§ 0. Introduction

Recently, A. Morimoto [5] proved that every isometry of a compact Riemannian manifold of positive dimension has not the pseudo-orbit tracing property, and that if a homeomorphism of a compact metric space has the pseudo-orbit tracing property then \( E_{\varphi} = O_{\varphi} \) (see § 1 for definition). The purpose of this paper is to show that every distal homeomorphism of a compact connected metric space has not the pseudo-orbit tracing property.

The author benefited from reading the papers by A. Morimoto [5, 6].

§ 1. Definitions

Let \( \varphi: X \to X \) be a (self-) homeomorphism of a compact metric space \( X \) with distance function \( d \). A sequence of points \( \{x_i\}_{i \in (a, b)} \) \((-\infty \leq a < b \leq \infty) \) is called a \( \delta \)-pseudo-orbit of \( \varphi \) if \( d(\varphi(x_i), x_{i+1}) < \delta \) for \( i \in (a, b - 1) \). A sequence \( \{x_i\} \) is called to be \( \varepsilon \)-traced by \( x \in X \) if \( d(\varphi^i(x), x_i) < \varepsilon \) holds for \( i \in (a, b) \). We say \( (X, \varphi) \) to have the pseudo-orbit tracing property (abbrev. P.O.T.P.) if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that every \( \delta \)-pseudo-orbit of \( \varphi \) can be \( \varepsilon \)-traced by some point \( x \in X \). The system \( (X, \varphi) \) is said to be minimal if a \( \varphi \)-invariant closed set \( K \) is necessarily \( K = \emptyset \) or \( K = X \). Let \( A \) be a subset of the integer group \( \mathbb{Z} \). Then \( A \) is syndetic if there is a finite subset \( K \) of \( \mathbb{Z} \) with \( \mathbb{Z} = K + A \). Let \( x \in X \). Then \( x \) is an almost periodic point if \( \{n \in \mathbb{Z} : \varphi^n(x) \in U\} \) is a syndetic set for all neighborhoods \( U \) of \( x \). Let \( (X, \varphi) \) be distal, that is, if \( \inf_{n \in \mathbb{Z}} d(\varphi^n(x), \varphi^n(y)) = 0 \) then \( x = y \). Then every \( x \in X \) is an almost periodic point and the converse is true (p. 36 of [2]). It is clear that every equi-continuous homeomorphism has this property and is hence distal. But the converse does not hold. To check this for example, let \( T^2 \) be a 2-dimensional torus...
and define a homeomorphism \( \varphi : T^2 \to T^2 \) by 
\[
\varphi(x, x) = (\alpha + x, nx + x)
\]
\((x, x) \in T^2\) where \( \alpha \in T^1 \) and \( 0 \neq n \in \mathbb{Z} \). Then it will be easily checked that \( \varphi \) is distal but not equi-continuous. A point \( x \in X \) is said to be non-wandering (with respect to \( \varphi \)) if for every neighborhood \( U \) of \( x \), there is an \( n > 0 \) with \( U \cap \varphi^n(U) \neq \emptyset \). The set of all nonwandering points is called the nonwandering set and denoted by \( \Omega(\varphi) \). Since \( X \) is compact, we get \( \Omega(\varphi) \neq \emptyset \). If in particular \((X, \varphi)\) is distal, then it is easily proved that \( \Omega(\varphi) = X \) since every \( x \in X \) is almost periodic. We know (cf. p. 132 of [7]) that there is always a Borel probability measure \( \mu \) on \( X \) which is preserved by \( \varphi \) and \( \varphi^{-1} \), and (cf. p. 135 of [7]) that if \((X, \sigma)\) is minimal then \( \mu(U) > 0 \) for all non-empty open set \( U \).

The set \( 2^X \) of all closed non-empty subsets of \( X \) will be a compact metric space by the distance function \( \delta \) defined by
\[
\delta(A, B) = \max\{\max_{a \in A} d(a, b), \max_{b \in B} d(a, b)\} \quad (A, B \in 2^X)
\]
where \( d(A, B) = \inf_{a \in A} d(a, b) \) (cf. p. 45 of [4]). We denote by \( \text{Orb}^\delta(\varphi) \) the set of all \( \delta \)-pseudo-orbits of \( \varphi \) and by \( \text{Orb}^{\delta^2}(\varphi) \) the set of all \( A \in 2^X \), for which there is \( \{x_i\} \in \text{Orb}^\delta(\varphi) \) such that \( A = \text{Cl} \{x_i : i \in \mathbb{Z}\} \), \( \text{Cl} \) denoting the closure. Let \( E_\varphi \) denote the set of all \( A \in 2^X \) such that for every \( \epsilon > 0 \) there is \( A, \in \text{Orb}^\epsilon(\varphi) \) with \( \delta(A, A_\epsilon) < \epsilon \). An element \( A \) of \( E_\varphi \) is called an extended orbit of \( \varphi \). On the other hand, we define \( O_\varphi = \text{Cl} \{O_\varphi(x) : i \in \mathbb{Z}\} \) where \( O_\varphi(x) = \text{Cl} \{\varphi^i(x) : i \in \mathbb{Z}\} \). We can easily see that \( E_\varphi \) is closed in \( 2^X \) and that \( O_\varphi \subset E_\varphi \) holds.

§ 2. Results

Throughout this section, \( X \) will be a compact metric space with distance function \( d \) and \( \varphi \) will be a self-homeomorphism of \( X \).

**Theorem.** Assume that \( X \) is connected. If \((X, \varphi)\) is distal, then \((X, \varphi)\) has not P.O.T.P.

**Lemma 1.** If \((X, \varphi)\) has P.O.T.P., for every \( \epsilon > 0 \) and every \( x_0 \in \Omega(\varphi) \) there is a point \( y \in X \) and an integer \( k = k(x_0, \epsilon) > 0 \) such that \( O_\varphi(y) \subset U_\epsilon(x_0) \).

**Proof.** Since \( x_0 \in \Omega(\varphi) \), for \( \delta > 0 \) with \( \delta < \epsilon \) there are a point \( x \in X \) and an integer \( k > 0 \) such that \( x \) and \( \varphi^i(x) \) belong to \( U_{\frac{\delta}{2}}(x_0) \). Now, set \( x_{n+1} = \varphi^i(x) \) for \( n \in \mathbb{Z} \) and \( 0 \leq i < k \). Obviously, \( \{x_i : i \in \mathbb{Z}\} = \{\cdots, x, \varphi(x), \cdots, \varphi^{k-1}(x), \cdots\} \in \text{Orb}^\delta(\varphi) \). Hence we can find a point \( y \in X \) such that \( d(\varphi^i(y), \)

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\( x_i < \varepsilon \) for \( i \in \mathbb{Z} \). In particular, \( d(\varphi^{n_k}(y), x_{n_k}) < \varepsilon \) and hence \( d(\varphi^{n_k}(y), x) < \varepsilon \) for \( n \in \mathbb{Z} \). Therefore we have \( O_{\varphi}(y) \subseteq U_i(x_i) \).

**Corollary 1.** Assume that \( X \) is connected and not one point. If \( (X, \varphi) \) is minimal, then \( (X, \varphi) \) has not P.O.T.P.

**Proof.** Let \( \varepsilon = \text{diameter}\ (X)/3 \) and assume that \( (X, \varphi) \) has P.O.T.P. By Lemma 1 we have that for some \( x_0 \in X \) there are \( y \in X \) and \( k > 0 \) with \( O_{\varphi}(y) \subseteq O_{\varphi}(x_0) \). Since \( X \) is connected, \( O_{\varphi}(y) = O_{\varphi}(y) = X \) and so diameter \( (X) \leq 2\varepsilon \). This is a contradiction.

**Corollary 2.** If \( (X, \varphi) \) is minimal, then \( E_\varphi = O_{\varphi} \).

**Proof.** It is proved by A. Morimoto that every \( A \in E_\varphi \) is \( \varphi \)-invariant \( (\varphi(A) = A) \). In fact, for every \( \varepsilon > 0 \) there is \( \varepsilon > \varepsilon_t > 0 \) such that \( d(\varphi(x), \varphi(y)) < \varepsilon \) when \( d(x, y) < \varepsilon_t \). By definition we can find \( \{x_i\} \in \text{Orb}^t(\varphi) \) with \( \overline{d}(A, \text{Cl}(x_i)) < \varepsilon_t \). Set \( y_i = \varphi(x_i) \) for \( i \in \mathbb{Z} \), then \( d(y_i, x_{i+1}) < \varepsilon_t \) and so \( \overline{d}(\text{Cl}(x_i), \text{Cl}(y_i)) < \varepsilon_t \). It is clear that \( d(\varphi(y_i), y_{i+1}) < \varepsilon \) for \( i \in \mathbb{Z} \). Hence, \( \{y_i\} \in \text{Orb}^t(\varphi) \). Let \( A' = \text{Cl}(x_i) \). Then \( \overline{d}(A', \varphi(A')) < \varepsilon_t \) and since \( \overline{d}(A, A') < \varepsilon_t \) we get \( \overline{d}(\varphi(A), \varphi(A')) < \varepsilon_t \). Therefore

\[
\overline{d}(\varphi(A), A) < \overline{d}(\varphi(A), \varphi(A')) + \overline{d}(\varphi(A'), A') + \overline{d}(A', A) < 3\varepsilon
\]

and so \( \overline{d}(\varphi(A), A) = 0 \); i.e. \( \varphi(A) = A \). Therefore we get \( E_\varphi = \{X\} = O_{\varphi} \).

**Lemma 2.** If \( (X, \varphi) \) has P.O.T.P., for every integer \( k > 0 \), \( (X, \varphi^k) \) has also P.O.T.P.

**Proof.** For every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that \( \{x_i\} \in \text{Orb}^t(\varphi) \) is \( \varepsilon \)-traced by a point in \( X \). Take \( \{y_i\} \in \text{Orb}^t(\varphi) \) and put \( x_{n_k+i} = \varphi^i(y_n) \) for \( n \in \mathbb{Z} \) and \( 0 \leq i \leq k - 1 \). Obviously, \( \{x_i\} \in \text{Orb}^t(\varphi) \). Hence there is \( y \in X \) with \( d(\varphi^i(y), x_{n_k+i}) < \varepsilon \) for \( i \in \mathbb{Z} \). In particular, \( d(\varphi^i(y), y_n) = d(\varphi^{n_k+i}(y), x_{n_k}) < \varepsilon \) for \( n \in \mathbb{Z} \). This completes the proof of Lemma 2.

**Lemma 3.** Let \( (X, \varphi) \) be distal. Then for every \( x \in X \), \((O_{\varphi}(x), \varphi)\) is minimal.

**Proof.** Since every \( x \in X \) is almost periodic under \( \varphi \), for a neighborhood \( U \) of \( x \) there is a finite set \( K = \{n_1, \cdots, n_k\} \) of \( \mathbb{Z} \) such that \( Z = A + K \) where \( A = \{n \in \mathbb{Z}: \varphi^n(x) \in U\} \). Hence \( O_{\varphi}(x) = \text{Cl}(\varphi^n(x): n \in A) \cup \text{Cl}(\varphi^{n+k}(x): n \in A) \cup \cdots \cup \text{Cl}(\varphi^{n+k}(x): n \in A) \). Let \( y \in O_{\varphi}(x) \). Then \( O_{\varphi}(y) \cap U \neq \emptyset \). This implies that \( x \in O_{\varphi}(y) \). Hence \( O_{\varphi}(x) = O_{\varphi}(y) \).
Remark 1. If \((X, \varphi)\) is distal and topologically transitive, then it is clearly minimal (by Lemma 3).

We shall now give a proof of the theorem.

Assuming that \((X, \varphi)\) has P.O.T.P., we shall draw a contradiction. To do this, let \(\varepsilon = \text{diameter}(X)/9\). Then there is \(\delta > 0\) with \(\delta < \varepsilon\) such that every \(\{z_t\} \in \text{Orb}^t(\varphi)\) is \(\varepsilon\)-traced by a point of \(X\). Lemma 1 insures us that for \(y \in \Omega(\varphi)\) there are \(y \in X\) and \(k > 0\) with \(O_{\varphi}(y) \subset U(y)\). Put \(\psi = \varphi^k\). Then \((X, \psi)\) has P.O.T.P. (by Lemma 2) and is distal. Since \(X\) is connected and compact, we can take a sequence of points \(\{p_i\}_{i=1}^N\) in \(X\) such that \(p_i = y\), \(d(p_i, p_{i+1}) < \delta/2\) for \(1 \leq i \leq N - 1\) and such that \(\bigcup_{i=1}^N U_{\psi}(p_i) = X\). Since \((X, \psi)\) is distal, every point of \(X\) is almost periodic. Hence for \(1 \leq i \leq N\) there is an integer \(c(i) > 0\) such that \(d(p_i, \psi^c(p_i)) < \delta/2\). Let us put

\[
x_i = \psi^{-i}(p_i) \quad (i < 0)
\]
\[
x_i = \psi^i(p_i) \quad (0 \leq i \leq c(1) - 1)
\]
\[
x_{c(1) + i} = \psi^i(p_c) \quad (0 \leq i \leq c(2) - 1)
\]
\[
\ldots
\]
\[
x_{c(1) + \cdots + c(N - 1) + i} = \psi^i(p_N) \quad (0 \leq i \leq c(N) - 1)
\]
\[
x_{c(1) + \cdots + c(N) + i} = \psi^i(p_{N-1}) \quad (0 \leq i \leq c(N - 1) - 1)
\]
\[
\ldots
\]
\[
x_{c(1) + 2c(2) + \cdots + 2c(N - 1) + c(N) + i} = \psi^i(p_i) \quad (i \geq 0).
\]

Obviously, \(\{x_i\}_{i \in \mathbb{Z}} \in \text{Orb}^t(\psi)\) and \(\overline{d}(\text{Cl} \{x_i\}, X) < \delta\). By assumption, there is \(z \in X\) with \(d(\psi^i(z), x_i) < \varepsilon\) \((i \in \mathbb{Z})\) so that \(\overline{d}(O_{\psi}(z), X) < \delta + \varepsilon < 2\varepsilon\), and in particular

\[
d(\psi^i(z), \psi^i(p_i)) = d(\psi^i(z), \psi^i(y)) < \varepsilon \quad (i < 0),
\]
\[
d(\psi^{i+1}(z), \psi^i(p_i)) = d(\psi^{i+1}(z), \psi^i(y)) < \varepsilon \quad (i \geq 0)
\]

where \(c = c(1) + c(N) + 2 \sum_{i=1}^{N-1} c(i)\). This implies that

\[
\psi^i(z) \in U_i(\psi^i(y)) \subset U_i(O_{\psi}(y)) \quad (i < 0),
\]
\[
\psi^{i+1}(z) \in U_i(\psi^i(y)) \subset U_i(O_{\psi}(y)) \quad (i \geq 0)
\]

where \(U_i(O_{\psi}(y)) = \bigcup_{h \in O_{\psi}(y)} U_i(h)\). Put \(O_{\psi}(z) = \text{Cl} \{\psi^i(z) : i < 0\}\) and \(O_{\psi}(z) = \text{Cl} \{\psi^i(z) : i > 0\}\). Then we have that \(O_{\psi}(z) \subset U_i(O_{\psi}(y))\) and \(\psi^iO_{\psi}(z) \subset U_i(O_{\psi}(y))\). Since \(O_{\psi}(z) \cup O_{\psi}(z) = O_{\psi}(z)\), by Baire’s theorem either \(O_{\psi}(z)\) or \(O_{\psi}(z)\) has non-empty interior in the set \(O_{\psi}(z)\).

Let \(\mu\) be a \(\psi\)-invariant Borel probability measure of \(O_{\psi}(z)\). Since
\((O_\psi(z), \psi)\) is minimal by Lemma 3, every non-empty open set in \(O_\psi(z)\) has \(\mu\)-positive measure. When the interior of \(O_\psi(z)\) in \(O_\psi(z)\) is non-empty, it is easy to see that \(O_\psi(z) = \psi O_\psi(z)\) and so \(O_\psi(z) = O_\psi(z)\). Indeed, assume \(\psi^{-1}O_\psi(z) \subseteq O_\psi(z)\). Then \(V = \bigcap_{k \geq 0} \psi^{-k}O_\psi(z)\) does not contain the interior of \(O_\psi(z)\) in \(O_\psi(z)\). Hence \(\mu(O_\psi(z)\setminus V) > 0\). Since \(O_\psi(z) = \bigcup_{k \geq 0} \psi^{-k}[O_\psi(z)\setminus 0^{-1}O_\psi(z)] \cup V\), we get \(\mu(O_\psi(z)\setminus \psi^{-1}O_\psi(z)) > 0\), thus contradicting \(\mu(O_\psi(z)) \leq 1\). If the interior of \(O_\psi(z)\) in \(O_\psi(z)\) is non-empty; i.e. \(\mu(O_\psi(z)) > 0\), then it follows that \(O_\psi(z) = O_\psi(z)\). Obviously \(O_\psi(z) = \psi^{-1}O_\psi(z)\). In any case we get \(O_\psi(z) \subset U_i(O_\psi(y))\) so that \(O_\psi(z) \subset U_i(O_\psi(y)) \subset U_i(y)\) (because \(O_\psi(y) \subset U_i(y)\)). Since \(2\epsilon > d(O_\psi(z), X) = \max_{x \in X} d(O_\psi(z), x)\), we have \(X = U_i(O_\psi(z))\) from which \(X = U_i(y)\); i.e. diameter \((X) \leq 8\epsilon\). This is a contradiction.

**Remark 2.** We know (Application 2 of [1]) that every (group) automorphism \(\sigma\) of a zero-dimensional compact metric group \(X\) has P.O.T.P. If \((X, \sigma)\) has zero topological entropy (the existence of such automorphisms is known), then we can prove (cf. Lemma 14 of [1]) that \(X\) contains a sequence \(X = X_0 \supset X_1 \supset \cdots\) of completely \(\sigma\)-invariant normal subgroups such that \(\cap X_n\) is trivial and for every \(n \geq 0\), \(X_n/X_{n+1}\) is a finite group. Hence for \(x, y \in X\) (\(x \neq y\)) there is \(n > 0\) such that \(xy^{-1} \notin x_n\). Since \(\sigma(X_n) = x_n\) for all \(j \in Z\), we get easily \(\sigma'(xy^{-1}) \notin X_n\) (\(j \in Z\)), which implies that \(d(\sigma'(x), \sigma'(y)) > d(\sigma'(x)X_n, \sigma'(y)X_n) > 0\) (the distance function \(d\) is a translation invariant metric of \(X\)). Since \(X_n/X_{n+1}\) is a finite group, we get \(\inf d(\sigma'(x), \sigma'(y)) > 0\); i.e. \((X, \sigma)\) is distal. Therefore every zero-dimensional automorphism with zero topological entropy is distal and has P.O.T.P. This shows that the assumption of connectedness in the theorem can not drop out.

**References**

Department of Mathematics
Tokyo Metropolitan University
Fukazawa, Setagaya-ku
Tokyo 158, Japan