad - [g_2(y, x) - g_1(y, x)] f_X(y) f_Y(x)\}\, dx\, dy \\
&\geqq \int_y \int_{x \geqq y} [g_1(x, y) - g_2(x, y)][f_X(x) f_Y(y) - f_X(y) f_Y(x)]\, dx\, dy =: LB
\end{aligned}$$

where the inequality follows from (a), which holds in all cases.

(i): From (b) and the fact that $X \geqq_{\text{lr}} Y$, we have $LB \geqq 0$, and therefore $\boldsymbol{E}[\Delta g_{12}(X, Y)] \geqq 0$.

(iv) and (vi): The proof goes through as in Shanthikumar and Yao (1991), Theorems 3.4 and 4.3 respectively, using the fact that $\boldsymbol{E}[\Delta g_{12}(X, Y)] \geqq LB$.

(ii) It is easy to check that if g_1 and g_2 satisfy conditions (a), (b), and (c), and h is any increasing convex function, then $h(g_1(x, y))$ and $h(g_2(x, y))$ satisfy (a) and (b), and therefore, from (i), $\boldsymbol{E}[h(g_1(X, Y))] \geqq \boldsymbol{E}[h(g_2(X, Y))]$.

(iii), (v) and (vii): The proof is similar to that of (ii).

$\Leftarrow$: Let $g_2(x, y) = g_1(y, x)$. then (a), (c), and (d) hold with equality. That (b) holds is equivalent to $g_1 \in \mathscr{G}_{\text{lr}}$, that (g) holds is equivalent to $g_1 \in \mathscr{G}_{\text{hr}}$, and that (g) and (h) hold is equivalent to $g_1 \in \mathscr{G}_{\text{st}}$. The result then follows from Lemma 1.

3. Scheduling application

Consider the following scheduling problem. There are n jobs to be scheduled on a single machine to minimize the total cost, $TC = \sum_{i=1}^n f_i(C_i)$, where C_i is the completion time of job i, and f_i is its cost function. We say that f_i is steeper than f_i, $f_i \geqq_s f_j$, $f_i - f_j$ is non-decreasing. We assume the cost functions and processing times are agreeable in the sense that if $\boldsymbol{E}X_i \leqq \boldsymbol{E}X_j$ then $f_i \geqq_s f_j$ for all i and j, where X_i is the processing time for job i. For example, the total cost might be the weighted flowtime, with agreeable weights. Then we have the following, where SEPT (LEPT) means shortest- (longest-) processing-time-first.

Theorem 2.

(i) If the processing times are likelihood-ratio ordered, and f_i is increasing for all i, then the total cost is stochastically minimized (maximized) by SEPT (LEPT).

(ii) If the processing times are hazard-rate ordered, and f_i is increasing for all i, then the total cost is minimized (maximized) in the increasing convex sense by SEPT (LEPT).

(iii) If the processing times are stochastically ordered, and f_i is increasing and concave for all i, then the expected total cost is minimized (maximized) by SEPT (LEPT).

Proof. Suppose policy π schedules job j immediately following job i, where $EX_i \geqq EX_j$. Let π' be the same as π except for interchanging jobs i and j. We condition on the processing times of all the other jobs besides the interchanged jobs. Then the total costs under policies π and π' are

$$TC_\pi := z + f_i(w + X_i) - f_j(w + X_i + X_j) + h(X_i + X_j) =: g_1(X_i, X_j),$$
$$TC_{\pi'} := z + f_j(w + X_j) + f_i(w + X_i + X_j) + h(X_i + X_j) =: g_2(X_i, X_j),$$

where z is the total cost of the jobs scheduled before job $i(j)$ under $\pi(\pi')$, w is the completion time of the job scheduled immediately before job $i(j)$ under $\pi(\pi')$, and $h(X_i + X_j)$ is the total cost of the jobs scheduled after job $j(i)$ under $\pi(\pi')$. Then one can easily check that $f_i \leqq_s f_j$ and f_l increasing for all l implies that conditions (a), (b), (c), (d), (e), and (g) hold (but not (f)), and that if f_l is also concave for all l, then condition (h) holds as well. Thus, $TC_{\pi'}$ will be no greater than TC_π in the appropriate sense by Theorem 1. The result follows using successive interchanges.

It is also possible to show Theorem 2 using the approach of Chang and Yao (1990) by using arrangement-increasing functions instead of symmetric functions (Chang, personal communication, 1991). See also Frenk (1991a, b) for related work with weaker ordering relations.

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