On sequences of lattice packings

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In this note we establish theorems on compactness of lattice packings.

I. Introduction

1. Let \( \mathbf{a}^1, \ldots, \mathbf{a}^d \) be a linearly independent set of vectors in \( d \)-dimensional euclidean space \( \mathbb{R}^d \). A set \( \Lambda \) is called a \( d \)-dimensional lattice with basis \( \mathbf{a}^1, \ldots, \mathbf{a}^d \) if its elements are all vectors of the form \( \lambda^1 \mathbf{a}^1 + \ldots + \lambda^d \mathbf{a}^d \) where the \( \lambda^i \) are integers. The determinant of \( \Lambda \), \( d(\Lambda) \), is defined by

\[
  d(\Lambda) = |\det(\mathbf{a}^1, \ldots, \mathbf{a}^d)|.
\]

This definition is, in fact, independent of the basis taken for \( \Lambda \).

2. An infinite sequence of lattices \( \{\Lambda_n\} \) is said to converge to a lattice \( \Lambda \) if each \( \Lambda_n \) has a basis \( \mathbf{a}^1_n, \ldots, \mathbf{a}^d_n \) and \( \Lambda \) has a basis \( \mathbf{a}^1, \ldots, \mathbf{a}^d \) such that \( \lim_{n \to \infty} \mathbf{a}^j_n = \mathbf{a}^j \) (1 \( \leq \) \( j \) \( \leq \) \( d \)). With this definition of convergence, \( d(\Lambda) \) becomes a continuous function, for \( \Lambda_n \to \Lambda \) implies \( d(\Lambda_n) \to d(\Lambda) \). Mahler's selection (compactness) theorem for lattices states (see Mahler [4]):

Let \( \{\Lambda_n\} \) be an infinite sequence of lattices satisfying the following two conditions: there are constants \( K_1 \) and \( K_2 \) such that
(i) $d(\Lambda_n) \leq K_1$ for all lattices $\Lambda_n$,

(ii) $\|a\| \geq K_2 > 0$ for every $a \neq 0 \in \Lambda_n$.

Then $\{\Lambda_n\}$ contains a convergent subsequence.

In other words it is possible to select from a bounded sequence of lattices a subsequence which tends to a lattice $\Lambda$. (A sequence of lattices is said to be bounded if it satisfies (i) and (ii).)

3. Let $S$ be a point-set in $E^d$. A lattice $\Lambda$ is called $S$-admissible if no point of $\Lambda$ except possibly 0 is an inner point of $S$. The critical determinant $\Delta(S)$ of $S$ is defined as the infimum of $d(\Lambda)$ taken over all $S$-admissible lattices $\Lambda$. A critical lattice for $S$ is one which is $S$-admissible and for which $d(\Lambda) = \Delta(S)$.

For the existence of a critical lattice of a set Mahler [5] has the following theorem.

Let $S$ be a point set in $E^d$ and let $0 < \Delta(S) < \infty$. Then $S$ possesses a critical lattice if and only if there exists a bounded infinite sequence of $S$-admissible lattices $\{\Lambda_n\}$ such that

$$\lim_{n \to \infty} d(\Lambda_n) = \Delta(S).$$

4. Let $S$ be a set and $P = \{P_n\}$ a sequence of points in $E^d$. If the sets $\{S+P_n\}$ do not overlap (that is, interiors do not meet) then $P_n$ is said to provide a packing for $S$. When $\{P_n\}$ is a lattice $\Lambda$, then we have an $(S, \Lambda)$ lattice packing. For a central symmetric convex set $S$ the following relationship holds: $\Lambda$ is a lattice packing for $S$ if and only if $\Lambda$ is admissible for $2S$.

We say that $\{S+P_n\}$ cover the whole space if each point of space lies in at least one of the sets of $\{S+P_n\}$.

5. A distance function on bounded subsets of $E^d$ is defined by

$$d(S_1, S_2) = \inf\{\epsilon \geq 0 : S_1 \subset N(S_2, \epsilon), S_2 \subset N(S_1, \epsilon)\},$$

where $N(S, \epsilon)$ is the $\epsilon$-neighbourhood of $S$. This distance function defines a metric on the
closed bounded subsets of $\mathbb{R}^d$. For this metric, Blaschke has the following convergence theorem (Blaschke [1]).

Let $\{S_n\}$ be an infinite sequence of compact convex subsets of $\mathbb{R}^d$ which are bounded; that is, contained in some solid sphere. Then $\{S_n\}$ contains a subsequence which converges to a compact convex set.

II. Sequence of packings

Let $L = \{(S, \Lambda_S) : S$ is central symmetric convex set and $\Lambda_S$ is a packing lattice of $S\}$.

Now $L$ can be viewed as a subset of the cartesian product $A \times B$ where $A = \{S\}$ and $B = \{\Lambda_S\}$, and so to have the subspace topology derived from the product (metric) topology on $A \times B$. Thus $(S_n, \Lambda_{S_n})$ tends to, say, $(S_0, \Lambda_{S_0})$ if and only if $S_n$ tends to $S_0$ and $\Lambda_{S_n}$ to $\Lambda_{S_0}$ in the sense defined in 2 and 5 of the introduction. We ask when is $L$ compact? If $L = A \times B$, that is, every element of $A \times B$ is a lattice packing, then by Tychonoff's product theorem $L$ is compact if and only if $A$ and $B$ are compact. But this is a very special case. It occurs if every element of $\{\Lambda_S\}$ provides a packing for every element of $\{S\}$. In the theorem below the above question is answered for the case that $L$ is a map; that is, there is a map $\phi : A \rightarrow B$ such that $(S, \Lambda_S) \in L$ if and only if $\Lambda_S = \phi(S)$.

**THEOREM 1.** Let $A$ and $B$ be compact and $\phi : A \rightarrow B$ as described above. Then $L$ is compact if and only if $\phi$ is continuous.

Proof. Let $\phi$ be continuous and $\{(S_n, \Lambda_{S_n})\}$ be a sequence in $L$. Because $A$ is compact $\{S_n\}$ has a convergent subsequence $\{S_{n_k}\}$ which converges to $S$, say. Since $\phi$ is continuous $\Lambda_{S_{n_k}} = \phi(S_{n_k})$ converges to $\Lambda_S = \phi(S)$.
Thus \( \left\{ \left( S_{n_k}, A_{S_{n_k}} \right) \right\} \) is a convergent subsequence of the original sequence. Therefore \( L \) is compact.

On the other hand let \( L \) be compact and \( \{ S_n \} \) be a sequence in \( A \) converging to \( S \). We have to show that \( A_{S_n} = \phi(S_n) \) tends to \( A_S = \phi(S) \).

Suppose, by way of contradiction, that there is a subsequence \( \{ S_{n_k} \} \) such that no \( A_{S_{n_k}} \) is within a certain distance of \( A_S \). Since \( L \) is compact, there is a subsequence \( \left\{ \left( S_{n_{k_l}}, A_{S_{n_{k_l}}} \right) \right\} \) of \( \left\{ \left( S_{n_k}, A_{S_{n_k}} \right) \right\} \) which converges. Now \( S_{n_{k_l}} \) tends to \( S \) and since each \( A_{S_{n_{k_l}}} \) does not lie within a certain distance of \( A_S \), \( \left\{ A'_{S_{n_{k_l}}} \right\} \) tends to \( A'_S \neq A_S \). However, this sequence tends to \( \left( S, A'_S \right) \) and \( \left( S, A'_S \right) \in L \) which contradicts \( L \) being a map. Therefore \( \phi \) is continuous.

An interesting illustration of the theorem occurs in discussion of Voronoi domains. In Groemer [2] it is shown that corresponding to a lattice \( \Lambda \) there is a unique Voronoi domain \( V(\Lambda) \);

\[
V(\Lambda) = \{x \in \mathbb{E}^d : \|x\| \leq \|x-g\| \text{ for all } g \in \Lambda \}.
\]

Let \( A \) be a compact set of Voronoi domains and \( B \) the corresponding compact set of lattices. Then \( L = \{ (V(\Lambda), \Lambda) : V(\Lambda) \in A \} \) is compact for Groemer has essentially shown \( V(\Lambda) \to \Lambda \) is continuous.

**REMARK 1.** Theorem 1 remains true if we replace the sequence of lattice packings by sequence of lattice coverings or by the same token the theorem is true for a sequence of any lattice distribution of sets \( S \) in \( \mathbb{E}^d \).

**REMARK 2.** In Theorem 1 we showed that the compactness of \( \{ S \} \) and \( \{ A_S \} \) and the continuity of \( \phi : \{ S \} \to \{ A_S \} \) implies that \( \{ (S, A_S) \} \) is compact. Similarly we can show that the compactness of \{ S \} and
{(S, Λ_S)} implies the compactness of \{Λ_S\}, or the compactness of \{Λ_S\} and \{S, Λ_S\} implies the compactness of \{S\}.

**REMARK 3.** By definition, the density function for a packing (S, Λ) is

\[
\lim_{\rho(B) \to \infty} \frac{\tau(B, S, Λ, z)}{V(B)} = \frac{V(S)}{d(Λ)},
\]

where \(V(S)\) is the volume of \(S\), \(B\) is an arbitrary bounded convex body, \(\rho(B)\) is the radius of largest sphere contained in \(B\), \(z\) is an arbitrary point, and \(\tau(B, S, Λ, z)\) is the total volume of the bodies \(S + x\) with \(x \in Λ + z\) and \(S + x \subset B\). (For particulars see Lekkerkerker [3, page 165].)

It can be seen that this density function is continuous in (S, Λ); so in a compact lattice packing there is a packing which has maximum density. In particular let \(Λ_0\) be a packing lattice for a given convex set \(S\). Put \(Λ = \{Λ : (S, Λ) \text{ is a packing and such that } d(Λ) \leq d(Λ_0)\}\). Then \(L = \{(S, Λ) : Λ \in Λ\}\) is a compact set of packings, which assumes a densest packing for \(S\).

### III. Sequences of admissible lattices

Let \(Λ\) be \(S\)-admissible. Then \((S, Λ)\) will be called a pair. If \(Λ\) is also critical for \(S\) then \((S, Λ)\) is called a critical pair. A sequence \((S_n, Λ_n)\) is called bounded if \(\{S_n\}\) and \(\{Λ_n\}\) are bounded.

**THEOREM 2.** Let \((S_n, Λ_n)\) be a bounded sequence of pairs where the \(S_n\) are closed convex. Then \((S_n, Λ_n)\) has a convergent subsequence such that its limit \((S, Λ)\) is a pair. If the \((S_n, Λ_n)\) are also critical pairs, then the limit of this subsequence is a critical pair.

**Proof.** (i) Since \(\{S_n\}\) and \(\{Λ_n\}\) are bounded we may then select a subsequence of pairs \((S_{n_k}, Λ_{n_k})\), a convex set \(S\) and a lattice \(Λ\) such that \(S_{n_k} \to S\), \(Λ_{n_k} \to Λ\). We show \(Λ\) is \(S\)-admissible. Suppose not. Then there exists \(x \in \text{int } S\), \(x \neq 0\), such that \(x \notin Λ\). Since the \(S_{n_k}\) tend to \(S\) and since the \(S_{n_k}\) and \(S\) are convex, there exist a number \(N\)
and a neighbourhood $U$ of $x$ such that

(1) \[ n_k > N \implies x \in U \subset \text{int } S_{n_k}. \]

Also we have

(2) \[ \lim_{n_k \to \infty} \Lambda_{n_k} = \Lambda. \]

From (1) and (2) we obtain a contradiction to $\Lambda_{n_k}$ being $S_{n_k}$-admissible.

Therefore $\Lambda$ is $S$-admissible; that is, $(S, \Lambda)$ is a pair.

(ii) Suppose further that the $(S_n, \Lambda_n)$ are critical. Then $\Lambda(S_n) = d(\Lambda_n)$. Thus

\[
\begin{align*}
d(\Lambda) &= \lim_{n_k \to \infty} d(\Lambda_{n_k}) = \lim_{n_k \to \infty} \Lambda(S_{n_k}) \\
&= \Lambda(S)
\end{align*}
\]

by Mahler's continuity result on critical lattices.

Therefore $\Lambda$ is critical for $S$.

**COROLLARY.** Let $\{S_n\}$ be a bounded sequence of central symmetric convex sets and $\{\Lambda_n\}$ a bounded sequence of lattices. If $\{(S_n, \Lambda_n)\}$ is a sequence of pairs then the sequence of lattice packings $\{(S_n, \Lambda_n)\}$ is compact.

**REMARK.** In Mahler [5] it has essentially been shown that the sequence of pairs $\{S, \Lambda_n\}$ is compact where $S$ is a point set such that $0 < \Delta(S) < \infty$ and $\{\Lambda_n\}$ is bounded.

This is a special case of the sequence of pairs $\{S_n, \Lambda_n\}$ for we can simply put $S_n = S$ for all $n$. But in Theorem 2, $S_n$ cannot be extended to non-convex sets, for example to star shaped sets. In the theorem we have used Blaschke's compactness theorem for convex sets and the continuity of the $\Delta$ function. To extend the theorem to star-shaped bodies we would want a compactness theorem corresponding to Blaschke's while retaining the continuity of the $\Delta$ function. Now Mahler [4] has shown that for $\Delta$ to
be continuous for star bodies, a topology \( \kappa \) must be used. This topology \( \kappa \) is strictly finer than the usual Hausdorff-Blaschke topology for closed sets, say \( \tau \). It is known that \( \tau \) is a compact topology. The following result from topology shows that \( \kappa \) cannot be compact.

If \( \tau \) is a coarser topology than \( \kappa \) on a set \( X \), and if \( \tau \) is Hausdorff and \( \kappa \) compact, then \( \tau = \kappa \) (see, for instance, Rudin [7, p. 61]).

On the other hand, Theorem 1 can be extended to non-convex sets \( S_n \) provided \( \{ S_n \} \) is compact. For instance, Melzak [6] has obtained a compactness theorem for a certain class of star-shaped bodies in \( E^3 \). For this family Theorem 1 is valid.

References


