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SUFFICIENT CONDITIONS FOR THE OSCILLATION OF DELAY AND NEUTRAL DELAY EQUATIONS

BY

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ABSTRACT. We established sufficient conditions for the oscillation of all solutions of the delay differential equation

$$\dot{x}(t) + px(t - \tau) - qx(t - \sigma) = 0$$

and of the neutral delay differential equation

$$\frac{d^n}{dt^n}[x(t) - px(t - \tau)] + qx(t - \sigma) = 0$$

where p, q, τ and σ are nonnegative constants and n is an odd natural number.

1. **Introduction.** Our aim in this paper is to establish sufficient conditions for the oscillation of all solutions of the delay differential equation (DDE)

$$\dot{x}(t) + px(t - \tau) - qx(t - \sigma) = 0$$

with positive and negative coefficients and nonnegative delays and of the neutral delay differential equations (NDDE) of order n

$$\frac{d^n}{dt^n}[x(t) - px(t - \tau)] + qx(t - \sigma) = 0,$$

where n is odd and

$$0 \leqq p \leqq 1, q > 0 \text{ and } \tau, \sigma \geqq 0.$$

Our results are substantial improvements of known results in [1], [2] and [4].

As is customary, a solution of a differential equation is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or eventually negative.

In this paper, unless otherwise specified, when we write a functional inequality we assume that it is satisfied eventually for all large t.

2. **Oscillations of Delay Equations.** Consider the DDE

$$(1) \qquad \dot{x}(t) + px(t - \tau) - qx(t - \sigma) = 0,$$

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where the coefficients p and q and the delays τ and σ are nonnegative real numbers. In [1] Arino, Ladas and Sficas established the following result giving sufficient conditions for all solutions of Eq. (1) to oscillate.

THEOREM 1. *Consider the* DDE (1) *and assume that the following conditions are satisfied.*

$$p > q \geqq 0 \tag{2}$$

$$\tau \geqq \sigma \geqq 0 \tag{3}$$

$$q(\tau - \sigma) \leqq 1 \tag{4}$$

and

$$(p - q)\tau > \frac{1}{e}. \tag{5}$$

Then, every solution of Eq. (1) *oscillates.*

REMARK 1. Recently, Györi [2] improved Theorem 1 by replacing condition (5) by

$$(p - q)[\tau + q(\tau - \sigma)^2] > \frac{1}{e}. \tag{5'}$$

Our aim in this paper is to further improve Theorem 1 by replacing condition (5) by the following hypothesis:

(H_1) There exists a nonnegative integer N such that every solution of the DDE

$$\dot{y}(t) + \sum_{i=0}^{N} (p - q)q^i(\tau - \sigma)^i y(t - \tau - i\sigma) = 0 \tag{6}$$

oscillates.

REMARK 2. Hypothesis (H_1) is, for example, satisfied when

$$(p - q) \sum_{i=0}^{\infty} q^i(\tau - \sigma)^i(\tau + i\sigma) > \frac{1}{e}. \tag{7}$$

This is because when (7) holds, then there exists a positive integer N such that

$$(p - q) \sum_{i=0}^{N} q^i(\tau - \sigma)^i(\tau + i\sigma) > \frac{1}{e}. \tag{7'}$$

Now by a result of Hunt and Yorke [3], Condition (7′) implies that every solution of Eq. (6) oscillates.

Another condition which implies that every solution of Eq. (6) oscillates is, see [5],

$$(7'') \qquad (p - q)q^{(N/2)}(\tau - \sigma)^{(N/2)}\left[(N + 1)\tau + \frac{N(N + 1)}{2}\sigma\right] > \frac{1}{e}.$$

For $N = 0$, both Conditions (7′) and (7″) reduce to Condition (5). For $N = 1$, Condition (7′) reduces to

$$(p - q)[\tau + q(\tau - \sigma)(\tau + \sigma)] > \frac{1}{e}$$

which is already better than Györi's Condition (5′). So it is clear that replacing Condition (5) of Theorem 1 by the Hypothesis (H_1) produces a substantial improvement of Theorem 1.

Next, we state and prove Theorem 2 which is our extension of Theorem 1.

THEOREM 2. *Consider the* DDE (1) *and assume that Conditions* (2), (3) *and* (4) *and Hypothesis* (H_1) *are satisfied. Then every solution of Eq.* (1) *oscillates.*

PROOF. Assume, for the sake of contradiction, that Eq. (1) has an eventually positive solution $x(t)$. That is, there exists $t_0 \geqq 0$ such that $x(t) > 0$, for $t \geqq t_0$. Set

$$(8) \qquad z(t) = x(t) - q\int_{t-\tau}^{t-\sigma} x(s)ds, \ t \geqq t_0 + \tau.$$

Then

$$(9) \qquad \dot{z}(t) = -(p - q)x(t - \tau) < 0, \ t \geqq t_0 + \tau.$$

It follows that $z(t)$ is a decreasing function and so either

$$(10) \qquad \lim_{t\to\infty} z(t) = -\infty$$

or

$$(11) \qquad \lim_{t\to\infty} z(t) \equiv L \in \mathbf{R}.$$

First, we will show that (10) is impossible. Otherwise, $x(t)$ would be unbounded and so there should exist a $t_1 \geqq t_0 + \tau$ such that

$$z(t_1) < 0$$

and

$$x(t_1) = \max_{s \leqq t_1} x(s).$$

Then (8) would imply that

$$0 > z(t_1) = x(t_1) - q\int_{t_1-\tau}^{t_1-\sigma} x(s)ds \geqq x(t_1)[1 - q(\tau - \sigma)] \geqq 0$$

which is impossible. Thus (11) holds. Now it is not difficult to show (for example, by direct substitution) that $z(t)$ is, itself, a solution of Eq. (1). That is,

$$\dot{z}(t) + pz(t-\tau) - qz(t-\sigma) = 0. \tag{12}$$

We claim that the limit L of $z(t)$ is zero. Otherwise, (12) yields

$$\lim_{t\to\infty} \dot{z}(t) = -(p-q)L \neq 0$$

and so $\lim_{t\to\infty} z(t) = \infty$ or $\lim_{t\to\infty} z(t) = -\infty$ which contradicts (11). We have established that

$$\lim_{t\to\infty} z(t) = 0 \tag{13}$$

which in view of the decreasing nature of $z(t)$ implies that

$$z(t) > 0. \tag{14}$$

Also, in view of (8),

$$x(t) > z(t). \tag{15}$$

From (8) and (9) we see that

$$\dot{z}(t) + (p-q)z(t-\tau) + (p-q)q\int_{t-2\tau}^{t-\tau-\sigma} x(s)ds = 0,\ t \geqq t_0 + 2\tau$$

and by induction we find that for $n = 1, 2, \ldots, N$

$$\begin{aligned} \dot{z}(t) &+ (p-q)z(t-\tau) + (p-q)\sum_{i=1}^{n} q^i \\ &\times \int_{t-2\tau}^{t-\tau-\sigma}\int_{\xi_1-\tau}^{\xi_1-\sigma}\cdots\int_{\xi_{i-1}-\tau}^{\xi_{i-1}-\sigma} z(\xi_i)d\xi_i \ldots d\xi_1 \\ &+ (p-q)q^{n+1}\int_{t-2\tau}^{t-\tau-\sigma}\int_{\xi_1-\tau}^{\xi_1-\sigma}\cdots\int_{\xi_n-\tau}^{\xi_n-\sigma} \\ &\times x(\xi_{n+1})d\xi_{n+1}\ldots d\xi_1 = 0,\ t \geqq t_0 + (N+2)\tau. \end{aligned} \tag{16}$$

By using (15) and the monotonic character of $z(t)$ (replacing $z(s)$ by its value at the upper-limit of integration) we find from (16) that $z(t)$ satisfies the delay inequality

$$\dot{z}(t) + \sum_{i=0}^{N} (p-q)q^i(\tau-\sigma)^i z(t-\tau-i\sigma) \leqq 0. \tag{17}$$

But by Proposition 2 in [1], under the Hypothesis (H_1) the delay inequality (17) cannot have an eventually positive solution. This contradicts (14) and completes the proof of Theorem 2.

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3. **Oscillation of Neutral Differential Equations.** Consider the neutral delay differential equation of order n

$$\frac{d^n}{dt^n}[x(t) - px(t - \tau)] + qx(t - \sigma) = 0 \tag{18}$$

where n is an odd natural number and

$$0 \leqq p \leqq 1, q > 0 \text{ and } \tau, \sigma > 0. \tag{19}$$

Furthermore, we will assume that the following hypothesis holds.

(H_2) There exists a nonnegative integer N such that every solution of the DDE

$$y^{(n)}(t) + \sum_{i=0}^{N} qp^i y(t - \sigma - i\tau) = 0 \tag{20}$$

oscillates.

The main result in this section is the following:

THEOREM 3. *Consider the* NDDE (18) *where n is odd and assume that condition* (19) *and Hypothesis* (H_2) *are satisfied. Then every solution of Eq.* (18) *oscillates.*

PROOF. Assume, for the sake of contradiction, that Eq. (18) has an eventually positive solution $x(t)$. Set

$$z(t) = x(t) - px(t - \tau). \tag{21}$$

Then

$$z^{(n)}(t) = -qx(t - \sigma) < 0. \tag{22}$$

Thus $z^{(n-1)}(t)$ decreases and either

$$\lim_{t\to\infty} z^{(n-1)}(t) = -\infty \tag{23}$$

or

$$\lim_{t\to\infty} z^{(n-1)}(t) \equiv L \in \mathbf{R}. \tag{24}$$

We will prove that (24) holds and that $L = 0$. In fact, if (23) were true then

$$\lim_{t\to\infty} z(t) = -\infty \tag{25}$$

and in particular,

$$z(t) < 0$$

which implies that

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$$x(t) < px(t - \tau) \leqq x(t - \tau).$$

Hence $x(t)$ is bounded which contradicts (25). Thus (24) holds. We now claim that $L = 0$. To this end, observe that $z(t)$ is an n-times differentiable solution of (18). That is,

$$z^{(n)}(t) - pz^{(n)}(t - \tau) + qz(t - \sigma) = 0. \tag{26}$$

If $L \neq 0$, then $\lim_{t\to\infty} z(t) \neq 0$ and so

$$\lim_{t\to\infty} \frac{d}{dt}[z^{(n-1)}(t) - pz^{(n-1)}(t - \tau)] \neq 0$$

which contradicts (24). Thus $L = 0$. Next, we will show that

$$\lim_{t\to\infty} z(t) = 0. \tag{27}$$

In fact, by integrating both sides of (22) from t_1 to t and letting $t \to \infty$ we find, that

$$z^{(n-1)}(t_1) = q \int_{t_1}^{\infty} x(s - \sigma)ds,$$

which shows that $x \in L^1[t_1, \infty)$. Then $z \in L^1[t_1, \infty)$ and as z is monotonic it follows that (27) holds. Also as $z^{(n-1)}(t)$ decreases to zero it follows that

$$z^{(n-1)}(t) > 0. \tag{28}$$

From (27) and (28) and the fact that n is odd it follows that

$$z(t) > 0. \tag{29}$$

We also have

$$z(t) \leqq x(t). \tag{30}$$

From (21) and (22) we see that

$$z^{(n)}(t) + qz(t - \sigma) + qpx(t - \sigma - \tau) = 0$$

and by induction

$$z^{(n)}(t) + \sum_{i=0}^{N} qp^i z(t - \sigma - i\tau) + qp^{N+1}x[t - \sigma - (N + 1)\tau] = 0.$$

Hence,

$$z^{(n)}(t) + \sum_{i=0}^{N} qp^i z(t - \sigma - i\tau) \leqq 0. \tag{31}$$

From (24), with $L = 0$, and (27) it follows that for $i = 1, 2, \ldots, n - 1$

$$\lim_{t\to\infty} z^{(i)}(t) = 0. \tag{32}$$

Integrating (31) from t to ∞ repeatedly n times, we find that

$$\sum_{i=0}^{N} \int_{t}^{\infty} qp^{i} \frac{(t-s)^{n-1}}{(n-1)!} z(s-\sigma-i\tau)ds \leqq z(t). \tag{33}$$

But by a result of Philos [7], if inequality (33) has an eventually positive solution $z(t)$ then the corresponding equation

$$\sum_{i=0}^{N} \int_{t}^{\infty} qp^{i} \frac{(t-s)^{n-1}}{(n-1)!} y(s-\sigma-i\tau)ds = y(t). \tag{34}$$

also has an eventually positive solution $y(t)$. It follows then that Eq. (20) has the eventually positive solution $y(t)$. This contradicts Hypothesis (H_2) and completes the proof of the Theorem.

REMARK 3. For $N = 0$, Hypothesis (H_2) is equivalent to the condition

$$q^{(1/n)}\frac{\sigma}{n} > \frac{1}{e}. \tag{35}$$

Indeed when $N = 0$, Eq. (20) reduces to

$$y^{(n)}(t) + qy(t-\sigma) = 0 \tag{36}$$

and (35) is a necessary and sufficient condition for all solutions of Eq. (36) to oscillate. See [6].

Thus we have the following corollary of Theorem 3.

COROLLARY 1. *Assume that conditions* (19) *and* (35) *are satisfied and suppose that n is an odd natural number. Then, every solution of Eq.* (18) *oscillates.*

The above corollary was established in [4].

REMARK 4. Assume that $n = 1$ and that

$$\sum_{i=0}^{\infty} qp^{i}(\sigma + i\tau) > \frac{1}{e}. \tag{37}$$

Then, there exists N sufficiently large such that

$$\sum_{i=0}^{N} qp^{i}(\sigma + i\tau) > \frac{1}{e}. \tag{38}$$

But (38) is a sufficient condition for all solutions of Eq. (20) to oscillate. See [3]. Thus Condition (37) implies that the Hypothesis (H_2) is satisfied. We have just established the following corollary of Theorem 3.

COROLLARY 2. *Assume that Conditions* (19) *and* (37) *are satisfied. Then every solution of the* NDDE

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$$\frac{d}{dt}[x(t) - px(t-\tau)] + qx(t-\sigma) = 0$$

oscillates.

The above corollary was proved by Györi [2].

REMARK 5. In addition to condition (37), each of the following two conditions which were established in [6] implies that the Hypothesis (H_2) is satisfied.

(i) $$\max_{0 \leqq i \leqq N} \left[\frac{\sigma + i\tau}{n}(qp^i)^{(1/n)}\right] > \frac{1}{e}$$

(ii) $$\left[q\frac{p^{N+1} - 1}{p - 1}\right]^{(1/n)} \frac{\sigma}{n} > \frac{1}{e} \text{ and } 0 \leqq p < 1.$$

REFERENCES

1. O. Arino, G. Ladas and Y. G. Sficas, *On oscillations of some retarded differential equations*, SIAM J. Math. Anal. (to appear).

2. I. Györi, *Oscillations of retarded differential equations of the neutral and the mixed types* (to appear).

3. B. R. Hunt and J. A. Yorke, *When all solution of* $x' = -\sum_{i=1}^{n} q_i(t)x(t - T_i(t))$ *oscillate*, J. Differential Equations **53** (1984), pp. 139-145.

4. G. Ladas and Y. G. Sficas, *Oscillations of higher-order neutral equations*, J. Austral. Math. Soc. Ser. B **27** (1986), pp. 502-511.

5. G. Ladas and I. P. Stavroulakis, *Oscillations caused by several retarded and advanced arguments*, J. Differential Equations **44** (1982), pp. 134-152.

6. ———, *Oscillations of differential equations of the mixed type*, J. Math. Phys. Sciences **18** (1984), pp. 245-262.

7. C. G. Philos, *On the existence of nonoscillatory solutions tending to zero at* ∞ *for differential equations with positive delays*, Archiv. Math. **36** (1980), pp. 168-178.

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