Can. J. Math. Vol. **46** (2), 1994 pp. 284–297

OSCILLATION AND NONOSCILLATION PROPERTIES OF NEUTRAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We obtain a number of new conditions for oscillation of the first order neutral delay equation with nonconstant coefficients of the form

$$\frac{d}{dt}[x(t) - px(t-\tau)] + \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) = 0.$$

Comparison results are also given as well as conditions for the existence of nonoscillatory solutions.

1. **Introduction.** In this paper we mainly consider the neutral delay differential equations of the form

$$\frac{d}{dt}[x(t) - px(t-\tau)] + \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) = 0 \tag{1.1}$$

where $p \in [0,1]$, $q_i\ (i = 1,\dots,n) \in C\big([t_0,\infty),[0,\infty)\big)$ and $\tau,\ \sigma_i\ (i = 1,\dots,n) \in (0,\infty)$.

Our aim here is to establish some new sufficient conditions for oscillation and existence of nonoscillatory solutions of equation (1.1). As corollaries, some results are derived which yield sufficient and necessary conditions for oscillation. We also obtain some comparison criteria and give some explicit conditions for oscillation. Similar results are obtained for delay equations with varible delays.

Let $r = \max\{\tau,\sigma_1,\dots,\sigma_n\}$. By a solution of (1.1) we mean a function $x \in C\big([t_1 - r,\infty), R\big)$ for some $t_1 \geq t_0$, such that $x(t) - px(t-\tau)$ is continuously differentiable for $t \geq t_1$ and such that (1.1) is satisfied for $t \geq t_1$.

As is customary, a solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory. For some recent results in oscillation theory, see [1–15] and the references cited therein. For completeness, we cite the following results:

RESULT 1 [1, 14]. Let $0 \leq p \leq 1$ and $q_i > 0\ (i = 1,\dots,n)$ be constants. Then (1.1) is oscillatory if and only if

$$\lambda(1 - pe^{-\lambda\tau}) + \sum_{i=1}^{n} q_i e^{-\lambda\sigma_i} = 0 \tag{1.2}$$

Research supported by NSERC-Canada.
Received by the editors May 19, 1992; revised May 9 1993.
AMS subject classification: 34K15.
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https://doi.org/10.4153/CJM-1994-013-9 Published online by Cambridge University Press

has no real root.

RESULT 2 [11]. Let $0 \leq p \leq 1$ be a constant, $\sigma_i = i\tau$, $i = 1, \ldots, n$, and $q_i(t)$, $i = 1, \ldots, n$ be τ-periodic functions. Then (1.1) is oscillatory if and only if

$$\lambda(1 - pe^{-\lambda\tau}) + \sum_{i=1}^{n} \frac{1}{\tau}\Big(\int_0^\tau q_i(s)\,ds\Big)e^{-\lambda\sigma_i} = 0 \tag{1.3}$$

has no real root.

RESULT 3 [4]. (This corrects the mistake in [9]). Consider the case $n = 1$ in (1.1) where $q_1(t) := q(t)$, $\sigma_1 := \sigma$, and p is replaced by $p(t) \in C\big([t_0, \infty), [0, \infty)\big)$.

i) Assume $p(t)$ is bounded, $p(t^* + n\tau) \leq 1$ for a $t^* \geq t_0$ and $n = 0, 1, 2, \ldots$, and $q(t) \geq q > 0$, $t \geq t_0$. If

$$\inf_{\mu > 0, t \geq T}\left[p(t-\sigma)\,\frac{q(t)}{q(t-\tau)}\,e^{\mu\tau} + \frac{1}{\mu}q(t)e^{\mu\sigma}\right] > 1,$$

then (1.1) is oscillatory.

ii) Assume there exists a $\mu^* > 0$ such that

$$\sup_{t \geq T}\left[p(t-\sigma)\,\frac{q(t)}{q(t-\tau)}e^{\mu^*\tau} + \frac{1}{\mu^*}q(t)e^{\mu^*\sigma}\right] \leq 1.$$

Then (1.1) has a positive solution.

In Section 2 we will obtain sufficient conditions for oscillation of (1.1) and for the existence of a nonoscillatory solution for the neutral equation (1.1). These conditions cover Result 1 for the constant coefficient case and improve Result 2 for the periodic coefficient case. They also yield some sufficient and necessary conditions for oscillation even for a class of equations with aperiodic coefficients. In Section 3, the above criteria for oscillation are developed for delay equations with variable delays, which substantially improve the conjecture of Hunt and Yorke [12]. Based on these results, in Section 4, we derive some comparison criteria for oscillation and existence of nonoscillatory solutions, and in Section 5, we obtain some explicit conditions for oscillation.

Before stating the main results we introduce the following lemmas which will be used in the proofs.

LEMMA 1.1. *Let $a > 0$, $b > 0$, and $f(t) \geq 0$ be a locally integrable function on $[0, \infty)$. Assume both the limits*

$$I_1 = \lim_{t\to\infty} \frac{1}{a}\int_t^{t+a} f(s)\,ds \quad \textit{and} \quad I_2 = \lim_{t\to\infty}\frac{1}{b}\int_t^{t+b} f(s)\,ds$$

exist and are finite. Then $I_1 = I_2$.

PROOF. We prove it by contradiction. Without loss of generality we only consider the case that $I_1 > I_2$. Choose a positive integer n such that $anI_1 - b(m+1)I_2 > 1$, where

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$m = [an/b]$ is the integer part of an/b. This is possible since $I_1 > I_2$ and $b[an/b] \le an$. Choose $T \ge 0$ so large that for $t \ge T$

$$\frac{1}{a}\int_t^{t+a} f(s)\,ds > I_1 - \frac{1}{n+m+1}$$

and

$$\frac{1}{b}\int_t^{t+b} f(s)\,ds < I_2 + \frac{1}{n+m+1}.$$

This implies that

$$\int_{T+bm}^{T+an} f(s)\,ds > anI_1 - bmI_2 - \frac{n+m}{n+m+1} > bI_2 + \frac{1}{n+m+1}.$$

But this is impossible since

$$(T+an)-(T+bm) = an - [an/b]b \le b,$$

and hence

$$\int_{T+bm}^{T+an} f(s)\,ds \le \int_{T+bm}^{T+(m+1)b} f(s)\,ds \le bI_2 + \frac{1}{n+m+1}.$$

This gives a contradition. ∎

With a similar proof we get the following generalization of Lemma 1.1.

LEMMA 1.2. *Let* $0 < a_1 \le a(t) \le a_2 < \infty$, $0 < b_1 \le b(t) \le b_2 < \infty$, *and let* $f(t) \ge 0$ *be a locally integrable function on* $[0,\infty)$. *Assume both the limits*

$$I_1 = \lim_{t\to\infty} \frac{1}{a(t)}\int_t^{t+a(t)} f(s)\,ds \quad \text{and} \quad I_2 = \lim_{t\to\infty}\frac{1}{b(t)}\int_t^{t+b(t)} f(s)\,ds$$

exist. Then $I_1 = I_2$.

2. **Criteria for neutral equations.** In this section we are concerned with the equation

$$\frac{d}{dt}[x(t) - px(t-\tau)] + \sum_{i=1}^{n} q_i(t)x(t-\sigma_i) = 0 \tag{2.1}$$

where

$$q_i(t) \in C\big([t_0,\infty),[0,\infty)\big), \quad i = 1,\ldots,n,$$
$$p \in [0,1], \quad \tau,\sigma_i \in (0,\infty), \quad i = 1,\ldots,n. \tag{2.2}$$

Denote $r = \max\{\tau,\sigma_1,\ldots,\sigma_n\}$.

The following lemma is needed in the proof.

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LEMMA 2.1. *In addition to (2.2) assume*

$$q_1(t) \geq q > 0 \tag{2.3}$$

and

$$q_i(t-\tau) \leq q_i(t), \quad t \geq t_0 + r, \quad i = 1, \ldots, n. \tag{2.4}$$

Let $x(t)$ be an eventually positive solution of (2.1), and let

$$z(t) = x(t) - px(t-\tau). \tag{2.5}$$

Then eventually $z(t) > 0$, $z'(t) < 0$, and

$$z'(t) - pz'(t-\tau) + \sum_{i=1}^{n} q_i(t) z(t-\sigma_i) \leq 0. \tag{2.6}$$

PROOF. From (2.1) it is easy to see that $z(t) > 0$, $z'(t) < 0$ eventually. From (2.4) and (2.5)

$$\begin{aligned} z'(t) - pz'(t-\tau) &= -\sum_{i=1}^{n} [q_i(t)x(t-\sigma_i) - pq_i(t-\tau)x(t-\tau-\sigma_i)] \\ &\leq -\sum_{i=1}^{n} q_i(t)[x(t-\sigma_i) - px(t-\tau-\sigma_i)] \\ &= -\sum_{i=1}^{n} q_i(t) z(t-\sigma_i). \end{aligned}$$

Thus (2.6) is true eventually. ∎

THEOREM 2.1. *Assume (2.2)–(2.4) hold, and for all $\mu > 0$, and $\ell = \tau, \sigma_1, \ldots, \sigma_n$*

$$\liminf_{t\to\infty} \Big[pe^{\mu\tau} + \frac{1}{\ell\mu} \sum_{i=1}^{n} e^{\mu\sigma_i} \int_t^{t+\ell} q_i(s)\, ds \Big] > 1. \tag{2.7}$$

Then (2.1) is oscillatory.

PROOF. Assume (2.1) has an eventually positive solution $x(t)$. Define $z(t)$ as given by (2.5). Then by Lemma 2.1 there exists a $T \geq t_0$ such that $z(t) > 0$, $\dot{z}(t) < 0$, and (2.6) holds for $t \geq T$. Let $w(t) = -\frac{z'(t)}{z(t)}$, $t \geq T$. Then $w(t) > 0$, $t \geq T$, and (2.6) becomes

$$w(t) \geq pw(t-\tau) \exp\Big(\int_{t-\tau}^{t} w(s)\, ds \Big) + \sum_{i=1}^{n} q_i(t) \exp\Big(\int_{t-\sigma_i}^{t} w(s)\, ds \Big)$$

for $t \geq T + r$, where $r = \max\{\tau, \sigma_1, \ldots, \sigma_n\}$.

We now define a sequence of functions $\{w_k(t)\}$ for $k = 1, 2, \ldots$, and $t \geq T$, and a sequence of numbers $\{\mu_k\}$ for $k = 1, 2, \ldots$, as follows:

$$w_1(t) \equiv 0, \quad t \geq T$$

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and for $k = 1, 2, \ldots, t \geq T + kr$

$$(2.8) \qquad w_{k+1}(t) = pw_k(t-\tau)\exp\Big(\int_{t-\tau}^{t} w_k(s)\,ds\Big) + \sum_{i=1}^{n} q_i(t)\exp\Big(\int_{t-\sigma_i}^{t} w_k(s)\,ds\Big);$$

and $\mu_1 = 0$, and for $k = 1, 2, \ldots$

$$(2.9) \qquad \mu_{k+1} = \inf_{t\geq T} \min_{\ell=\tau,\sigma_1,\ldots,\sigma_n} \Big\{p\mu_k e^{\mu_k\tau} + \frac{1}{\ell}\sum_{i=1}^{n} e^{\mu_k\sigma_k}\int_t^{t+\ell} q_i(s)\,ds\Big\}.$$

We claim that the following inequalities hold:

i) $0 = \mu_1 < \mu_2 < \cdots$;

ii) $w_k(t) \leq w(t)$ for $t \geq T + (k-1)r$ and $k = 1, 2, \ldots$;

iii) $\frac{1}{\ell}\int_t^{t+\ell} w_k(s)\,ds \geq \mu_k$ for $t \geq T + (k+1)r$, $k = 1, 2, \ldots$, and $\ell = \tau, \sigma_1, \ldots, \sigma_n$.

In fact, since $\mu_2 > \mu_1 = 0$, and $w_1(t) < w(t)$ for $t \geq T$, by induction we see i) and ii) are true. We now show that iii) also holds. Clearly iii) is true for $k = 1$. Assume iii) is true for some k. Then (2.8) and (2.9) imply that for $t \geq T + kr$, $\ell = \tau, \sigma_1, \ldots, \sigma_n$

$$\begin{aligned}\frac{1}{\ell}\int_t^{t+\ell} w_{k+1}(s)\,ds &= \frac{p}{\ell}\int_t^{t+\ell} w_k(s-\tau)\exp\Big(\int_{s-\tau}^{s} w_k(\theta)\,d\theta\Big)\,ds \\ &\quad + \frac{1}{\ell}\sum_{i=1}^{n}\int_t^{t+\ell} q_i(s)\exp\Big(\int_{s-\sigma_i}^{s} w_k(\theta)\,d\theta\Big)\,ds \\ &\geq p\mu_k e^{\mu_k\tau} + \frac{1}{\ell}\sum_{i=1}^{n} e^{\mu_k\sigma_i}\int_t^{t+\ell} q_i(s)\,ds \\ &\geq \inf_{t\geq T}\min_{\ell=\tau,\sigma_1,\ldots,\sigma_n}\Big\{p\mu_k e^{\mu_k\tau} + \frac{1}{\ell}\sum_{i=1}^{n} e^{\mu_k\sigma_i}\int_t^{t+\ell} q_i(s)\,ds\Big\} \\ &= \mu_{k+1}.\end{aligned}$$

Hence iii) holds.

Let $\mu^* = \lim_{k\to\infty}\mu_k$. From (2.7) and (2.9) there exists an $\alpha > 1$ such that $\mu_{k+1} \geq \alpha\mu_k$, $k = 1, 2, \ldots$, and this means that $\mu^* = \infty$.

By ii) and iii) we have that $\lim_{t\to\infty}\int_t^{t+\sigma_1} w(s)\,ds = \infty$, and so

$$\limsup_{t\to\infty}\int_t^{t+\frac{\sigma_1}{2}} w(s)\,ds = \infty.$$

Integrating both sides of the equation $w(t) = -\frac{z'(t)}{z(t)}$ from t to $t + \frac{\sigma_1}{2}$ for t sufficiently large we get

$$\frac{z(t)}{z(t+\frac{\sigma_1}{2})} = \exp\Big(\int_t^{t+\frac{\sigma_1}{2}} w(s)\,ds\Big).$$

Thus

$$(2.10) \qquad \limsup_{t\to\infty}\frac{z(t)}{z(t+\frac{\sigma_1}{2})} = \infty.$$

Since

$$z'(t) = -\sum_{i=1}^{n} q_i(t)x(t-\sigma_i) \leq -qx(t-\sigma_1) \leq -qz(t-\sigma_1),$$

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integrating both sides from $t+\sigma_1/2$ to $t+\sigma_1$ and using the decreasing nature of $z(t)$ we find for t sufficiently large

$$0 < z(t+\sigma_1) \le z\Big(t+\frac{\sigma_1}{2}\Big) - \frac{q\sigma_1}{2} z(t).$$

Thus

$$\frac{z(t)}{z(t+\frac{\sigma_1}{2})} \le \frac{2}{q\sigma_1}$$

contradicting (2.10). This completes the proof. ∎

THEOREM 2.2. *Assume (2.2) holds and there exist a* $\mu^* > 0$ *and a* $T \ge t_0$ *such that for* $\ell = \tau, \sigma_1, \ldots, \sigma_n$

$$\sup_{t\ge T}\Big[pe^{\mu^*\tau} + \frac{1}{\ell\mu^*}\sum_{i=1}^{n} e^{\mu^*\sigma_i}\int_t^{t+\ell} q_i(s)\,ds\Big] \le 1, \tag{2.11}$$

Then (2.1) has a positive solution on $[T+r,\infty)$.

PROOF. First we claim that the integral equation

$$v(t) = pv(t-\tau)\exp\Big(\int_{t-\tau}^{t} v(s)\,ds\Big) + \sum_{i=1}^{n} q_i(t)\exp\Big(\int_{t-\sigma_i}^{t} v(s)\,ds\Big) \tag{2.12}$$

possesses a positive solution on $[T+r,\infty)$. To this end set

$$v_1(t) \equiv 0, \quad t \ge T,$$

and for $k = 1, 2, \ldots$

(2.13)

$$v_{k+1}(t) = \begin{cases} pv_k(t-\tau)\exp\big(\int_{t-\tau}^{t} v_k(s)\,ds\big) + \sum_{i=1}^{n} q_i(t)\exp\big(\int_{t-\sigma_i}^{t} v_k(s)\,ds\big), & t \ge T+r \\ \beta_{k+1}(t), & T \le t < T+r \end{cases}$$

where $\{\beta_k\}$ are given function sequence satisfying

i) $\beta_k \in C^2\big([T,T+r),[0,\infty)\big)$ with $\beta_k' \ge 0$ and $\beta_k''(t) \ge 0$, $t \in [T,T+r)$, $k = 1,2,\ldots,$

ii) $\beta_k(t) = 0$, $t \in [T, T+r-\tau]$, $\beta_k(T+r) = v_k(T+r)$, and $\beta_k(t)$ are increasing in k for $t \in [T+r-\tau, T+r)$ and $k = 1, 2, \ldots,$

iii) for every $\ell = \tau, \sigma_1, \ldots, \sigma_n$, for $t \in [T+r-\ell, T+r)$, $k = 1, 2, \ldots,$

$$\int_t^{T+r} \beta_k(s)\,ds \le \int_{t+\ell}^{T+r+\ell} v_k(s)\,ds.$$

Obviously, $v_1(t) \le v_2(t) \le \cdots$. By induction we will show that for $k = 1, 2, \ldots$ and $\ell = \tau, \sigma_1, \ldots, \sigma_n$

$$\frac{1}{\ell}\int_t^{t+\ell} v_k(s)\,ds \le \mu^*, \quad t \ge T. \tag{2.14}$$

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In fact, (2.14) is true for $k = 1$. Assume (2.14) is true for some k. Then from (2.11) and (2.13) we have for $t \geq T + r$, $\ell = \tau, \sigma_1, \ldots, \sigma_n$

(2.15)
$$\frac{p}{\ell}\int_t^{t+\ell} v_k(s-\tau)\exp\Big(\int_{s-\tau}^{s} v_k(\theta)\,d\theta\Big)\,ds + \frac{1}{\ell}\sum_{i=1}^{n}\int_t^{t+\ell} q_i(s)\exp\Big(\int_{s-\sigma_i}^{s} v_k(\theta)\,d\theta\Big)\,ds$$
$$\leq p\mu^* e^{\mu^*\tau} + \frac{1}{\ell}\sum_{i=1}^{n} e^{\mu^*\sigma_i}\int_t^{t+\ell} q_i(s)\,ds \leq \mu^*$$

For every $\ell = \tau, \sigma_1, \ldots, \sigma_n$, for $t \in [T + r - \ell, T + r)$, from (2.15) and condition iii) for $\{\beta_k\}$,

$$\frac{1}{\ell}\int_t^{t+\ell} v_{k+1}(t) = \frac{1}{\ell}\Big[\int_t^{T+r}\beta_{k+1}(s)\,ds + \int_{T+r}^{t+\ell} v_{k+1}(s)\,ds\Big]$$
$$\leq \frac{1}{\ell}\int_{T+r}^{T+r+\ell} v_k(s)\,ds \leq \mu^*.$$

From the monotonic property of $\beta_{k+1}(t)$ with respect to t we see that (2.14) also holds for $t \in [T, T + r - \ell)$, $\ell = \tau, \sigma_1, \ldots, \sigma_n$.

Let $v(t) = \lim_{k\to\infty} v_k(t)$. Then $v(t) \equiv 0$, $t \in [T, T + r - \tau]$, $v(t)$ is increasing on $[T + r - \tau, T + r)$ and for $t \geq T$ and $\ell = \tau, \sigma_1, \ldots, \sigma_n$,

$$\frac{1}{\ell}\int_t^{t+\ell} v(s)\,ds \leq \mu^*.$$

Taking limits as $k \longrightarrow \infty$ on both sides of (2.13), from Lebesgue monotone convergence theorem we see that $v(t)$ satisfies (2.12) for $t \geq T + r$. It is also easy to see that $v(t)$ is well-defined on $[T, \infty)$. In fact, by condition ii) of $\{\beta_k\}$,

$$v(T + r) = \sum_{i=1}^{n} q_i(T + r)\exp\Big(\int_{T+r-\sigma_i}^{T+r} v(s)\,ds\Big)$$
$$\leq \sum_{i=1}^{n} q_i(T + r)e^{\mu^*\sigma_i} := M < \infty,$$

and hence $v(t) \leq M$ for $t \in [T, T + r]$. If $v(t^*) = \infty$ for some $t^* > T + r$, then choose an integer m such that $t^* - m\tau \in [T + r - \tau, T + r)$. By (2.12) we have $v(t^* - m\tau) = \infty$. This is impossible. Furthermore, from condition i) of $\{\beta_k\}$ we get that $v(t)$ is continuous on $[T, T + r]$, and in view of (2.12) we see that $v(t)$ is continuous on the whole interval $[T, \infty)$. Thus $v(t)$ is a positive solution of (2.12) on $[T + r, \infty)$. Set

$$x(t) = \exp\Big(-\int_{T+r}^{t} v(s)\,ds\Big), \quad t \geq T + r.$$

Then $x(t)$ is a positive solution of (2.1). ■

REMARK 2.1. Theorems 2.1 and 2.2 partially improve the criteria given by Result 3 since in (2.7) and (2.11) the "integral averages" of functions are used instead of the functions themselves. Both the theorems are sharp since together they give the following result which extends Results 1 and 2.

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THEOREM 2.3. *Assume (2.2)–(2.4) hold, and* $\sum_{i=1}^{n} e^{\mu\sigma_i}\int_t^{t+\ell} q_i(s)\,ds$ *is a nondecreasing function in* t *for* $\ell=\tau,\sigma_1,\ldots,\sigma_n$. *Then (2.1) is oscillatory if and only if for all* $\mu>0$, *and for* $\ell=\tau,\sigma_1,\ldots,$ *or* σ_n

$$\lim_{t\to\infty}\Big[pe^{\mu\tau}+\frac{1}{\ell\mu}\sum_{i=1}^{n}e^{\mu\sigma_i}\int_t^{t+\ell}q_i(s)\,ds\Big]>1.$$

PROOF. Denote

$$f(t,\mu,\ell)=pe^{\mu\tau}+\frac{1}{\ell\mu}\sum_{i=1}^{n}e^{\mu\sigma_i}\int_t^{t+\ell}q_i(s)\,ds.$$

From the condition we see that $\lim_{t\to\infty}f(t,\mu,\ell)$ exists for $\ell=\tau,\sigma_1,\ldots,\sigma_n$. By Lemma 1.1 we have

$$\lim_{t\to\infty}f(t,\mu,\tau)=\lim_{t\to\infty}f(t,\mu,\sigma_1)=\cdots=\lim_{t\to\infty}f(t,\mu,\sigma_n).$$

In this case for $\ell=\tau,\sigma_1,\ldots,\sigma_n$

$$\lim_{t\to\infty}f(t,\mu,\ell)=\liminf_{t\to\infty}f(t,\mu,\ell)=\sup_{t\ge T}f(t,\mu,\ell).$$

The conclusion is then immediate from Theorems 2.1 and 2.2. ∎

As a special case, we have the following corollary.

COROLLARY 2.4. *Assume (2.2) and (2.3) hold, and there exists an* $\ell>0$ *such that* $\tau=m_0\ell,\sigma_i=m_i\ell$ *for some integers* m_i, $i=0,\ldots,n$. *Furthermore,*

$$q_i(t)=g_i(t)+h_i(t),\quad i=1,\ldots,n$$

where $g_i(t)$ *are* ℓ*-periodic functions with* $\frac{1}{\ell}\int_t^{t+\ell}g_i(s)\,ds=g_i^*$, $h_i(t)$ *are nondecreasing functions with* $\lim_{t\to\infty}h_i(t)=h_i^*$, $i=1,\ldots,n$. *Then (2.1) is oscillatory if and only if for all* $\mu>0$,

$$pe^{\mu\tau}+\frac{1}{\mu}\sum_{i=1}^{n}e^{\mu\sigma_i}(q_i^*+h_i^*)>1.$$

If $q_i(t)\equiv 0$ and $h_i(t)$ are constants, $i=1,\ldots,n$, then Corollary 2.4 becomes Result 1; if $h_i(t)\equiv 0$, $i=1,\ldots,n$, then Corollary 2.4 gives an extension to Result 2 since the requirement $\sigma_i=i\tau, i=1,\ldots,n$ is improved here.

Theorems 2.1 and 2.2 will also yield necessary and sufficient conditions for some equations other than those satisfying the hypotheses of Theorem 2.3. To see this, we give the following example.

EXAMPLE 2.1. Consider the equation

$$\frac{d}{dt}[x(t)-px(t-\tau)]+\Big[1-\frac{1}{t}(1-\sin t)\Big]x(t-\sigma)=0,\quad t\ge 2 \tag{2.16}$$

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where $0 \leq p \leq 1, \tau, \sigma > 0$. Let $q(t) = 1 - \frac{1}{t}(1 - \sin t)$. It is easy to see that for all $\ell > 0$

$$\frac{1}{\ell}\int_t^{t+\ell} q(s)\,ds = \frac{1}{\ell}\int_t^{t+\ell}\Big[1 - \frac{1}{s}(1 - \sin s)\Big]\,ds \longrightarrow 1, \quad \text{as } t \longrightarrow \infty,$$

and

$$\sup_{t\geq T}\frac{1}{\ell}\int_t^{t+\ell} q(s)\,ds = 1$$

According to Theorems 2.1 and 2.2, (2.16) is oscillatory if and only if for all $\mu > 0$

$$pe^{\mu\tau} + \frac{1}{\mu}e^{\mu\sigma} > 1.$$

3. **Criteria for delay equations.** Now we are going to extend the criteria in Section 2 to the delay equation with variable delays

$$\frac{d}{dt}x(t) + \sum_{i=1}^{n} q_i(t)x\big(t - \sigma_i(t)\big) = 0 \tag{3.1}$$

under the assumptions

(H1) $q_i(t), \sigma_i(t) \in C\big([t_0, \infty), [0, \infty)\big)$, $t - \sigma_i(t) \longrightarrow \infty$, as $t \longrightarrow \infty$, $i = 1, \ldots, n$;

(H2) $0 \leq q_i(t) \leq q^*$, $0 \leq \sigma_i(t) \leq \sigma^*$, $i = 1, \ldots, n$, where $q^*, \sigma^* > 0$.

The following is a conjecture of Hunt and Yorke in [12] which was recently established by Chen and Huang in [3].

RESULT 4. Under the assumptions (H1) and (H2), if for all $\mu > 0$

$$\liminf_{t\longrightarrow\infty}\Big\{\frac{1}{\mu}\sum_{i=1}^{n} q_i(s)e^{\mu\sigma_i(t)}\Big\} > 1,$$

then (3.1) is oscillatory.

Employing the method in Section 2 we may obtain this as a consequence of the result below. Again, since the "integral average" technique is involved, they greatly improve Result 4. Under certain circumstances, they also yield necessary and sufficient conditions.

Denote

$$f\big(t, \mu, \ell(t)\big) = \frac{1}{\mu\ell(t)}\sum_{i=1}^{n}\int_t^{t+\ell(t)} q_i(s)e^{\mu\sigma_i(s)}\,ds.$$

THEOREM 3.1. *Let (H1) and (H2) hold, and for all $\mu > 0$*

$$\liminf_{t\longrightarrow\infty} f\big(t, \mu, \sigma_i(t)\big) > 1, \quad i = 1, \ldots, n.$$

Then (3.1) is oscillatory.

THEOREM 3.2. *Let (H1) and (H2) hold, and there exist $\mu > 0$and $T \geq t_0$ such that*

$$\sup_{t\geq T} f\big(t, \mu^*, \sigma_i(t)\big) \leq 1, \quad i = 1, \ldots, n.$$

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Then (3.1) has a positive solution on $[T_1, \infty)$ *for some* $T_1 > T$.

THEOREM 3.3. *Let (H1) and (H2) hold with* $0 \leq \sigma_* \leq \sigma_i(t) \leq \sigma^*$. *Assume* $f\big(t, \mu, \sigma_i(t)\big)$ *is a nondecreasing function in t for* $\mu > 0$ *and* $i = 1, \ldots, n$. *Then (3.1) is oscillatory if and only if* $\lim_{t\to\infty} f\big(t, \mu, \sigma_i(t)\big) > 1$ *for all* $\mu > 0$ *and some* $i = 1, \ldots,$ *or, n.*

The proofs are basically the same as those of Theorems 2.1–2.3. Here we only give an outline of the proof of Theorem 3.1. Note also that Lemma 1.2 is needed in the proof of Theorem 3.3.

PROOF OF THEOREM 3.1. Assume (3.1) has an eventually positive solution $x(t)$. From (3.1) there exists $t_0 \geq 0$ such that $x(t) > 0, x'(t) \leq 0$, $t \geq t_0$. Denote a sequence $\{t_k\}_{k=0}^{\infty}$ by

$$t_k = \sup\Big\{t : \min_{i=1,\ldots,n} \{t - \sigma_i(t)\} \leq t_{k-1}\Big\}$$

Then $t_k \geq t_{k-1}$, $k \geq 1$. Let $w(t) = -\frac{x'(t)}{x(t)}$. From (3.1)

$$w(t) = \sum_{i=1}^{n} q_i(t) \exp\Big(\int_{t-\sigma_i(t)}^{t} w(s)\, ds\Big), \quad t \geq t_1. \tag{3.2}$$

We now define a sequence of functions $\{w_k(t)\}$ and a sequence of numbers $\{\mu_k\}$ as follows:

$$w_1(t) \equiv 0, \quad t \geq t_0$$

and for $k = 1, 2, \ldots$

$$w_{k+1} = \sum_{i=1}^{n} q_i(t) \exp\Big(\int_{t-\sigma_i(t)}^{t} w_k(s)\, ds\Big), \quad t \geq t_k;$$

and $\mu_1 = 0$, and for $k = 1, 2, \ldots$

$$\mu_{k+1} = \inf_{t\geq t_0} \min_{\ell=\sigma_1,\ldots,\sigma_n} \Big\{ \frac{1}{\ell(t)} \sum_{i=1}^{n} \int_t^{t+\ell(t)} q_i(s) e^{\mu_k \sigma_i(s)}\, ds \Big\}.$$

Similar to the proof of Theorem 2.1 we can show that for $\ell(t) = \sigma_1(t), \ldots, \sigma_n(t)$

$$\frac{1}{\ell(t)} \int_t^{t+\ell(t)} w_k(s)\, ds \geq \mu_k \longrightarrow \infty, \quad \text{as } k \longrightarrow \infty$$

and $w(t) \geq w_k(t)$ for $t \geq t_k$, $k = 1, 2, \ldots$. Hence for $\ell(t) = \sigma_1(t), \ldots, \sigma_n(t)$

$$\frac{1}{\ell(t)} \int_t^{t+\ell(t)} w(s)\, ds \longrightarrow \infty, \quad \text{as } t \longrightarrow \infty. \tag{3.3}$$

From Theorem 1 in [11] we get that

$$\liminf_{t\to\infty} \max_{1\leq i\leq n} \{q_i(t)\sigma_i(t)\} > 0,$$

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and that if we let $q(t) = q_j(t), \sigma(t) = \sigma_j(t)$ be such that for each t

$$q_j(t)\sigma_j(t) = \max\{q_i(t)\sigma_i(t), i = 1, \ldots, n\},$$

then $0 < q_* \leq q(t) \leq q^*$ and $0 < \sigma_* \leq \sigma(t) \leq \sigma^*$. From (3.3), $\frac{1}{\sigma(t)} \int_t^{t+\sigma(t)} w(s)\,ds \longrightarrow \infty$ as $t \longrightarrow \infty$, hence $\int_t^{t+\sigma^*} w(s)\,ds \longrightarrow \infty$ as $t \longrightarrow \infty$. Therefore,

$$\limsup_{t\longrightarrow\infty} \int_t^{t+\sigma_*/2} w(s)\,ds = \infty.$$

Noting that

$$x'(t) = -\sum_{i=1}^n q_i(t)x\big(t - \sigma_i(t)\big) \leq -q(t)x\big(t - \sigma(t)\big) \leq -q_* x(t - \sigma_*),$$

the rest of the proof is similar to that of Theorem 2.1. ∎

REMARK 3.1. It can be shown that Theorems 3.1–3.3 still hold if we replace (H2) by:

(H3) there exists a nonempty subset I of the set $\{1, \ldots, n\}$ such that $\sigma(t) = \min\{\sigma_i(t), i \in I\}$ satisfying $t - \sigma(t) \longrightarrow \infty$ as $t \longrightarrow \infty$,

$$\liminf_{t\longrightarrow\infty} \sigma(t) = \sigma_0 > 0$$

and

$$\liminf_{T\longrightarrow\infty} \int_{t-\sigma_0}^t \sum_{i\in I} q_i(s)\,ds > 0.$$

The corresponding results are improvement of Theorem 2 in [3].

4. **Comparison results.** Using the above theorems we can derive some comparison results for oscillation and for the existence of nonoscillatory solutions for a pair of equations. Here we will only mention the results based on the theorems in Section 2. Parallel results based on the theorems in Section 3 are obtained in a similar fashion. This is left to the interested reader.

Consider two equations

$$\frac{d}{dt}[x(t) - px(t-\tau)] + \sum_{i=1}^n q_i(t)x(t - \sigma_i) = 0 \tag{4.1}$$

and

$$\frac{d}{dt}[x(t) - \tilde{p}x(t-\tilde{\tau})] + \sum_{i=1}^n \tilde{q}_i(t)x(t - \tilde{\sigma}_i) = 0. \tag{4.2}$$

Assume (2.2)–(2.4) hold for both the sets $\{q_i(t)\}_{i=1}^n$ and $\{\tilde{q}_i(t)\}_{i=1}^n$. Furthermore, assume the conditions for $q_i(t)$, $i = 1, \ldots, n$, in Corollary 2.4 hold.

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THEOREM 4.1. *i) Suppose (4.1) is oscillatory, and* $\tilde{p} \geq p$, $\tilde{\tau} \geq \tau$, *and for all* $\mu > 0$, $\ell = \tilde{\tau}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_n$

$$\liminf_{t \to \infty} \frac{1}{\ell} \sum_{i=1}^{n} e^{\mu \tilde{\sigma}_i} \int_t^{t+\ell} \tilde{q}_i(s)\, ds \geq \sum_{i=1}^{n} e^{\mu \sigma_i} (q_i^* + h_i^*). \tag{4.3}$$

Then (4.2) is oscillatory.

ii) Suppose (4.2) is oscillatory, and $\tilde{p} \leq p$, $\tilde{\tau} \leq \tau$, *and there exists a* $T \geq t_0$ *such that for all* $\mu > 0$, $\ell = \tilde{\tau}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_n$

$$\sup_{t \geq T} \frac{1}{\ell} \sum_{i=1}^{n} e^{\mu \tilde{\sigma}_i} \int_t^{t+\ell} \tilde{q}_i(s)\, ds \leq \sum_{i=1}^{n} e^{\mu \sigma_i} (q_i^* + h_i^*). \tag{4.4}$$

Then (4.1) is oscillatory.

iii) Suppose (4.2) is nonoscillatory, and $\tilde{p} \geq p$, $\tilde{\tau} \geq \tau$, *and for all* $\mu > 0$, $\ell = \tilde{\tau}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_n$, *(4.3) holds. Then (4.1) has a nonoscillatory solution.*

iv) Suppose (4.1) has a nonoscillatory solution, and $\tilde{p} \leq p$, $\tilde{\tau} \leq \tau$, *and there exists a* $T \geq t_0$ *such that for all* $\mu > 0$, $\ell = \tilde{\tau}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_n$, *(4.4) holds. Then (4.2) has a nonoscillatory solution.*

PROOF. i) Since (4.1) is oscillatory, by Corollary 2.4 we have for all $\mu > 0$,

$$pe^{\mu\tau} + \frac{1}{\mu} \sum_{i=1}^{n} e^{\mu\sigma_i} (q_i^* + h_i^*) > 1.$$

Then (4.3) gives that for all $\mu > 0$, $\ell = \tilde{\tau}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_n$

$$\liminf_{t \to \infty} \Big[\tilde{p} e^{\mu\tilde{\tau}} + \frac{1}{\ell\mu} \sum_{i=1}^{n} e^{\mu\tilde{\sigma}_i} \int_t^{t+\ell} \tilde{q}_i(s)\, ds \Big] > 1.$$

By Theorem 2.1, (4.2) is oscillatory.

ii) If not, (4.1) has a nonoscillatory solution. By Corollary 2.4 there exists a $\mu^* > 0$ such that

$$pe^{\mu^*\tau} + \frac{1}{\mu^*} \sum_{i=1}^{n} e^{\mu^*\sigma_i} (q_i^* + h_i^*) \leq 1.$$

Then (4.4) gives that for all $\mu > 0$, $\ell = \tilde{\tau}, \tilde{\sigma}_1, \ldots, \sigma_n$

$$\sup_{t \geq T} \Big[\tilde{p} e^{\mu^*\tilde{\tau}} + \frac{1}{\ell\mu^*} \sum_{i=1}^{n} e^{\mu^*\tilde{\sigma}_i} \int_t^{t+\ell} \tilde{q}_i(s)\, ds \Big] \leq 1.$$

By Theorem 2.2, (4.2) has a nonoscillatory solution, contradicting the assumption.

iii) and iv) are the converses of i) and ii). ∎

Theorem 4.1 provides criteria for oscillation and for existence of nonoscillatory solutions for a certain class of equations by means of a comparison with equations having constant, periodic or even aperiodic coefficients. The criteria are given by investigating the "integral averages" of the coefficients q_i over an interval of length ℓ, and can be easily verified. It is clear that Theorem 4.1 substantially improves Theorems 2 and 3 in [12] under the conditions (2.2)–(2.4) since the latter are only very special cases of parts ii) and iv) in Theorem 4.1.

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5. **Explicit conditions for oscillation.** We can also obtain some explicit conditions for oscillation from Theorems 2.1 and 3.1. As an example, we mention the result derived from Theorem 2.1.

THEOREM 5.1. *Assume (2.2)–(2.4) hold, and for* $\ell = \tau, \sigma_1, \ldots, \sigma_n$

$$\liminf_{t\to\infty} \sum_{i=1}^{n} \sum_{k=0}^{\infty} \Big(\frac{1}{\ell} \int_t^{t+\ell} q_i(s)\, ds\Big) p^k (k\tau + \sigma_i) > \frac{1}{e}. \tag{5.1}$$

Then (2.1) is oscillatory.

PROOF. We show that (2.7) is true for all $\mu > 0$, and then (2.1) is oscillatory by Theorem 2.1.

For any $\mu > 0$, if $pe^{\mu\tau} \geq 1$, then (2.7) is obviously true. So we only consider the values of μ such that $pe^{\mu\tau} < 1$. From a well-known inequality we see that for $\mu > 0$,

$$e^{\mu(k\tau+\sigma_i)} \geq e\mu(k\tau + \sigma_i). \tag{5.2}$$

So we have

$$\begin{aligned}\frac{1}{\mu} \sum_{i=1}^{n} \Big(\frac{1}{\ell} \int_t^{t+\ell} q_i(s)\, ds\Big) e^{\mu\sigma_i} (1 - pe^{\mu\tau})^{-1} &= \frac{1}{\mu} \sum_{i=1}^{n} \sum_{k=0}^{\infty} \Big(\frac{1}{\ell} \int_t^{t+\ell} q_i(s)\, ds\Big) p^k e^{\mu(k\tau+\sigma_i)} \\ &\geq \sum_{i=1}^{n} \sum_{k=0}^{\infty} \Big(\frac{1}{\ell} \int_t^{t+\ell} q_i(s)\, ds\Big) p^k e (k\tau + \sigma_i).\end{aligned}$$

Then (5.1) implies that for $\ell = \tau, \sigma_1, \ldots, \sigma_n$

$$\liminf_{t\to\infty} \frac{1}{\mu} \sum_{i=1}^{n} \Big(\frac{1}{\ell} \int_t^{t+\ell} q_i(s)\, ds\Big) e^{\mu\sigma_i} (1 - pe^{\mu\tau})^{-1} > 1$$

or

$$\liminf_{t\to\infty} \frac{1}{\ell\mu} \sum_{i=1}^{n} e^{\mu\sigma_i} \int_t^{t+\ell} q_i(s)\, ds > 1 - pe^{\mu\tau}.$$

Hence (2.7) holds for all $\mu > 0$, and $\ell = \tau, \sigma_1, \ldots, \sigma_n$, and the theorem is proved. ∎

Theorem 5.1 gives a sharp condition for oscillation in the sense that for the constant coefficient case it coincides with our recent result in [5] and it is better than the corresponding results in [2, 6, 7, 8, 13].

REFERENCES

1. O. Arino and I. Györi, *Necessary and sufficient condition for oscillation of a neutral differential system with several delays*, J. Differential Equations **81**(1989), 98–105.

2. Q. Chuanxi and G. Ladas, *Oscillations of first-order neutral equations with variable coefficients*, Monatsh. Math. **109**(1990), 103–111.

3. S. Chen and Q. Huang, *On a conjecture of Hunt and Yorke*, J. Math. Anal. Appl. **159**(1991), 469–484.

4. Q. Chuanxi, G. Ladas, B. G. Zhang and T. Zhao, *Sufficient conditions for oscillation and existence of positive solutions*, Applicable Anal. **35**(1990), 187–194.

5. L. Erbe and Qingkai Kong, *Some necessary and sufficient conditions for oscillation of neutral differential equations*, preprint.

https://doi.org/10.4153/CJM-1994-013-9 Published online by Cambridge University Press

6. M. K. Grammatikopoulos, E. A. Grove and G. Ladas, *Oscillation of first order neutral delay differential equations*, J. Math. Anal. Appl. **120**(1986), 510–520.

7. M. K. Grammatikopoulos, G. Ladas and A. Meimaridou, *Oscillation and asymptotic behavior of higher order neutral equations with variable coefficients*, Chinese Ann. Math. **3**(1988), 322–337.

8. M. K. Grammatikopoulos, G. Ladas and Y. G. Sficas, *Oscillation and asymptotic behavior of neutral equations with variable coefficients*, Rad. Mat. **2**(1986), 279–303.

9. E. A. Grove, M. R. S. Kulenović and G. Ladas, *Sufficient conditions for oscillation and nonoscillation of neutral equations*, J. Differential Equations **68**(1987), 373–382.

10. ———, *A Myskis-type comparison result for neutral equations*, Math. Nachr. **146**(1990), 195–206.

11. Q. Huang and S. Chen, *Oscillation of neutral differential equations with periodic coefficients*, Proc. Amer. Math. Soc. **110**(1990), 997–1001.

12. B. R. Hunt and J. A. Yorke, *When all solutions of* $x' = -\sum q_i x(t - T_i(t))$ *oscillate*, J. Differential Equations **53**(1984), 139–145.

13. G. Ladas and Y. G. Sficas, *Oscillations of neutral delay differential equations*, Canad. Math. Bull. **29**(1986), 438–445.

14. ———, *Oscillation of higher-order neutral equations*, J. Austral. Math. Soc. Ser. (B) **27**(1986), 502–511.

15. G. Ladas, Ch. G. Philos and Y. G. Sficas, *Oscillation in neutral equations with periodic coefficients*, Proc. Amer. Math. Soc. **113**(1991), 123–134.

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