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# Regular Dilations on Kreĭn Spaces

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*Abstract.* For bounded operators on Kreĭn spaces, isometric or unitary dilations always exist. We prove that any minimal isometric or unitary dilation has a precise geometrical structure. Moreover, a bounded operator $T$ has a unique minimal unitary dilation if and only if $T$ and $T^*$ have unique minimal isometric dilation if and only if $T$ is either contractive or expansive and $T^*$ is either contractive or expansive. Passing to the bi-dimensional case, a minimal unitary extension (in short, m.u.e.) $U = (U_1, U_2)$ is obtained for a pair $V = (V_1, V_2)$ of commuting bounded isometries on a Kreĭn space. There is a link with the one-dimensional case: if $U$ is an m.u.e. for $V$, then $U_1U_2$ is an m.u.e. for $V_1V_2$. Also, if $(V_1V_2)^*$ is either contractive or expansive, then $V$ has a unique minimal unitary extension. A minimal regular isometric dilation is obtained for a commuting pair $T = (T_1, T_2)$ of bounded operators on a Kreĭn space such that $T_1$, $T_2$ are contractions and $T$ is a bidisc contraction or $T_1$, $T_2$ are expansions and $T$ is a bidisc expansion. The existence of a minimal unitary extension is used to provide a minimal regular unitary dilation for $T$. Discussions about uniqueness and geometric structure conclude the article.

## 1 Introduction

One of the most fruitful directions of research in order to develop a suitable spectral theory for nonselfadjoint operators was opened by the theorem of Sz.-Nagy [56] on the existence of a unitary dilation for every contraction operator on a Hilbert space. The matrix construction for such dilations, proposed by Schäffer in [52], was the starting point to obtain their precise geometrical structure (cf. [29, 58, 61]).

The problem of finding isometric or unitary dilations for families of commuting contractions was proposed by Sz.-Nagy and solved in the case when the family in discussion is double commuting (cf. [57, 60]). Ando [2] proved that every pair of commuting contractions has isometric dilation. Unfortunately, Ando's result cannot be extended for arbitrary families of more than two contractions, according to the example given by Parrot [46].

Later developments show that the problem of finding a unitary dilation for a family $T = (T_\omega)_{\omega\in\Omega}$ of commuting contractions on a Hilbert space $\mathfrak{H}$ can be reduced, by the Naimark theorem [45], to the possibility of extending the function

$$\mathbb{Z}_+^\Omega \ni n \mapsto T^n \in \mathcal{B}(\mathfrak{H})$$

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to a positive definite one on $\mathbb{Z}^\Omega$. It was the idea of Brehmer [9] to consider the *regular extension*

$$\mathbb{Z}^\Omega \ni n \mapsto (T^{n^-})^* T^{n^+} \in \mathcal{B}(\mathfrak{H})$$

and to obtain the so-called *regular unitary dilations*. Their systematic study was completed and simplified later by Sz.-Nagy [59] and Halperin [30, 31]. In the multi-variable case, even supposing that the existence is assured, minimal isometric or unitary dilations are, in general, not unique. However, the minimality condition ensures the uniqueness of a regular isometric or unitary dilation. Regular dilation results have been used to provide models for commuting multi-operators [13, 47, 49, 63], in connection with intertwining liftings [26], von Neumann inequalities [6], [25], operator moment problems [50], Markov processes [41], completely contractive representations of product systems of correspondences [55], or in the context of right LCK semigroups [37].

The large class of applications involving dilation theory, for example, in operator interpolation problems, optimization, control, and systems theory (excellent references are given by the survey of Shalit [53] or by the books of Foiaş–Frazho [22], Foiaş–Frazho–Gohberg–Kaashoek [23] and Rosenblum–Rovnyak [51]), but also in prediction theory [35] motivate our work.

It is natural to assume that such a theory on spaces with indefinite metric (in particular on Kreĭn or Pontryagin spaces) will provide at least a similar set of applications. We should mention in this context that operators on Kreĭn have been used recently, for example, in machine learning [39, 40] or frame theory [17, 34, 38].

The following section (Section 2) is devoted to some preliminary facts concerning Kreĭn spaces, their Kreĭn subspaces, and bounded operators on such objects. Basic facts on the theory of Kreĭn spaces and operators on them are given in [3, 8, 32, 36]; to see also the excellent monograph [27].

One variable dilation theory on Kreĭn spaces is the subject of Section 3. The indefinite case started with the theorem of Davis [14] proving that every bounded operator on a Hilbert space $\mathfrak{H}$ has a unitary dilation on a Kreĭn space $\mathfrak{K}$ containing $\mathfrak{H}$ as a regular subspace. The result holds true even if we suppose that $\mathfrak{H}$ is a (more general) Kreĭn space, as showed by Dijskma–Langer–de Snoo [16] using Carathéodory-type representations for holomorphic operator functions, or by Constantinescu–Gheondea [10] following a Schäffer-type matrix construction. Geometric structure results are obtained for any minimal isometric or unitary dilation (Theorem 3.1). In such a generality, a minimal isometric dilation of a bounded operator $T$ is unique (up to a unitary equivalence) if and only if $T$ is either contractive or expansive [28]. We prove that $T$ has a unique minimal unitary dilation if and only if $T$ and $T^*$ have unique minimal isometric dilations.

The main result of Section 4, the existence of a minimal unitary extension for every commuting pair of bounded isometries on a Kreĭn space, is based on a matrix construction similar to the one given by the author in [48, Theorem 3.3.1]. The Hilbert space case was obtained by Itô [33] (cf. also Brehmer [9] and Douglas [18]). The problem of finding conditions for the uniqueness of a minimal unitary extension reduces to the unidimensional case by the observation in [7] (on Hilbert spaces), and extended here, that $U_1U_2$ is a minimal unitary extension for $V_1V_2$ if $U = (U_1, U_2)$ is a minimal unitary extension for $V = (V_1, V_2)$.

Two-variable dilation theory on Kreĭn spaces has also been considered earlier. The first result in this generalized context has been obtained by Azizov, Barsukov, and Dijksma in [4]. We should also remark that, in the Hilbert space case, Andō's theorem [2] was proved to be equivalent with the commutant lifting theorem given by Sz.-Nagy and Foiaş [62]. In the indefinite case, several versions of this last mentioned theorem have been obtained by Alpay [1], Baidiuk and Hassi [5], Constantinescu and Gheondea [10, 11], Dritschel [19], Dritschel and Rovnyak [21], or Dijksma, Dritschel, Marcantognini, and de Snoo [15]. Some of these proofs could lead to different Andō-type dilations. An excellent survey on this topic is presented in the paper [20] of Dritschel. There are attempts to a several (more than two) variable dilation theory on Kreĭn spaces (see, e.g., [42]). The theory of unitary extensions for pairs of Kreĭn space isometries has been initiated in the paper of Marcantognini and Moran [43].

The last section contains structure results for the minimal (regular) isometric dilation provided that such a dilation exists. The geometric structure given by Theorem 5.3 is the indefinite correspondent of some Hilbert space results appeared in [54] (for double commuting contractions) or, more generally, in [24] (for commuting contractive pairs having regular dilation). The most important result of the article is the existence of a minimal regular isometric dilation for every commuting pair $T = (T_1, T_2)$ of bounded operators on a Kreĭn space such that $T_1$, $T_2$ are contractions and $T$ is a bidisc contraction or $T_1$, $T_2$ are expansions and $T$ is a bidisc expansion (Theorem 5.6). If the conditions above are satisfied, a minimal regular isometric dilation is unique up to a unitary equivalence (Theorem 5.8). The unitary extension (obtained in Section 4) for a regular isometric dilation provides a regular unitary dilation (Corollary 5.7).

We remark that similar results also hold true for finite families of more than two commuting operators. These topics will be treated elsewhere.

## 2 Preliminaries on Kreĭn spaces

### 2.1 Kreĭn spaces, regular subspaces, and operators

A *Kreĭn space* is a complex linear space $\mathfrak{K}$ equipped with a Hermitian sesquilinear form $\langle\cdot,\cdot\rangle_{\mathfrak{K}}$ and having a decomposition

$$\mathfrak{K} = \mathfrak{K}_+ \oplus \mathfrak{K}_- , \tag{2.1}$$

where $(\mathfrak{K}_\pm, \pm\langle\cdot,\cdot\rangle_{\mathfrak{K}})$ are Hilbert spaces ("$\oplus$" denotes an orthogonal direct sum). Decomposition (2.1) is said to be a *fundamental decomposition* of the Kreĭn space $\mathfrak{K}$ and, in general, it is not unique. It induces on $\mathfrak{K}$ a Hilbert space structure: if $P_\pm$ are the orthogonal projections onto $\mathfrak{K}_\pm$ and $J = P_+ - P_-$ (called a *fundamental symmetry* or *signature operator*) then $\mathfrak{K}$ becomes a Hilbert space (denoted $\mathfrak{K}_J$) when equipped with the inner product

$$\mathfrak{K}_J \times \mathfrak{K}_J \ni (x, y) \mapsto [x, y]_J := \langle Jx, y\rangle_{\mathfrak{K}} \in \mathbb{C}.$$

The strong topology of this Hilbert space is independent of the choice of a fundamental decomposition and is usually called the *Mackey topology of* $\mathfrak{K}$. All topological notions on a Kreĭn space are to be understood with respect to this strong topology.

The cardinal numbers

$$\kappa^{\pm}(\mathfrak{K}) = \dim_{alg}(\mathfrak{K}_{\pm})$$

are the *positive* (respectively, *negative*) *indices of* $\mathfrak{K}$ and are also independent of the choice of a fundamental decomposition. The *rank of indefiniteness* of $\mathfrak{K}$ is $\kappa(\mathfrak{K}) = \min \kappa^{\pm}(\mathfrak{K})$.

A *subspace* $\mathfrak{H}$ of a Kreĭn space $\mathfrak{K}$ is a closed linear manifold of $\mathfrak{K}$. It is *positive* (respectively, *negative*) if

$$\langle h, h \rangle_{\mathfrak{K}} \geq 0 \qquad (\text{respectively, } \langle h, h \rangle_{\mathfrak{K}} \leq 0),$$

for every $h \in \mathfrak{K}$. A positive (respectively, negative) subspace is said to be *maximal positive* (respectively, *maximal negative*) if it is not contained in a larger positive (respectively, negative) subspace. It is called *uniformly positive* (respectively, *uniformly negative*) if, for a certain $\delta_J > 0$,

$$\langle h, h \rangle_{\mathfrak{K}} \geq \delta_J \|h\|_J^2 \quad (\text{respectively, } \langle h, h \rangle_{\mathfrak{K}} \leq -\delta_J \|h\|_J^2), \quad h \in \mathfrak{H}.$$

Similarly, one defines *maximal uniformly positive* (respectively, *maximal uniformly negative*) subspaces.

The *orthogonal subspace* of $\mathfrak{H}$ is $\mathfrak{H}^{\perp} = \{k \in \mathfrak{K} \mid \langle h, k \rangle = 0,\ h \in \mathfrak{H}\}$. For each pair $(\mathfrak{M}, \mathfrak{N})$ of subspaces in $\mathfrak{K}$, we use the notation $\mathfrak{M} \perp \mathfrak{N}$ if $\mathfrak{M} \subset \mathfrak{N}^{\perp}$, and $\mathfrak{M} \oplus \mathfrak{N}$ if the sum $\mathfrak{M} + \mathfrak{N}$ is closed, orthogonal, and direct. $\mathfrak{H}$ is said to be *regular* (or *ortho-complemented*) if $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{H}^{\perp}$.

***Proposition 2.1*** *The following conditions are equivalent:*

(i) $\mathfrak{H}$ *is regular.*

(ii) $\mathfrak{H}$ *is a Kreĭn space in the inner product inherited from* $\mathfrak{K}$.

(iii) *There exists a fundamental symmetry* $J$ *on* $\mathfrak{K}$ *such that* $J\mathfrak{H} \subset \mathfrak{H}$ *(hence* $J\mathfrak{H} = \mathfrak{H}$*).*

Condition (ii) justifies the use of the term *Kreĭn subspace* for any regular subspace.

If $T : \mathfrak{D}(T) \subset \mathfrak{K}_1 \to \mathfrak{K}_2$ is a densely defined linear operator between Kreĭn spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$, then its *Kreĭn adjoint* $T^* : \mathfrak{D}(T^*) \subset \mathfrak{K}_2 \to \mathfrak{K}_1$ is uniquely determined by the relation

$$\langle x, T^* y \rangle_{\mathfrak{K}_1} = \langle Tx, y \rangle_{\mathfrak{K}_2}, \quad x \in \mathfrak{D}(T),\ y \in \mathfrak{D}(T^*).$$

The Kreĭn adjoint $T^*$ and the Hilbert adjoint $T^{\times}$ computed relative to fundamental symmetries $J_1$ (on $\mathfrak{K}_1$) and $J_2$ (on $\mathfrak{K}_2$) are related by $T^* = J_1 T^{\times} J_2$. If $T$ belongs to $\mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2)$ (the set of all bounded linear operators between $\mathfrak{K}_1$ and $\mathfrak{K}_2$), then $T^* \in \mathcal{B}(\mathfrak{K}_2, \mathfrak{K}_1)$. Note that any fundamental symmetry $J$ on a Kreĭn space $\mathfrak{K}$ belongs to $\mathcal{B}(\mathfrak{K})$ and $J^* = J^{\times} = J^{-1} = J$.

A linear operator $V : \mathfrak{D}(V) \subset \mathfrak{K}_1 \to \mathfrak{K}_2$ is *isometric* if $\langle Vx, Vy \rangle_{\mathfrak{K}_2} = \langle x, y \rangle_{\mathfrak{K}_1}$, $x$, $y \in \mathfrak{K}_1$. An isometric operator $U$ between Kreĭn spaces $\mathfrak{K}_1$ and $\mathfrak{K}_2$ is said to be *unitary* if $\mathfrak{D}(U) = \mathfrak{K}_1$ and $\mathfrak{R}(U) = \mathfrak{K}_2$.

An everywhere defined isometry is bounded if and only if its range is closed. Then, the range is a regular subspace. We deduce that any unitary operator is continuous. However, a densely defined Kreĭn space isometry may fail to have a continuous extension.

Two subspaces $\mathfrak{M}_1$ (of $\mathfrak{K}_1$) and $\mathfrak{M}_2$ (of $\mathfrak{K}_2$) are said to be *isometrically isomorphic* if there exists a boundedly invertible isometric operator $U : \mathfrak{M}_1 \to \mathfrak{M}_2$. In this situation, $\mathfrak{M}_1$ is regular if and only if $\mathfrak{M}_2$ is regular. Note that two regular subspaces are isometrically isomorphic if and only if they have the same positive (respectively, negative) indices. Let us finally remark that definite Kreĭn spaces (their rank of indefiniteness is null) $\mathfrak{K}$ can be characterized by the fact that every unitary operator $U$ on $\mathfrak{K}$ is power bounded ($\sup_n \|U^n\| < \infty$).

## 2.2 Hardy-type Kreĭn spaces and operator extensions

If $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_n$ is a finite family of mutually orthogonal regular subspaces of a Kreĭn space $\mathfrak{K}$, then the subspace $\mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots \oplus \mathfrak{M}_n$ is clearly regular. The result remains no longer true if the family is infinite. For a finite or infinite family $\{\mathfrak{K}_n\}_{n\geq 0}$ of Kreĭn spaces we can, however, compute their *external orthogonal direct sum* as the set $\bigoplus_{n\geq 0} \mathfrak{K}_n$ of all sequences $k = \{k_n\}_{n\geq 0}$ with $\sum_{n\geq 0} \|k_n\|_n^2 < \infty$ (here, for each $n$, the norm $\|\cdot\|_n$ is computed relative to a given *fundamental decomposition of* $\mathfrak{K}_n$). It becomes a Kreĭn space relative to the indefinite inner product $\langle \{h_n\}_n, \{k_n\}_n \rangle := \sum_{n\geq 0} \langle h_n, k_n \rangle_{\mathfrak{K}_n}$, $\{h_n\}_n, \{k_n\}_n \in \bigoplus_{n\geq 0} \mathfrak{K}_n$. If $\mathfrak{K}$ is a Kreĭn space, then the external orthogonal direct sum of a family of identical copies of $\mathfrak{K}$ can be obviously identified with the Hardy-type Kreĭn space $H^2_{\mathfrak{K}}(\mathbb{T})$, of functions on the torus

$$z \mapsto f(z) = \sum_{n\geq 0} z^n k_n, \quad \text{with } \sum_{n\geq 0} \|k_n\|^2 < \infty.$$

If our family is doubly indexed, we obtain similarly $H^2_{\mathfrak{K}}(\mathbb{T}^2)$.

The following set of bounded operators will be frequently used in our constructions.

Let $\mathfrak{K}, \mathfrak{K}_1, \mathfrak{K}_2$ be given Kreĭn spaces:

- the multiplication by the independent variable $z$ on the Hardy-type space $H^2_{\mathfrak{K}}(\mathbb{T})$ :

$$(T_z f)(z) := z f(z), \quad z \in \mathbb{T},\ f \in H^2_{\mathfrak{K}}(\mathbb{T}),$$

  has the adjoint $T_z^*$ given by

$$(T_z^* f)(z) := \bar{z}(f(z) - f(0)), \quad z \in \mathbb{T},\ f \in H^2_{\mathfrak{K}}(\mathbb{T});$$

- the pair $(T_{z_1}, T_{z_2})$ of multiplications by coordinate functions $z_1$ and $z_2$ on $H^2_{\mathfrak{K}}(\mathbb{T}^2)$ – defined similarly;
- any $T \in \mathcal{B}(\mathfrak{K})$ can be extended to a bounded operator $[T]$ on $H^2_{\mathfrak{K}}(\mathbb{T})$ by

$$([T]f)(z) := T(f(z)), \quad z \in \mathbb{T},\ f \in H^2_{\mathfrak{K}}(\mathbb{T});$$

- any $T \in \mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2)$ can be extended to $[T]_0 \in \mathcal{B}(\mathfrak{K}_1, H^2_{\mathfrak{K}_2}(\mathbb{T}))$ by

$$([T]_0 k_1)(z) := z^0 T k_1, \quad z \in \mathbb{T},\ k_1 \in \mathfrak{K}_1;$$

its adjoint is given by

$$[T]_0^* f = T^*(f(0)), \quad f \in H^2_{\mathfrak{K}_2}(\mathbb{T});$$

- finally, any $T \in \mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2)$ can be extended to a bounded operator $[T]_i \in \mathcal{B}(H^2_{\mathfrak{K}_1}(\mathbb{T}), H^2_{\mathfrak{K}_2}(\mathbb{T}^2))$ by

$$([T]_i f)(z_1, z_2) := T(f(z_i)), \quad z_i \in \mathbb{T},\ f \in H^2_{\mathfrak{K}_1}(\mathbb{T}),\ i = 1, 2.$$

Moreover,

$$([T]_i^* f)(z) = T^*(f(ze_i)), \quad z \in \mathbb{T},\ f \in H^2_{\mathfrak{K}_2}(\mathbb{T}^2),\ i = 1, 2$$

(here $e_1 := (1, 0)$ and $e_2 := (0, 1)$).

Their joint properties are mentioned in the following.

***Proposition 2.2*** *Let $\mathfrak{K}, \mathfrak{K}_1, \mathfrak{K}_2, \mathfrak{K}_1', \mathfrak{K}_2'$ be given Kreĭn spaces. Then,*

(i) *the map*

$$\mathcal{B}(\mathfrak{K}) \ni T \mapsto [T] \in \mathcal{B}(H^2_{\mathfrak{K}}(\mathbb{T}))$$

*is a $*$-algebra homomorphism;*

(ii) *the map*

$$\mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2) \ni T \mapsto [T]_0 \in \mathcal{B}(\mathfrak{K}_1, H^2_{\mathfrak{K}_2}(\mathbb{T}))$$

*is linear; moreover, for $T \in \mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2)$, $[S]_0 T = [ST]_0$ (when $S \in \mathcal{B}(\mathfrak{K}_2, \mathfrak{K}_2')$), $[S]_0^*[T]_0 = S^*T$ (when $S \in \mathcal{B}(\mathfrak{K}_1', \mathfrak{K}_2)$), and*

$$[S]_0[T]_0^* f = ST^*(f(0))z^0,\ f \in H^2_{\mathfrak{K}_2}(\mathbb{T})$$

*(when $S \in \mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2')$);*

(iii) *the map*

$$\mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2) \ni T \mapsto [T]_i \in \mathcal{B}(H^2_{\mathfrak{K}_1}(\mathbb{T}), H^2_{\mathfrak{K}_2}(\mathbb{T}^2))$$

*is linear; moreover, for $T \in \mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2)$, $[S]_i^*[T]_i = [S^*T]$ (when $S \in \mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2)$) and*

$$([S]_i[T]_i^* f)(z_1, z_2) = ST^*(f(z_i e_i)), \quad z_i \in \mathbb{T},\ f \in H^2_{\mathfrak{K}_2}(\mathbb{T}^2),\ i = 1, 2$$

*(when $S \in \mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2')$);*

(iv) *let $T \in \mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2)$. Then,*
- $[T]T_z = T_z[T]$ *and* $T_z[T]^* = [T]^*T_z$ *(when $\mathfrak{K}_1 = \mathfrak{K}_2$);*
- $[T]_0^* T_z = 0$;
- $[T]_i T_z = T_{z_i}[T]_i$, $T_z[T]_i^* = [T]_i^* T_{z_i}$ *and* $[T]_i^* T_{z_{3-i}} = 0,\ i = 1, 2$;

(v) *let $T \in \mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2)$ and $S \in \mathcal{B}(\mathfrak{K}_1', \mathfrak{K}_2')$. Then,*
- $[S][T]_0 = [ST]_0$ *(when $\mathfrak{K}_1' = \mathfrak{K}_2' = \mathfrak{K}_2$);*
- $[S]_i[T] = [ST]_i,\ i = 1, 2$ *(when $\mathfrak{K}_1 = \mathfrak{K}_2 = \mathfrak{K}_1'$);*
- $[S]_1^*[T]_2 = [S]_2^*[T]_1 = [S^*]_0[T^*]_0^*$ *(when $\mathfrak{K}_2' = \mathfrak{K}_2$);*
- $[S]_1[T]_0 = [S]_2[T]_0$ *(when $\mathfrak{K}_1' = \mathfrak{K}_2$);*
- $[T]_i[S]_0 k_1 = z_1^0 z_2^0 TSk_1, \quad k_1 \in \mathfrak{K}_1,\ i = 1, 2$ *(when $\mathfrak{K}_1 = \mathfrak{K}_2'$).*

## 3 Isometric and unitary dilations for bounded operators

It is well known that any bounded self-adjoint operator $A$ on a Kreĭn space $\mathfrak{A}$ can be factorized into the form

$$A = BB^*,$$

for a certain operator $B \in \mathcal{B}(\mathfrak{B}, \mathfrak{A})$ with zero kernel on a Kreĭn space $\mathfrak{B}$.

By a *defect operator* for $T \in \mathcal{B}(\mathfrak{K}_1, \mathfrak{K}_2)$, $\mathfrak{K}_1$ and $\mathfrak{K}_2$ Kreĭn spaces, we mean an operator $D_T \in \mathcal{B}(\mathfrak{D}_T, \mathfrak{K}_1)$ with zero kernel on a Kreĭn space $\mathfrak{D}_T$ (called its *defect space*) such that

$$I - T^* T = D_T D_T^*. \tag{3.1}$$

Let $T \in \mathcal{B}(\mathfrak{H})$, $\mathfrak{H}$ Kreĭn space. An *isometric* (respectively, *unitary*) *dilation* of $T$ is a bounded isometric (respectively, unitary) operator $U$ on a Kreĭn space $\mathfrak{K} \supset \mathfrak{H}$ satisfying

$$\langle T^n h, h' \rangle_{\mathfrak{H}} = \langle U^n h, h' \rangle_{\mathfrak{K}}, \quad h, h' \in \mathfrak{H},\ n \in \mathbb{Z}_+. \tag{3.2}$$

An isometric (respectively, unitary) dilation $U \in \mathcal{B}(\mathfrak{K})$ of $T \in \mathcal{B}(\mathfrak{H})$ is said to be *minimal* if $\mathfrak{K} = \bigvee_{n \geq 0} U^n \mathfrak{H}$ (respectively, $\mathfrak{K} = \bigvee_{n=-\infty}^{\infty} U^n \mathfrak{H}$).

For bounded operators $T \in \mathcal{B}(\mathfrak{H})$, minimal isometric (respectively, unitary) dilations always exist [10, 16]. A Schäffer-like matrix construction is still possible in these generalized settings [10].

Define $\mathfrak{K}_+ = \mathfrak{H} \oplus H^2_{\mathfrak{D}_T}(\mathbb{T})$. Then, an isometric dilation $V$ of $T$ on $\mathfrak{K}_+$ is given by the representation

$$V = \begin{pmatrix} T & 0 \\ [D_T^*]_0 & T_z \end{pmatrix}. \tag{3.3}$$

A minimal unitary dilation $U$ of $T$ on the Kreĭn space $\mathfrak{K} = H^2_{\mathfrak{D}_{T^*}}(\mathbb{T}) \oplus \mathfrak{H} \oplus H^2_{\mathfrak{D}_T}(\mathbb{T})$ can be built in terms of a *Julia operator* or *elementary rotation* (to see [11, 12]) for $T$, i.e., a unitary operator of the form

$$\begin{pmatrix} T & D_{T^*} \\ D_T^* & L \end{pmatrix} \in \mathcal{B}(\mathfrak{H} \oplus \mathfrak{D}_{T^*}, \mathfrak{H} \oplus \mathfrak{D}_T).$$

More precisely,

$$U = \begin{pmatrix} T_z^* & 0 & 0 \\ [D_{T^*}^*]_0^* & T & 0 \\ [L]_0 [I_{\mathfrak{D}_{T^*}}]_0^* & [D_T^*]_0 & T_z \end{pmatrix}. \tag{3.4}$$

As regarding the geometry of minimal dilations, we could mention the following theorem.

***Theorem 3.1*** *Let $V \in \mathcal{B}(\mathfrak{K}_+)$ (respectively, $U \in \mathcal{B}(\mathfrak{K})$) be any minimal isometric (respectively, unitary) dilation of $T \in \mathcal{B}(\mathfrak{H})$.*

(a) (i) $\mathfrak{L} = \overline{(V - T)\mathfrak{H}}$ *is wandering for $V$ (i.e., $V^n \mathfrak{L} \perp V^m \mathfrak{L}$, $n, m \geq 0$, $n \neq m$), regular, and isometrically isomorphic with $\mathfrak{D}_T$;*

(ii) *$M_+(\mathfrak{L}) = \bigvee_{n\geq 0} V^n\mathfrak{L}$ is regular and*

$$\mathfrak{K}_+ = \mathfrak{H} \oplus M_+(\mathfrak{L}); \tag{3.5}$$

(b) (i) *$\mathfrak{L} = \overline{(U-T)\mathfrak{H}}$ and $\mathfrak{L}^* = \overline{(U^*-T^*)\mathfrak{H}}$ are wandering for $U$, regular, and isometrically isomorphic, respectively, with $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$;*

(ii) *$M_+(\mathfrak{L}) = \bigvee_{n\geq 0} U^n\mathfrak{L}$ and $M_-(\mathfrak{L}^*) = \bigvee_{n\leq 0} U^n\mathfrak{L}^*$ are regular and*

$$\mathfrak{K} = M_-(\mathfrak{L}^*) \oplus \mathfrak{H} \oplus M_+(\mathfrak{L}). \tag{3.6}$$

**Proof** All the orthogonality properties involved here can be easily checked following the definition of an isometric or unitary dilation and, therefore, we shall omit the details.

Observe firstly that a successive application of the formula

$$Vh = Th + (V-T)h, \quad h \in \mathfrak{H}$$

will lead to

$$V^n h = T^n h + \sum_{k=0}^{n-1} V^k(V-T)T^{n-k-1}h, \quad h \in \mathfrak{H},\ n \in \mathbb{N}. \tag{3.7}$$

Hence, (3.5) holds. Consequently, $M_+(\mathfrak{L})$ is regular and, since $M_+(\mathfrak{L}) = \mathfrak{L} \oplus VM_+(\mathfrak{L})$, $\mathfrak{L}$ is also regular. Moreover,

$$\begin{aligned}\langle (V-T)h, (V-T)h'\rangle_{\mathfrak{K}_+} &= \langle h, h'\rangle_{\mathfrak{H}} - \langle Th, Th'\rangle_{\mathfrak{H}} \\ &= \langle (I - T^*T)h, h'\rangle_{\mathfrak{H}} \\ &= \langle D_T^* h, D_T^* h'\rangle_{\mathfrak{D}_T}, \quad h, h' \in \mathfrak{H}\end{aligned}$$

shows that the map

$$\mathfrak{L} \ni (V-T)h \mapsto D_T^* h \in \mathfrak{D}_T$$

is well defined ($\mathfrak{D}_T$ is a Kreĭn space), isometric, densely defined, and with dense range. It is also injective (since $\mathfrak{L}$ is regular) and, therefore, $\kappa^\pm(\mathfrak{L}) = \kappa^\pm(\mathfrak{D}_T)$. Deduce that $\mathfrak{L}$ and $\mathfrak{D}_T$ are isometrically isomorphic and the proof of $(a)$ is complete.

To obtain (3.6), we apply (3.7) for $(U, T)$ and then for $(U^*, T^*)$ instead of $(V, T)$. Since $M_-(\mathfrak{L}^*)$ and $M_+(\mathfrak{L})$ are regular we show, as before, that $\mathfrak{L}$ and $\mathfrak{L}^*$ are regular. A similar argument as for the minimal isometric dilation allows us to conclude that $\mathfrak{L}$ and $\mathfrak{L}^*$ are isometrically isomorphic, respectively, with $\mathfrak{D}_T$ and $\mathfrak{D}_{T^*}$. ∎

***Corollary 3.2*** *Let $U \in \mathcal{B}(\mathfrak{K})$ be any minimal unitary dilation of $T \in \mathcal{B}(\mathfrak{H})$. Then,*

$$\mathfrak{K}_+ = \bigvee_{n\geq 0} U^n\mathfrak{H}$$

*is a regular subspace of $\mathfrak{K}$, invariant to $U$, and $V_+ = U|_{\mathfrak{K}_+}$ is a minimal isometric dilation of $T$.*

*Similarly,*

$$\mathfrak{K}_- = \bigvee_{n\leq 0} U^n\mathfrak{H}$$

*is a regular subspace of* $\mathfrak{K}$*, invariant to* $U^*$*, and* $V_- = U^*|_{\mathfrak{K}_-}$ *is a minimal isometric dilation of* $T^*$.

***Remark 3.3*** • Proposition 2.1 gives another condition, usually used as an axiom in the isometric or unitary dilations definition: if $U \in \mathcal{B}(\mathfrak{K})$ is any minimal isometric or unitary dilation of $T \in \mathcal{B}(\mathfrak{H})$, then $\mathfrak{H}$ is a regular subspace of $\mathfrak{K}$. Theorem above indicates the structure of its orthogonal complement. Therefore, (3.2) can be rewritten as

$$T^n = P_{\mathfrak{H}} U^n|_{\mathfrak{H}}, \quad n \geq 0,$$

where $P_{\mathfrak{H}}$ is the orthogonal projection onto $\mathfrak{H}$. Another consequence of (3.2) is

$$T^{*n} = P_{\mathfrak{H}} U^{*n}|_{\mathfrak{H}}, \quad n \geq 0.$$

• Let $V \in \mathcal{B}(\mathfrak{K}_+)$ be a minimal isometric dilation of $T \in \mathcal{B}(\mathfrak{H})$. Then, for any $h, h' \in \mathfrak{H}$ and $n \geq 0$,

$$\langle V^* h - T^* h, h' \rangle = \langle h, V h' \rangle - \langle h, T h' \rangle = 0$$

and

$$\begin{aligned}\langle V^* h - T^* h, V^n (V - T) h' \rangle = \langle h, V^{n+2} h' \rangle - \langle h, V^{n+1} T h' \rangle \\ - \langle T^* h, V^{n+1} h' \rangle + \langle T^* h, V^n T h' \rangle = 0.\end{aligned}$$

By the geometrical structure of $\mathfrak{K}_+$ given by (3.5), we deduce that $\mathfrak{H}$ is invariant to $V^*$ and $V^*|_{\mathfrak{H}} = T^*$.

• Let $U \in \mathcal{B}(\mathfrak{K})$ be a minimal unitary dilation of $T \in \mathcal{B}(\mathfrak{H})$. It is clear that $M_+(\mathfrak{L})$ is invariant to $U$ and $M_-(\mathfrak{L}^*)$ to $U^*$. Since $M_+(\mathfrak{L})$ and $M_-(\mathfrak{L})$ are regular, $U|_{M_+(\mathfrak{L})}$ and $U|_{M_-(\mathfrak{L}^*)}$ are *unilateral shifts* (according to the definition in [44]).

An operator $T \in \mathcal{B}(\mathfrak{H}, \mathfrak{K})$, $\mathfrak{H}, \mathfrak{K}$ Kreĭn spaces, is a *contraction* (respectively, *expansion*) if

$$\langle Th, Th \rangle_{\mathfrak{K}} \leq \langle h, h \rangle_{\mathfrak{H}} \quad \text{(respectively, } \langle Th, Th \rangle_{\mathfrak{K}} \geq \langle h, h \rangle_{\mathfrak{H}}), \quad h \in \mathfrak{H}$$

or, equivalently,

$$I - T^* T \geq 0 \quad \text{(respectively, } I - T^* T \leq 0).$$

Contractions (respectively, expansions) can be characterized by the fact that their defect spaces are Hilbert (respectively, anti-Hilbert) spaces.

Let $U \in \mathcal{B}(\mathfrak{K})$, $U' \in \mathcal{B}(\mathfrak{K}')$ be two minimal unitary dilations of $T \in \mathcal{B}(\mathfrak{H})$ and $V = U|_{\mathfrak{K}_+}$, $V' = U'|_{\mathfrak{K}'_+}$, where $\mathfrak{K}_+ = \bigvee_{n \geq 0} U^n \mathfrak{H}$, $\mathfrak{K}'_+ = \bigvee_{n \geq 0} U'^n \mathfrak{H}$ the corresponding minimal isometric dilations (according to Corollary 3.2). $V \in \mathcal{B}(\mathfrak{K}_+)$ and $V' \in \mathcal{B}(\mathfrak{K}'_+)$ are *unitarily equivalent* if there exists a unitary operator $\Phi : \mathfrak{K}_+ \to \mathfrak{K}'_+$ which intertwines $V$ and $V'$ (i.e., $\Phi V = V' \Phi$) and $\Phi|_{\mathfrak{H}} = I_{\mathfrak{H}}$. If, moreover, $\Phi$ can be extended to a unitary operator on $\mathfrak{K}$ onto $\mathfrak{K}'$ which intertwines $U$ and $U'$, then $U \in \mathcal{B}(\mathfrak{K})$ and $U' \in \mathcal{B}(\mathfrak{K}')$ are said to be *unitarily equivalent.*

A result by Gheondea and Popescu shows that minimal isometric dilations are, in general, not unique (up to a unitary equivalence).

***Theorem 3.4*** [28] *A bounded operator $T$ on a Kreĭn space $\mathfrak{H}$ has a unique minimal isometric dilation if and only if $T$ is either contractive or expansive.*

The same kind of conditions holds for minimal unitary dilations.

***Theorem 3.5*** *$T \in \mathcal{B}(\mathfrak{H})$ has a unique minimal unitary dilation if and only if $T$ is either contractive or expansive and $T^*$ is either contractive or expansive.*

**Proof** Assume that, for example, $T$ is contractive and $T^*$ is expansive. Then, $\mathfrak{D}_T$ (and hence also $H^2_{\mathfrak{D}_T}(\mathbb{T})$) is a Hilbert space and $\mathfrak{D}_{T^*}$ (and hence also $H^2_{\mathfrak{D}_{T^*}}(\mathbb{T})$) is an anti-Hilbert space.

Let $U$ be the minimal unitary dilation of $T$ given by (3.4) and acting on the Kreĭn space $\mathfrak{K} = H^2_{\mathfrak{D}_{T^*}}(\mathbb{T}) \oplus \mathfrak{H} \oplus H^2_{\mathfrak{D}_T}(\mathbb{T})$. If $U' \in \mathcal{B}(\mathfrak{K}')$ is any other minimal unitary dilation of $T$ then, according to (3.6), $\mathfrak{K}'$ has an orthogonal decomposition of the form $\mathfrak{K}' = M_-(\mathfrak{L}'^*) \oplus \mathfrak{H} \oplus M_+(\mathfrak{L}')$, with $\mathfrak{L}' = \overline{(U' - T)\mathfrak{H}}$ and $\mathfrak{L}'^* = \overline{(U'^* - T^*)\mathfrak{H}}$.

Since, for arbitrary finite sequences $\{h_n\}_n$, $\{g_n\}_n$ of vectors in $\mathfrak{H}$, we have

$$\begin{aligned}\langle \sum_{n\geq 0} U'^n (U' - T)h_n, \sum_{m\geq 0} U'^m (U' - T)g_m\rangle_{\mathfrak{K}'} &= \sum_{n\geq 0} \langle (U' - T)h_n, (U' - T)g_n\rangle_{\mathfrak{K}'} \\ &= \sum_{n\geq 0} \langle (I - T^*T)h_n, g_n\rangle_{\mathfrak{H}} \\ &= \langle \sum_{n\geq 0} z^n D_T^* h_n, \sum_{m\geq 0} z^m D_T^* g_m\rangle_{H^2_{\mathfrak{D}_T}(\mathbb{T})},\end{aligned}$$

the mapping

$$M_+(\mathfrak{L}') \ni \sum_{n\geq 0} U'^n (U' - T)h_n \overset{\Phi_+}{\longmapsto} \sum_{n\geq 0} z^n D_T^* h_n \in H^2_{\mathfrak{D}_T}(\mathbb{T})$$

is well defined, and the linear operator $\Phi_+$ is isometric, densely defined, and with dense range. It can be uniquely extended to a unitary operator $\Phi_+ \in \mathcal{B}(M_+(\mathfrak{L}'), H^2_{\mathfrak{D}_T}(\mathbb{T}))$ (since $H^2_{\mathfrak{D}_T}(\mathbb{T})$) is a Hilbert space). Similarly, we can define a unitary operator $\Phi_- \in \mathcal{B}(M_-(\mathfrak{L}'^*), H^2_{\mathfrak{D}_{T^*}}(\mathbb{T}))$.

Then, $\Phi = \Phi_- \oplus I_{\mathfrak{H}} \oplus \Phi_+ \in \mathcal{B}(\mathfrak{K}', \mathfrak{K})$ is unitary, $\Phi|_{\mathfrak{H}} = I_{\mathfrak{H}}$ and $\Phi U' = U\Phi$. Moreover, $\Phi\mathfrak{K}'_+ = \mathfrak{K}_+$ and, therefore, $U$ and $U'$ are unitarily equivalent.

Conversely, suppose that, for example, $T$ is neither contractive nor expansive or, equivalently, the Kreĭn space $\mathfrak{D}_T$ is indefinite. If $Z \in \mathcal{B}(\mathfrak{D}_T)$ is a unitary operator with $\|Z^n\| \overset{n}{\longrightarrow} \infty$, then the matrix

$$U' = \begin{pmatrix} T_z^* & 0 & 0 \\ [D_{T^*}^*]_0^* & T & 0 \\ [L]_0[I_{\mathfrak{D}_{T^*}}]_0^* & [D_T^*]_0 & T_z[Z] \end{pmatrix}$$

defines a minimal unitary dilation of $T$ on $\mathfrak{K} = H^2_{\mathfrak{D}_{T^*}}(\mathbb{T}) \oplus \mathfrak{H} \oplus H^2_{\mathfrak{D}_T}(\mathbb{T})$. If $T$ would have a unique minimal unitary dilation, then $U'$ and the minimal unitary dilation $U$ on $\mathfrak{K}$ given by (3.4) would be unitarily equivalent via a unitary operator $\Phi \in \mathcal{B}(\mathfrak{K})$ which leaves invariant $H^2_{\mathfrak{D}_T}(\mathbb{T})$. We get a contradiction since

$\lim_{n\to\infty}\|(\Phi^*|_{H^2_{\mathfrak{D}_T}(\mathbb{T})} T_z\Phi|_{H^2_{\mathfrak{D}_T}(\mathbb{T})})^n\| = \lim_{n\to\infty}\|\Phi^*|_{H^2_{\mathfrak{D}_T}(\mathbb{T})} T_z^n\Phi|_{H^2_{\mathfrak{D}_T}(\mathbb{T})}\| < \infty$, while $\lim_{n\to\infty}\|(T_z[Z])^n\| = \lim_{n\to\infty}\|Z^n\| = \infty$. ∎

***Corollary 3.6*** *$T$ has a unique minimal unitary dilation if and only if $T$ and $T^*$ have unique minimal isometric dilations.*

## 4 Commuting isometric pairs and their unitary extensions

Using dilation theory, we can easily deduce that every bounded isometry $V$ on a Kreĭn space $\mathfrak{H}$ has a *unitary extension* $U$ on a Kreĭn space $\mathfrak{K}$ containing $\mathfrak{H}$ as a regular subspace which is *minimal* in the sense that

$$\mathfrak{K} = \bigvee_{n\in\mathbb{Z}} U^n\mathfrak{H}.$$

We just have to take the minimal isometric dilation of $V^*$ given by (3.3) and observe that it is a unitary operator $U^*$ on $\mathfrak{K} = \mathfrak{H} \oplus H^2_{\ker V^*}(\mathbb{T})$. More precisely, the linear operator given by the matrix

$$U = \begin{pmatrix} V & [P^{\mathfrak{H}}_{\ker V^*}]^*_0 \\ 0 & T_z^* \end{pmatrix} \tag{4.1}$$

is a minimal unitary extension of $V$.

***Remark 4.1*** • $U \in \mathcal{B}(\mathfrak{K})$ is a minimal unitary extension of $V \in \mathcal{B}(\mathfrak{H})$ if and only if $U^*$ is a minimal isometric dilation of $V^*$.

• If $V \in \mathcal{B}(\mathfrak{K}_+)$ is any minimal isometric dilation of $T \in \mathcal{B}(\mathfrak{H})$ and $U \in \mathcal{B}(\mathfrak{K})$ is any minimal unitary extension of $V$, then $U$ is a minimal unitary dilation of $T$.

• Suppose that $V$ is the minimal isometric dilation of $T$ given by (3.3) on the Kreĭn space $\mathfrak{K}_+ = \mathfrak{H} \oplus H^2_{\mathfrak{D}_T}(\mathbb{T})$. The minimal unitary extension $U$ given by (4.1) on the Kreĭn space $\mathfrak{K} = \mathfrak{H} \oplus H^2_{\mathfrak{D}_T}(\mathbb{T}) \oplus H^2_{\ker V^*}(\mathbb{T})$ is a minimal unitary dilation of $T$ and its matrix representation depends only on $T$ and on its corresponding defect operators (does not require the construction of a Julia operator or of an elementary rotation for $T$).

The following theorem explains the geometrical structure of minimal unitary extensions.

***Theorem 4.2*** *Let $V$ be a bounded isometry on a Kreĭn space $\mathfrak{H}$ and $U \in \mathcal{B}(\mathfrak{K})$ be any minimal unitary extension of $V$. Then,*

(i) $\mathfrak{L}^* = \overline{(U^* - V^*)\mathfrak{H}}$ *is wandering for $U$, regular, and isometrically isomorphic with* $\ker V^*$;

(ii) $M_-(\mathfrak{L}^*) = \bigvee_{n\le 0} U^n\mathfrak{L}^*$ *is a regular subspace of $\mathfrak{K}$ and*

$$\mathfrak{K} = \mathfrak{H} \oplus M_-(\mathfrak{L}^*);$$

(iii) $M_-(\mathfrak{L}^*)$ *is invariant to $U^*$ and $U^*|_{M_-(\mathfrak{L}^*)}$ is a unilateral shift.*

**Proof** The proof follows Theorem 3.1 $(a)$ with the observation that $U^*$ is a minimal isometric dilation of $V^*$. ∎

Following Remark 4.1 and Theorem 3.4, we can characterize the uniqueness of a minimal unitary extension.

***Theorem 4.3*** *Let $V \in \mathcal{B}(\mathfrak{H})$, $\mathfrak{H}$ Kreĭn space, be isometric. The following conditions are equivalent:*

(i) *$V$ has a unique minimal unitary extension;*
(ii) *$V^*$ is either contractive or expansive;*
(iii) $\ker V^*$ *is either uniformly positive or uniformly negative;*
(iv) $\ker V^*$ *is either Hilbert or anti-Hilbert space.*

*If $U \in \mathcal{B}(\mathfrak{K})$ is a minimal unitary extension of $V$, then $\mathfrak{K} \ominus \mathfrak{H}$ is a Hilbert or an anti-Hilbert space.*

A pair $T = (T_1, T_2)$ of bounded linear operators on a Kreĭn space $\mathfrak{H}$ is said to be *commuting* if $T_1T_2 = T_2T_1$. $T$ is called *double commuting* if $T_1$ commutes not only with $T_2$, but also with its adjoint $T_2^*$. By the end of this article, any pair of bounded operators acting on a Kreĭn space will be considered a commutative pair. In case the components $T_1$ and $T_2$ are clear from the context or they are not needed in the corresponding discussion, in order to avoid repetitions, we simply use the notation $T \in \mathcal{B}(\mathfrak{H})^2$ instead of $T = (T_1, T_2) \in \mathcal{B}(\mathfrak{H})^2$. If $n = (n_1, n_2) \in \mathbb{Z}^2$ and $T = (T_1, T_2) \in \mathcal{B}(\mathfrak{H})^2$, the notation $T^n = T_1^{n_1}T_2^{n_2}$ will be frequently used whenever the computations $T_1^{n_1}$ and $T_2^{n_2}$ make sense.

***Definition 4.1*** Let $V = (V_1, V_2)$ be a pair of commuting isometries in $\mathcal{B}(\mathfrak{H})$, $\mathfrak{H}$ a Kreĭn space. A *unitary extension* of $V$ is a commuting pair $U = (U_1, U_2)$ of unitary operators on a Kreĭn space $\mathfrak{K}$ containing $\mathfrak{H}$ as a regular subspace such that $U_1, U_2$ extend, respectively, $V_1, V_2$. $U$ is said to be *minimal* if, in addition,

$$\mathfrak{K} = \bigvee_{n \in \mathbb{Z}^2} U^n \mathfrak{H}.$$

Just to give an example, observe that the pair $(T_{z_1}, T_{z_2})$ of multiplications by coordinate functions $z_1$ and $z_2$ on a certain Hardy-type Kreĭn space $H^2_{\mathfrak{H}}(\mathbb{T}^2)$ can be extended by the commuting unitary pair $(M_{z_1}, M_{z_2})$,

$$f \overset{M_{z_i}}{\longmapsto} z_i f$$

on the $L^2$-type Kreĭn space $L^2_{\mathfrak{H}}(\mathbb{T}^2)$ introduced in an obvious manner. This unitary extension is minimal.

***Theorem 4.4*** *Let $V = (V_1, V_2)$ be a commuting pair of isometric operators in $\mathcal{B}(\mathfrak{H})$. Then, the pair $U = (U_1, U_2)$ given by the matrix representation*

$$U_1 = \begin{pmatrix} V_1 & [P^{\mathfrak{H}}_{\ker(V_1V_2)^*} V_2|_{\ker(V_1V_2)^*}]_0^* \\ 0 & [V_1(I - V_2V_2^*)|_{\ker(V_1V_2)^*}] + [V_2^*|_{\ker(V_1V_2)^*}]T_z^* \end{pmatrix} \tag{4.2}$$

*and*

$$U_2 = \begin{pmatrix} V_2 & [P^{\mathfrak{H}}_{\ker(V_1V_2)^*} V_1|_{\ker(V_1V_2)^*}]_0^* \\ 0 & [V_2(I - V_1V_1^*)|_{\ker(V_1V_2)^*}] + [V_1^*|_{\ker(V_1V_2)^*}]T_z^* \end{pmatrix} \tag{4.3}$$

*is a minimal unitary extension of* $V$ *on the Kreĭn space*

$$\mathfrak{K} = \mathfrak{H} \oplus H^2_{\ker(V_1V_2)^*}(\mathbb{T}).$$

**Proof** Standard computations with operator matrices (in view of Proposition 2.2) show that, in fact, formulas (4.2) and (4.3) define commuting unitary operators on $\mathfrak{K} = \mathfrak{H} \oplus H^2_{\ker(V_1V_2)^*}(\mathbb{T})$. Obviously, $U$ is a unitary extension of $V$.

It remains to show the minimality. We will actually prove an apparently stronger result, namely,

$$\mathfrak{K} = \bigvee_{n\leq 0} (U_1U_2)^n \mathfrak{H}.$$

To this end, let $h \in \mathfrak{H}$ and $n \in \mathbb{N}$ be arbitrary. We see that

$$(U_1U_2)^{*n}\left[(U_1U_2)^*\begin{pmatrix} h \\ 0\end{pmatrix} - \begin{pmatrix}(V_1V_2)^*h \\ 0\end{pmatrix}\right] = \begin{pmatrix} 0 \\ z^n P^{\mathfrak{H}}_{\ker(V_1V_2)^*} h\end{pmatrix},$$

hence,

$$H^2_{\ker(V_1V_2)^*}(\mathbb{T}) = \bigvee\left\{(U_1U_2)^{*(n+1)}\begin{pmatrix} h \\ 0\end{pmatrix} - (U_1U_2)^{*n}\begin{pmatrix}(V_1V_2)^*h \\ 0\end{pmatrix} \mid h \in \mathfrak{H}, n \geq 0\right\},$$

which proves our claim. ∎

***Remark 4.5*** The conclusion of the previous theorem also holds for arbitrary finite families $V = (V_1, V_2, \dots, V_n)$ of commuting bounded isometries on a Kreĭn space $\mathfrak{H}$. More precisely, the family $U = (U_1, U_2, \dots, U_n)$ given by

$$U_i = \begin{pmatrix} V_i & [P^{\mathfrak{H}}_{\ker(V_1\dots V_n)^*} W_i|_{\ker(V_1\dots V_n)^*}]^*_0 \\ 0 & [V_i(I - W_iW_i^*)|_{\ker(V_1\dots V_n)^*}] + [W_i^*|_{\ker(V_1\dots V_n)^*}]T_z^* \end{pmatrix},$$

where $W_i = \prod_{j\neq i} V_j$, $i = 1, 2, \dots, n$, is a minimal unitary extension of $V$ on the Kreĭn space

$$\mathfrak{K} = \mathfrak{H} \oplus H^2_{\ker(V_1\dots V_n)^*}(\mathbb{T}).$$

For the Hilbert space case, we refer to [48, Theorem 3.3.1].

***Proposition 4.6*** *If* $U = (U_1, U_2) \in \mathcal{B}(\mathfrak{K})^2$ *is a minimal unitary extension of the commuting isometric pair* $V = (V_1, V_2) \in \mathcal{B}(\mathfrak{H})^2$, *then* $U_1U_2$ *is a minimal unitary extension of* $V_1V_2$.

**Proof** It is clear that, under the given hypothesis, $U_1U_2$ is a unitary extension of $V_1V_2$. Since, for $m \leq n \leq 0$, $U_1^m U_2^n \mathfrak{H} \subset (U_1U_2)^m \mathfrak{H}$, we observe that

$$\mathfrak{K} = \bigvee_{m,n\leq 0} U_1^m U_2^n \mathfrak{H} \subset \bigvee_{m\leq 0} (U_1U_2)^m \mathfrak{H} \subset \mathfrak{K},$$

hence, $\mathfrak{K} = \bigvee_{m\leq 0}(U_1U_2)^m\mathfrak{H}$, and the minimality of the unitary extension $U_1U_2$ is proved. ∎

In view of Theorem 4.2, we can deduce a geometrical structure for the minimal unitary extension.

***Corollary 4.7*** *Let $U = (U_1, U_2) \in \mathcal{B}(\mathfrak{K})^2$ be any minimal unitary extension of the commuting isometric pair $V = (V_1, V_2) \in \mathcal{B}(\mathfrak{H})^2$. Then,*

(i) $\mathfrak{L}^* = \overline{((U_1U_2)^* - (V_1V_2)^*)\mathfrak{H}}$ *is wandering for $U_1U_2$, regular, and isometrically isomorphic with* $\ker(V_1V_2)^*$;

(ii) $M_-(\mathfrak{L}^*) = \bigvee_{n\le 0}(U_1U_2)^n\mathfrak{L}^*$ *is a regular subspace of $\mathfrak{K}$ and*

$$\mathfrak{K} = \mathfrak{H} \oplus M_-(\mathfrak{L}^*); \tag{4.4}$$

(iii) $M_-(\mathfrak{L}^*)$ *is invariant to $(U_1U_2)^*$ and $(U_1U_2)^*|_{M_-(\mathfrak{L}^*)}$ is a unilateral shift.*

Let $U = (U_1, U_2) \in \mathcal{B}(\mathfrak{K})^2$, $U' = (U_1', U_2') \in \mathcal{B}(\mathfrak{K}')^2$ be two minimal unitary extensions of the pair $V = (V_1, V_2)$ of commuting bounded isometries on $\mathfrak{H}$. $U$ and $U'$ are said to be *unitarily equivalent* if there exists a unitary operator $\Phi : \mathfrak{K} \to \mathfrak{K}'$ which intertwines $U_1$ and $U_1'$, respectively, $U_2$ and $U_2'$ and such that $\Phi|_{\mathfrak{H}} = I_{\mathfrak{H}}$.

***Theorem 4.8*** *Let $V = (V_1, V_2) \in \mathcal{B}(\mathfrak{H})^2$ be a commuting isometric pair such that $(V_1V_2)^*$ is either contractive or expansive. Then, $V$ has a unique minimal unitary extension.*

**Proof** Assume that, for example, $(V_1V_2)^*$ is contractive. Then, $\ker(V_1V_2)^*$ (and, hence, also $H^2_{\ker(V_1V_2)^*}$) is a Hilbert space.

Let $U = (U_1, U_2)$ be the minimal unitary extension of $V$ given by (4.2) and (4.3) and acting on the Kreĭn space $\mathfrak{K} = \mathfrak{H} \oplus H^2_{\ker(V_1V_2)^*}(\mathbb{T})$. If $U' = (U_1', U_2') \in \mathcal{B}(\mathfrak{K}')^2$ is any other minimal unitary extension of $V$, then, according to (4.4), $\mathfrak{K}'$ has an orthogonal decomposition of the form $\mathfrak{K}' = \mathfrak{H} \oplus M_-(\mathfrak{L}'^*)$, with $\mathfrak{L}'^* = \overline{((U_1'U_2')^* - (V_1V_2)^*)\mathfrak{H}}$.

Since, for arbitrary finite sequences $\{h_n\}_n$, $\{g_n\}_n$ of vectors in $\mathfrak{H}$, we have

$$\begin{aligned}&\langle \sum_{n\ge 0}(U_1'U_2')^{*n}((U_1'U_2')^* - (V_1V_2)^*)h_n, \sum_{m\ge 0}(U_1'U_2')^{*m}((U_1'U_2')^* - (V_1V_2)^*)g_m\rangle_{\mathfrak{K}'}\\ &\quad = \langle \sum_{n\ge 0} z^n(I - V_1V_2(V_1V_2)^*)h_n, \sum_{m\ge 0} z^m(I - V_1V_2(V_1V_2)^*)g_m\rangle_{H^2_{\ker(V_1V_2)^*}(\mathbb{T})},\end{aligned}$$

the mapping

$$\begin{aligned}M_-(\mathfrak{L}'^*) \ni \sum_{n\ge 0}(U_1'U_2')^{*n}((U_1'U_2')^*-(V_1V_2)^*)h_n &\overset{\phi_+}{\longmapsto}\\ &\sum_{n\ge 0} z^n(I - V_1V_2(V_1V_2)^*)h_n \in H^2_{\ker(V_1V_2)^*}(\mathbb{T})\end{aligned}$$

is well defined, and the linear operator $\Phi$ is isometric, densely defined, and with dense range. It can be uniquely extended to a unitary operator $\Phi \in \mathcal{B}(M_-(\mathfrak{L}'^*), H^2_{\ker(V_1V_2)^*}(\mathbb{T}))$ (since $H^2_{\ker(V_1V_2)^*}(\mathbb{T})$ is a Hilbert space).

Then, $\Phi = I_{\mathfrak{H}} \oplus \Phi_+ \in \mathcal{B}(\mathfrak{K}', \mathfrak{K})$ is unitary and $\Phi|_{\mathfrak{H}} = I_{\mathfrak{H}}$. Moreover, for any $h \in \mathfrak{H}$,

$$\Phi U_1'h = \Phi V_1h = V_1h = V_1\Phi h = U_1\Phi h,$$

$$\begin{aligned}&\Phi U_1'((U_1'U_2')^*-(V_1'V_2')^*)h\\ &\quad = \Phi[(I - V_1V_1^*)V_2^*h + ((U_1'U_2')^* - (V_1V_2)^*)V_1(I - V_2V_2^*)h]\end{aligned}$$

$$\begin{aligned} &= z^0(I - V_1V_1^*)V_2^*h + zV_1(I - V_2V_2^*)h \\ &= U_1(z^0(I - V_1V_2V_1^*V_2^*)h) \\ &= U_1\Phi((U_1'U_2')^* - (V_1V_2)^*)h, \end{aligned}$$

and, by a similar argument,

$$\Phi U_1'(U_1'U_2')^{*n}((U_1'U_2')^* - (V_1V_2)^*)h = U_1\Phi(U_1'U_2')^{*n}((U_1'U_2')^* - (V_1V_2)^*)h,$$

for any positive integer $n$. Conclude that $\Phi U_1' = U_1\Phi$ since they are continuous and coincide on a dense subset. By symmetry, we also obtain $\Phi U_2' = U_2\Phi$ and, therefore, $U$ and $U'$ are unitarily equivalent. ■

***Corollary 4.9*** *Let $V = (V_1, V_2)$ be a commuting isometric pair in $\mathcal{B}(\mathfrak{H})^2$. If $V_1V_2$ has a unique minimal unitary extension, then $V$ has a unique minimal unitary extension.*

## 5 Regular dilations for commuting pairs

***Definition 5.1*** Let $T = (T_1, T_2)$ be a commuting pair of bounded operators on a Kreĭn space $\mathfrak{H}$.

• An *isometric* (respectively, *unitary*) *dilation* of $T$ is a pair $U = (U_1, U_2)$ of bounded commuting isometric (respectively, unitary) operators on a Kreĭn space $\mathfrak{K}$ containing $\mathfrak{H}$ as a Kreĭn subspace and satisfying

(5.1) $$T^n = P_{\mathfrak{H}}U^n|_{\mathfrak{H}}, \quad n \in \mathbb{Z}_+^2.$$

• An isometric or unitary dilation $U = (U_1, U_2) \in \mathcal{B}(\mathfrak{K})^2$ of $T = (T_1, T_2) \in \mathcal{B}(\mathfrak{H})^2$ is said to be *regular* if

(5.2) $$(T^{n^-})^*T^{n^+} = P_{\mathfrak{H}}(U^{n^-})^*U^{n^+}|_{\mathfrak{H}}, \quad n \in \mathbb{Z}^2.$$

Here, for $n = (n_1, n_2) \in \mathbb{Z}^2$, the usual notations $n^+ := (\max\{n_1, 0\}, \max\{n_2, 0\})$ and $n^- := (-\min\{n_1, 0\}, -\min\{n_2, 0\})$ are used. Formula (5.2) is consistent with the dilation definition (5.1) which can be obtained for $n \in \mathbb{Z}_+^2$ (in this case $n^- = (0, 0)$ and $n^+ = n$). If (5.1) holds true, then (5.2) is actually equivalent with

$$T_1^{*m}T_2^n = P_{\mathfrak{H}}U_1^{*m}U_2^n|_{\mathfrak{H}}, \quad m, n \geq 0.$$

• An isometric (respectively, unitary) dilation $U \in \mathcal{B}(\mathfrak{K})^2$ of $T \in \mathcal{B}(\mathfrak{H})^2$ is called *minimal* if $\mathfrak{K} = \bigvee_{n\in\mathbb{Z}_+^2} U^n\mathfrak{H}$ (respectively, $\mathfrak{K} = \bigvee_{n\in\mathbb{Z}^2} U^n\mathfrak{H}$).

***Remark 5.1*** Suppose that the commuting pair $T \in \mathcal{B}(\mathfrak{H})^2$ has a minimal isometric (respectively, minimal regular isometric) dilation $V \in \mathcal{B}(\mathfrak{K}_+)^2$. Let $U \in \mathcal{B}(\mathfrak{K})^2$ be the minimal unitary extension of $V \in \mathcal{B}(\mathfrak{K}_+)^2$ as constructed in Theorem 4.4. Then, $U$ is a minimal unitary (respectively, minimal regular unitary) dilation of $T$.

As in the one-dimensional case, a minimal isometric dilation for a commuting pair $T$ ensures the existence of a co-isometric extension for $T^*$.

***Proposition 5.2*** *Let $V = (V_1, V_2) \in \mathcal{B}(\mathfrak{K}_+)^2$ be a minimal isometric dilation of the commuting pair $T = (T_1, T_2) \in \mathcal{B}(\mathfrak{H})^2$. Then, $\mathfrak{H}$ is invariant to $V^* = (V_1^*, V_2^*)$,*

$$T^nP_{\mathfrak{H}} = P_{\mathfrak{H}}V^n, \ n \in \mathbb{Z}_+^2 \quad \textit{and} \quad T^* = V^*|_{\mathfrak{H}}.$$

**Proof** For every $m, n \in \mathbb{Z}_+^2$ and $h \in \mathfrak{H}$, the dilation definition shows that

$$T^n P_{\mathfrak{H}}(V^m h) = T^{n+m} h = P_{\mathfrak{H}} V^{n+m} h = P_{\mathfrak{H}} V^n (V^m h).$$

Since $\{V^m h \mid m \in \mathbb{Z}_+^2, h \in \mathfrak{H}\}$ is dense in $\mathfrak{K}_+$ and $P_{\mathfrak{H}}, T, V$ are bounded, we actually deduce that

$$T^n P_{\mathfrak{H}} = P_{\mathfrak{H}} V^n.$$

We will prove that $V^{n*} h = T^{n*} h$, for every $n \in \mathbb{Z}_+^2$ and $h \in \mathfrak{H}$. To this end, let $m \in \mathbb{Z}_+^2$ and $h' \in \mathfrak{H}$. Then,

$$\langle V^{n*} h - T^{n*} h, V^m h' \rangle_{\mathfrak{K}_+} = \langle h, V^{n+m} h' \rangle_{\mathfrak{K}_+} - \langle T^{n*} h, T^m h' \rangle = 0.$$

Use again the minimality of $V$ to obtain that $V^{n*} h = T^{n*} h$. Consequently, $\mathfrak{H}$ is invariant to $V^*$ and $T^* = V^*|_{\mathfrak{H}}$. ∎

Let $T = (T_1, T_2)$ be a pair of commuting bounded operators on a Kreĭn space $\mathfrak{H}$. By a *defect operator* for $T$, we mean an operator $D_T \in \mathcal{B}(\mathfrak{D}_T, \mathfrak{H})$ with zero kernel on a Kreĭn space $\mathfrak{D}_T$ (called its *defect space*) such that

$$I - T_1^* T_1 - T_2^* T_2 + T_1^* T_2^* T_1 T_2 = D_T D_T^*.$$

$T$ is said to be a *bidisc contraction*, respectively, *bidisc expansion* if

$$\langle T_1 h, T_1 h \rangle_{\mathfrak{H}} + \langle T_2 h, T_2 h \rangle_{\mathfrak{H}} \le \langle h, h \rangle_{\mathfrak{H}} + \langle T_1 T_2 h, T_1 T_2 h \rangle_{\mathfrak{H}}, \quad h \in \mathfrak{H},$$

respectively,

$$\langle T_1 h, T_1 h \rangle_{\mathfrak{H}} + \langle T_2 h, T_2 h \rangle_{\mathfrak{H}} \ge \langle h, h \rangle_{\mathfrak{H}} + \langle T_1 T_2 h, T_1 T_2 h \rangle_{\mathfrak{H}}, \quad h \in \mathfrak{H}$$

or, equivalently, the defect space of $T$ is a Hilbert, respectively, an anti-Hilbert space.

In what follows, we shall use the notations $D_1 = D_{T_1}, D_2 = D_{T_2}, D = D_T$ for the defect operators and $\mathfrak{D}_1 = \mathfrak{D}_{T_1}, \mathfrak{D}_2 = \mathfrak{D}_{T_2}, \mathfrak{D} = \mathfrak{D}_T$ for the corresponding defect spaces.

As in the one-dimensional case, recall that a subspace $\mathfrak{L}$ is said to be *wandering* for a commuting isometric pair $V$ if $V^n \mathfrak{L} \perp V^m \mathfrak{L}$ for all $n, m \in \mathbb{Z}_+^2, n \neq m$.

Regarding the geometrical structure of a minimal regular isometric dilation, we could mention the following theorem.

***Theorem 5.3*** *Let $V = (V_1, V_2) \in \mathcal{B}(\mathfrak{K}_+)^2$ be a minimal regular isometric dilation of a commuting pair $T = (T_1, T_2) \in \mathcal{B}(\mathfrak{H})^2$. Then,*

(i) $\mathfrak{L}_1 = \overline{(V_1 - T_1)\mathfrak{H}}, \mathfrak{L}_2 = \overline{(V_2 - T_2)\mathfrak{H}}, \mathfrak{L} = \overline{(V_1 V_2 - V_1 T_2 - V_2 T_1 + T_1 T_2)\mathfrak{H}}$ *are regular, wandering, respectively, for $V_1, V_2, V$ and isometrically isomorphic, respectively, with $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}$;*

(ii) $M_+^1(\mathfrak{L}_1) = \bigvee_{m \ge 0} V_1^m \mathfrak{L}_1, M_+^2(\mathfrak{L}_2) = \bigvee_{n \ge 0} V_2^n \mathfrak{L}_2, M_+(\mathfrak{L}) = \bigvee_{p \in \mathbb{Z}_+^2} V^p \mathfrak{L}$ *are regular and*

$$\mathfrak{K}_+ = \mathfrak{H} \oplus M_+(\mathfrak{L}) \oplus M_+^1(\mathfrak{L}_1) \oplus M_+^2(\mathfrak{L}_2); \tag{5.3}$$

(iii) *$M_+^i(\mathfrak{L}_i)$ is invariant to $V_i$ and $V_i|_{M_+^i(\mathfrak{L}_i)}$ is a unilateral shift, $i = 1, 2$;*
(iv) *$M_+(\mathfrak{L})$ is invariant to $V$ and $V|_{M_+(\mathfrak{L})}$ is a pair of double commuting unilateral shifts.*

**Proof** As in the Hilbert space case [24], it is not hard to check that $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}$ are wandering, respectively, for $V_1$, $V_2$, $V$ and $\mathfrak{H}, M_+^1(\mathfrak{L}_1), M_+^2(\mathfrak{L}_2), M_+(\mathfrak{L})$ are pairwise orthogonal. Therefore, we prefer to omit the details.

Proceed similarly as in the proof of Theorem 3.1 to obtain

$$V_1^m h = T_1^m h + \sum_{k=0}^{m-1} V_1^k (V_1 - T_1) T_1^{m-k-1} h, \quad h \in \mathfrak{H},\ m \geq 1. \tag{5.4}$$

We use (5.4) in conjunction with the formulas

$$V_2 h = T_2 h + (V_2 - T_2) h, \quad h \in \mathfrak{H} \tag{5.5}$$

and

$$V_2(V_1 - T_1)h = (V_1 V_2 - V_1 T_2 - V_2 T_1 + T_1 T_2) h + (V_1 - T_1) T_2 h, \quad h \in \mathfrak{H}, \tag{5.6}$$

applied successively, to finally get

$$\begin{aligned} V_1^m V_2^n h = T_1^m T_2^n h &+ \sum_{0 \leq p \leq (m-1, n-1)} V^p (V_1 V_2 - V_1 T_2 - V_2 T_1 + T_1 T_2) T^{(m-1,n-1)-p} h \\ &+ \sum_{i=0}^{m-1} V_1^i (V_1 - T_1) T_1^{m-1-i} T_2^n h + \sum_{j=0}^{n-1} V_2^j (V_2 - T_2) T_1^m T_2^{n-1-j} h, \\ &\qquad h \in \mathfrak{H},\ m, n \in \mathbb{N}^*. \end{aligned}$$

More precisely, we firstly apply $V_2$ to (5.4) and then use (5.5) for $T_1^m h$ and (5.6) for $T_1^{m-k-1} h$, $k \in \{0, 1, \ldots, m-1\}$, instead of $h$. We obtain that

$$\begin{aligned} V_1^m V_2 h = T_1^m T_2 h + (V_2 - T_2) T_1^m h &+ \sum_{k=0}^{m-1} V_1^k (V_1 V_2 - V_1 T_2 - V_2 T_1 + T_1 T_2) T_1^{m-k-1} h \\ &+ \sum_{k=0}^{m-1} V_1^k (V_1 - T_1) T_1^{m-k-1} T_2 h. \end{aligned}$$

Following again (5.5) and (5.6) for the computation of the vectors $V_2 T_2 T_1^m h$ and, respectively, $V_2(V_1 - T_1) T_1^{m-k-1} T_2 h$, $k \in \{0, 1, \ldots, m-1\}$, another application of $V_2$ shows that

$$\begin{aligned} V_1^m V_2^2 h = T_1^m T_2^2 h &+ (V_2 - T_2) T_1^m T_2 h + V_2 (V_2 - T_2) T_1^m h \\ &+ \sum_{k=0}^{m-1} V_1^k V_2 (V_1 V_2 - V_1 T_2 - V_2 T_1 + T_1 T_2) T_1^{m-k-1} h \end{aligned}$$

$$+\sum_{k=0}^{m-1} V_1^k(V_1V_2 - V_1T_2 - V_2T_1 + T_1T_2)T_1^{m-k-1}T_2h$$
$$+\sum_{k=0}^{m-1} V_1^k(V_1 - T_1)T_1^{m-k-1}T_2^2h.$$

This iterative procedure is repeated $n$ times.

Since, obviously, $\mathfrak{K}_+$ contains $\mathfrak{H}, M_+(\mathfrak{L}), M_+^1(\mathfrak{L}_1)$, and $M_+^2(\mathfrak{L}_2)$, we deduce that

$$\mathfrak{K}_+ = \mathfrak{H} \vee M_+(\mathfrak{L}) \vee M_+^1(\mathfrak{L}_1) \vee M_+^2(\mathfrak{L}_2)$$

and, by orthogonality, (5.3) also holds.

It is then clear that $\mathfrak{H}, M_+(\mathfrak{L}), M_+^1(\mathfrak{L}_1), M_+^2(\mathfrak{L}_2)$ are all regular and, since $M_+(\mathfrak{L}) = \mathfrak{L} \oplus V_1V_2M_+(\mathfrak{L}) \oplus V_1(V_2M_+(\mathfrak{L}))^\perp \oplus V_2(V_1M_+(\mathfrak{L}))^\perp, M_+^1(\mathfrak{L}_1) = \mathfrak{L}_1 \oplus V_1M_+^1(\mathfrak{L}_1)$, $M_+^2(\mathfrak{L}_2) = \mathfrak{L}_2 \oplus V_2M_+^2(\mathfrak{L}_2)$, we obtain that $\mathfrak{L}, \mathfrak{L}_1, \mathfrak{L}_2$ are also regular.

By a similar argument as in the proof of Theorem 3.1, we can deduce that $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}$ are isometrically isomorphic, respectively, with $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}$.

It is obvious that $M_+(\mathfrak{L}), M_+^1(\mathfrak{L}_1), M_+^2(\mathfrak{L}_2)$ are invariant to $V_1, V_2$ and, respectively, $V$ and that $V_1|_{M_+^1(\mathfrak{L}_1)}$, $V_2|_{M_+^2(\mathfrak{L}_2)}$ are unilateral shifts. Since $M_+(\mathfrak{L}) = M_+^1(M_+^2(\mathfrak{L})) = M_+^2(M_+^1(\mathfrak{L}))$, we obtain that $V|_{M_+(\mathfrak{L})}$ is a pair of commuting unilateral shifts which, moreover, doubly commute.

To this aim, we firstly note that it is only necessary to prove that $(V_1|_{M_+(\mathfrak{L})})^*$ and $V_2|_{M_+(\mathfrak{L})}$ commute on the set $\{V_1^mV_2^nl \mid m, n \geq 0, l \in \mathfrak{L}\}$, which generates $M_+(\mathfrak{L})$.

Indeed, for $n \geq 0$,

$$\big((V_1|_{M_+(\mathfrak{L})})^*V_2\big)V_2^nl = (V_1|_{M_+(\mathfrak{L})})^*V_2^{n+1}l = 0 = \big(V_2(V_1|_{M_+(\mathfrak{L})})^*\big)V_2^nl,$$

since $V_2^nl \in M_+^2(\mathfrak{L}) = \ker(V_1|_{M_+(\mathfrak{L})})^*$. Also, in view of the fact that $V_1|_{M_+(\mathfrak{L})}$ is isometric (i.e., $(V_1|_{M_+(\mathfrak{L})})^*V_1|_{M_+(\mathfrak{L})} = I_{M_+(\mathfrak{L})}$), the following equalities

$$\big((V_1|_{M_+(\mathfrak{L})})^*V_2\big)V_1^mV_2^nl = (V_1|_{M_+(\mathfrak{L})})^*V_1V_1^{m-1}V_2^{n+1}l = V_1^{m-1}V_2^{n+1}l$$

and

$$\big(V_2(V_1|_{M_+(\mathfrak{L})})^*\big)V_1^mV_2^nl = V_2(V_1|_{M_+(\mathfrak{L})})^*V_1V_1^{m-1}V_2^nl = V_1^{m-1}V_2^{n+1}l$$

hold true for every $m > 0$ and $n \geq 0$. ∎

***Corollary 5.4*** *Let $V = (V_1, V_2) \in \mathcal{B}(\mathfrak{K})^2$ be a minimal regular isometric dilation of the commuting pair $T = (T_1, T_2) \in \mathcal{B}(\mathfrak{H})^2$. The following conditions are equivalent:*

(i) *$M_+^1(\mathfrak{L}_1)$ is invariant to $V_2$;*
(ii) *$M_+^2(\mathfrak{L}_2)$ is invariant to $V_1$;*
(iii) *$T$ is a bidisc isometry (i.e., $I - T_1^*T_1 - T_2^*T_2 + T_1^*T_2^*T_1T_2 = 0$).*

**Proof** The conclusion follows from the geometrical structure of $\mathfrak{K}_+$ given by the theorem above since

$$V_{3-i}V_i^n(V_i - T_i)h = V_i^n(V_1V_2 - V_1T_2 - V_2T_1 + T_1T_2)h + V_i^n(V_i - T_i)T_{3-i}h,$$
$$h \in \mathfrak{H},\ i = 1, 2,\ n \in \mathbb{N}$$

and $(V_1V_2 - V_1T_2 - V_2T_1 + T_1T_2)h = 0$, for all $h \in \mathfrak{H}$, if and only if $I - T_1^*T_1 - T_2^*T_2 + T_1^*T_2^*T_1T_2 = 0$.

For the Hilbert space case, we refer to [24]. ∎

For the rest of the article, we shall suppose that $T = (T_1, T_2)$ is a pair of commuting bounded operators on a Kreĭn space $\mathfrak{H}$ such that $T_1, T_2$ are both contractive and $T$ is a bidisc contraction or $T_1, T_2$ are both expansive and $T$ is a bidisc expansion. Equivalently, the defect spaces $\mathfrak{D}_1, \mathfrak{D}_2$, and $\mathfrak{D}$ are either Hilbert or anti-Hilbert spaces. Denote by $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|$ the Hilbert space norms, respectively, on $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}$.

***Remark 5.5*** (i) Observe firstly that

$$\begin{aligned} I - T_1^*T_1 - T_2^*T_2 + T_1^*T_2^*T_1T_2 \qquad & \\ &= D_1D_1^* - T_2^*(I - T_1^*T_1)T_2 \\ &= D_1D_1^* - (T_2^*D_1)(T_2^*D_1)^* \\ &= D_2D_2^* - (T_1^*D_2)(T_1^*D_2)^*. \end{aligned} \tag{5.7}$$

Use the inequality

$$\|D_1^*T_2h\|_1 \le \|D_1^*h\|_1 \quad (\text{respectively, } \|D_2^*T_1h\|_2 \le \|D_2^*h\|_2), \quad h \in \mathfrak{H}$$

to introduce a densely defined Hilbert space contraction on $\mathfrak{D}_1$ (respectively, $\mathfrak{D}_2$) by

$$R_2D_1^*h = D_1^*T_2h \quad (\text{respectively, } R_1D_2^*h = D_2^*T_1h), \quad h \in \mathfrak{H}, \tag{5.8}$$

which can be extended, by continuity, to the whole space. In fact, the maps above are (under our Kreĭn space terminology) contractions if $T$ is a bidisc contraction, respectively, expansions if $T$ is a bidisc expansion.

(ii) Taking into account the operators $R_1$ and $R_2$ (defined by (5.8)), formula (5.7) can be re-written as

$$DD^* = (D_1D_{R_2})(D_1D_{R_2})^* = (D_2D_{R_1})(D_2D_{R_1})^*$$

or, equivalently, as

$$\|D^*h\| = \|D_{R_2}^*D_1^*h\|_{\mathfrak{D}_{R_2}} = \|D_{R_1}^*D_2^*h\|_{\mathfrak{D}_{R_1}}, \quad h \in \mathfrak{H}.$$

Hence, the linear operators $U_1 : \mathfrak{D} \to \mathfrak{D}_{R_1}$ and $U_2 : \mathfrak{D} \to \mathfrak{D}_{R_2}$ given by

$$U_1D^*h = D_{R_1}^*D_2^*h \quad \text{and} \quad U_2D^*h = D_{R_2}^*D_1^*h, \quad h \in \mathfrak{H} \tag{5.9}$$

are well defined unitary operators.

The next construction of a regular isometric dilation is the main result of this section.

***Theorem 5.6*** *Let $T = (T_1, T_2)$ be a pair of commuting bounded operators on a Kreĭn space $\mathfrak{H}$ such that $T_1, T_2$ are both contractive and $T$ is a bidisc contraction or $T_1, T_2$ are both expansive and $T$ is a bidisc expansion. The pair $V = (V_1, V_2) \in \mathcal{B}(\mathfrak{K}_+)^2$ given by* *the matrix representation*

$$V_1 = \begin{pmatrix} T_1 & 0 & 0 & 0 \\ 0 & T_{z_1} & 0 & [U_1^* D_{R_1}^*]_2 \\ [D_1^*]_0 & 0 & T_z & 0 \\ 0 & 0 & 0 & [R_1] \end{pmatrix} \tag{5.10}$$

*and*

$$V_2 = \begin{pmatrix} T_2 & 0 & 0 & 0 \\ 0 & T_{z_2} & [U_2^* D_{R_2}^*]_1 & 0 \\ 0 & 0 & [R_2] & 0 \\ [D_2^*]_0 & 0 & 0 & T_z \end{pmatrix} \tag{5.11}$$

*is a minimal regular isometric dilation of T on the Kreĭn space*

$$\mathfrak{K}_+ = \mathfrak{H} \oplus H^2_{\mathfrak{D}}(\mathbb{T}^2) \oplus H^2_{\mathfrak{D}_1}(\mathbb{T}) \oplus H^2_{\mathfrak{D}_2}(\mathbb{T}).$$

**Proof** Direct computations with matrices show that, for $i = 1, 2$, $V_i$ is an isometric operator on $\mathfrak{K}_+$ if and only if $[D_i^*]_0^* T_z = 0$, $[U_i^* D_{R_i}^*]_{3-i}^* T_{z_i} = 0$, $T_i^* T_i + [D_i^*]_0^* [D_i^*]_0 = I_{\mathfrak{H}}$ and $[U_i^* D_{R_i}^*]_{3-i}^* [U_i^* D_{R_i}^*]_{3-i} + [R_i]^* [R_i] = I_{H^2_{\mathfrak{D}_{3-i}}(\mathbb{T})}$.

While the first two equalities hold true by Proposition 2.2 (iv), the last two are consequences of the conditions (ii), respectively, (iii) of the same proposition. Indeed, $[D_i^*]_0^* [D_i^*]_0 = D_i D_i^*$ and, hence, $T_i^* T_i + D_i D_i^* = I_{\mathfrak{H}}$, by (3.1). Also,

$$\begin{aligned} &[U_i^* D_{R_i}^*]_{3-i}^* [U_i^* D_{R_i}^*]_{3-i} + [R_i]^* [R_i] \\ &\quad = [(U_i^* D_{R_i}^*)^* U_i^* D_{R_i}^* + R_i^* R_i] && \text{(by Proposition 2.2 (i) and (ii))} \\ &\quad = [D_{R_i} D_{R_i}^* + R_i^* R_i] && \text{(since } U_i \text{ is unitary)} \\ &\quad = [I_{\mathfrak{D}_{3-i}}] = I_{H^2_{\mathfrak{D}_{3-i}}(\mathbb{T})}. && \text{(by (3.1))} \end{aligned}$$

Similarly, $V_1 V_2 = V_2 V_1$ if and only if $[U_1^* D_{R_1}^*]_2 [D_2^*]_0 = [U_2^* D_{R_2}^*]_1 [D_1^*]_0$, $[U_i^* D_{R_i}^*]_{3-i} T_z = T_{z_{3-i}} [U_i^* D_{R_i}^*]_2$, $T_z [R_i] = [R_i] T_z$ and $[D_i^*]_0 T_{3-i} = [R_{3-i}][D_i^*]_0$, $i = 1, 2$. The first condition follows by Proposition 2.2 (v) and (5.9):

$$[U_i^* D_{R_i}^*]_{3-i} [D_{3-i}^*]_0 h = z_1^0 z_2^0 U_i^* D_{R_i}^* D_{3-i}^* h = z_1^0 z_2^0 D^* h, \quad h \in \mathfrak{H}, i = 1, 2.$$

The following two conditions are consequences of Proposition 2.2 (iv). The last equality uses Proposition 2.2 (v) and formula (5.8):

$$[D_i^*]_0 T_{3-i} = [D_i^* T_{3-i}]_0 = [R_{3-i} D_i^*]_0 = [R_{3-i}][D_i^*]_0.$$

Moreover, by an inductive method, $V_2^{*n} V_1^m$ has the form

$$V_2^{*n} V_1^m = \begin{pmatrix} T_2^{*n} T_1^m & 0 & 0 & * \\ 0 & (T_{z_2})^{*n} (T_{z_1})^m & 0 & * \\ * & * & [R_2^{*n}](T_z)^m & * \\ 0 & 0 & 0 & (T_z)^{*n} [R_1^m] \end{pmatrix},$$

which proves that

$$T_2^{*n} T_1^m = P_{\mathfrak{H}} V_2^{*n} V_1^m |_{\mathfrak{H}}, \quad m, n \geq 0.$$

We can also obtain, by a similar argument, that

$$T_1^m T_2^n = P_{\mathfrak{H}} V_1^m V_2^n|_{\mathfrak{H}}, \quad m, n \geq 0.$$

Hence, $V$ is a regular isometric dilation of $T$.

It remains to prove the minimality. To this end, take $h \in \mathfrak{H}$ and observe that

$$(V_1 - T_1)h = (0, 0, [D_1^*]_0 h, 0).$$

Proceed inductively to show that

$$V_1^m (V_1 - T_1)h = (0, 0, (T_z)^n [D_1^*]_0 h, 0), \quad m \geq 0,$$

that is,

$$H^2_{\mathfrak{D}_1}(\mathbb{T}) = \bigvee_{m \geq 0} V_1^m (V_1 - T_1)\mathfrak{H}. \tag{5.12}$$

By symmetry, it also holds

$$H^2_{\mathfrak{D}_2}(\mathbb{T}) = \bigvee_{n \geq 0} V_2^n (V_2 - T_2)\mathfrak{H}. \tag{5.13}$$

Now, the relation

$$(V_1 V_2 - V_1 T_2 - V_2 T_1 + T_1 T_2)h = (0, z_1^0 z_2^0 D^* h, 0, 0)$$

applied successively gives

$$V_1^m V_2^n (V_1 V_2 - V_1 T_2 - V_2 T_1 + T_1 T_2)h = (0, z_1^m z_2^n D^* h, 0, 0), \quad m, n \geq 0,$$

that is,

$$H^2_{\mathfrak{D}}(\mathbb{T}^2) = \bigvee_{n \in \mathbb{Z}_+^2} V^n (V_1 V_2 - V_1 T_2 - V_2 T_1 + T_1 T_2)\mathfrak{H}. \tag{5.14}$$

Equations (5.12–5.14) show that the regular isometric dilation given by (5.10) and (5.11) is minimal. ■

Use Theorem 4.4, Remark 5.1, and Theorem 5.6 to obtain the following.

***Corollary 5.7*** *Let $T \in \mathcal{B}(\mathfrak{H})^2$ be as in Theorem 5.6 and $V \in \mathcal{B}(\mathfrak{K}_+)^2$ be the minimal regular isometric dilation of $T$ given by (5.10) and (5.11). Then, $T$ has a minimal regular unitary dilation $U \in \mathcal{B}(\mathfrak{K})^2$ given by (4.2) and (4.3) on the Kreĭn space*

$$\mathfrak{K} = \mathfrak{H} \oplus H^2_{\mathfrak{D}}(\mathbb{T}^2) \oplus H^2_{\mathfrak{D}_1}(\mathbb{T}) \oplus H^2_{\mathfrak{D}_2}(\mathbb{T}) \oplus H^2_{\ker(V_1 V_2)^*}(\mathbb{T}).$$

Let $V = (V_1, V_2) \in \mathcal{B}(\mathfrak{K}_+)^2$ and $V' = (V_1', V_2') \in \mathcal{B}(\mathfrak{K}_+')^2$ be two minimal regular isometric dilations of $T \in \mathcal{B}(\mathfrak{H})^2$. $V$ and $V'$ are said to be *unitarily equivalent* if there exists a unitary operator $\Phi : \mathfrak{K}_+ \to \mathfrak{K}_+'$ which intertwines $V_1$ and $V_1'$, respectively, $V_2$ and $V_2'$ and such that $\Phi|_{\mathfrak{H}} = I_{\mathfrak{H}}$.

***Theorem 5.8*** *Let $T = (T_1, T_2)$ be a pair of commuting bounded operators on a Kreĭn space $\mathfrak{H}$ such that $T_1, T_2$ are both contractive and $T$ is a bidisc contraction or $T_1, T_2$ are both expansive and $T$ is a bidisc expansion. Then, $T$ has a unique minimal regular isometric dilation (up to a unitary equivalence).*

**Proof** Let $V \in \mathcal{B}(\mathfrak{K}_+)^2$ be the minimal regular isometric dilation of $T \in \mathcal{B}(\mathfrak{H})^2$ given by (5.10) and (5.11) on $\mathfrak{K}_+ = \mathfrak{H} \oplus H^2_{\mathfrak{D}}(\mathbb{T}^2) \oplus H^2_{\mathfrak{D}_1}(\mathbb{T}) \oplus H^2_{\mathfrak{D}_2}(\mathbb{T})$. If $V' = (V_1', V_2') \in \mathcal{B}(\mathfrak{K}_+')^2$ is any other minimal regular isometric dilation of $T$ then, according to (5.3), $\mathfrak{K}'$ has an orthogonal decomposition of the form

$$\mathfrak{K}_+' = \mathfrak{H} \oplus M_+(\mathfrak{L}') \oplus M_+^1(\mathfrak{L}_1') \oplus M_+^2(\mathfrak{L}_2'),$$

with $\mathfrak{L}' = \overline{(V_1'V_2' - V_1'T_2 - V_2'T_1 + T_1T_2)\mathfrak{H}}$ and $\mathfrak{L}_i' = \overline{(V_i' - T_i)\mathfrak{H}}$, $i = 1, 2$.

The maps

$$M_+(\mathfrak{L}') \ni V_1'^m V_2'^n (V_1'V_2' - V_1'T_2 - V_2'T_1 + T_1T_2)h \overset{\Phi}{\longmapsto} z_1^m z_2^n D^* h \in H^2_{\mathfrak{D}}(\mathbb{T}^2),$$

$$M_+(\mathfrak{L}_1') \ni V_1'^m (V_1' - T_1)h \overset{\Phi_1}{\longmapsto} z^m D_1^* h \in H^2_{\mathfrak{D}_1}(\mathbb{T})$$

and

$$M_+(\mathfrak{L}_2') \ni V_2'^n (V_1' - T_2)h \overset{\Phi_2}{\longmapsto} z^n D_2^* h \in H^2_{\mathfrak{D}_2}(\mathbb{T})$$

are well defined and can be extended by linearity to densely defined isometries with dense ranges. Since $H^2_{\mathfrak{D}}(\mathbb{T}^2)$, $H^2_{\mathfrak{D}_1}(\mathbb{T})$, and $H^2_{\mathfrak{D}_2}(\mathbb{T})$ are either Hilbert or anti-Hilbert spaces, the applications above can be extended to unitary operators.

A routine check shows that $I_{\mathfrak{H}} \oplus \Phi \oplus \Phi_1 \oplus \Phi_2 : \mathfrak{K}_+' \to \mathfrak{K}_+$ is unitary and intertwines $V_1$ and $V_1'$, respectively, $V_2$ and $V_2'$. Hence, $V$ and $V'$ are unitarily equivalent. ∎

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