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ESSENTIAL COVERS AND COMPLEMENTS OF RADICALS

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We show that a radical has a semisimple essential cover if and only if it is hereditary and has a complement in the lattice of hereditary radicals. In 1971 Snider gave a full description of supernilpotent radicals which have a complement. Recently Beidar, Fong, Ke, and Shum have determined radicals with semisimple essential covers. Using their results, we are able to provide a lower radical representation of complemented subidempotent radicals. This completes Snider's description of hereditary complemented radicals.

In the context of radical theory the usefulness of the essential cover operator $\mathcal{E}$ has been known from Armendariz [2] and Rjabuhin [8], who showed that a semisimple class is closed under essential extension if and only if the corresponding radical class is hereditary. In 1970, Stewart [10] characterised semisimple radical classes in terms of subdirect sums of a finite set of finite fields. In 1983, Loi [7] showed that a radical class is semisimple if and only if it is closed under essential extensions (also see Gardner [6]). The last two results naturally lead one to consider the classification of the essential covers of radicals in terms of semisimplicity. In 1994, Birkenmeier [4] showed that the essential cover $\mathcal{E}\rho$ of a supernilpotent radical ρ is nearly a semisimple class: it is hereditary, closed under extensions, finite subdirect sums, arbitrary direct sums and products. Hence $\mathcal{E}\rho$ only lacks the requirement of being closed under arbitrary subdirect sums to become a semisimple class. Thus the question arises in [4]: which supernilpotent classes have semisimple essential covers? Imposing this seemingly mild extra condition on the essential cover $\mathcal{E}\rho$ has turned out to be very restrictive: none of the classical radicals have semisimple essential covers [5]. Recently, Beidar, Fong, Ke, and Shum [3] have fully described radicals having semisimple essential covers. Their description is reminiscent of Stewart's characterisation of radical semisimple classes [10].

Working on the same problem we found an alternative solution: radicals whose essential covers are semisimple classes, are exactly the hereditary radicals which have

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a complement in the lattice of hereditary radicals, and the latter ones have been determined by Snider [**9**]. However, Snider described explicitly only the complemented supernilpotent radicals. Thus by our result the lower radical representation of [**3**] for subidempotent radicals with semisimple essential cover applies to complemented subidempotent radicals, supplementing the description of Snider [**9**].

Rings will be always associative, not necessarily possessing a unity element. A radical will always mean a Kurosh-Amitsur radical. When the context is clear, we shall use the words "radical" and "radical class" interchangeably and denote the radical assignment and the radical class by the same symbol.

As usual, the lower, the upper and the semi-simple operators will be denoted by $\mathcal{L}$, $\mathcal{U}$, and $\mathcal{S}$, respectively. The *essential cover operator* $\mathcal{E}$ acting on a class ζ of rings is defined by

$$\mathcal{E}\zeta = \{\text{all rings } A \mid A \text{ has an essential ideal } I \text{ in } \zeta\}.$$

The class $\mathcal{E}\zeta$ is called the *essential cover* of the class ζ. If $\mathcal{E}\zeta = \zeta$, then the class ζ is said to be *closed under essential extensions*. If I is an essential ideal in a ring A, then we shall write $I \lhd \cdot A$.

For the notions and fundamental results of the radical theory the reader is referred to [**11**]. However, we list some well-known notions and results we shall frequently use in the sequel.

ADS-Theorem: For any radical γ, if $I \lhd A$ and A is any ring, then $\gamma(I) \lhd A$.

A radical class γ is *hereditary* (that is, $I \lhd A \in \gamma$ implies $I \in \gamma$) if and only if the corresponding semisimple class $\mathcal{S}\gamma$ is closed under essential extensions, [**2, 8**].

Every semisimple class is hereditary.

For hereditary radicals γ and δ, $\gamma \cap \delta = \{0\}$ is equivalent to the condition

$$\gamma(A) \cap \delta(A) = 0 \text{ for all rings A.}$$

For a given radical γ the radical $\overline{\gamma}$ is said to be the *radical supplementing* γ, if $\gamma(A) \cap \overline{\gamma}(A) = 0$ for all rings A and $\overline{\gamma}$ is the largest such radical.

For a hereditary radical γ the supplementing radical $\overline{\gamma}$ is the largest homomorphically closed subclass, $h\mathcal{S}\gamma$, of the semisimple class $\mathcal{S}\gamma$:

$$h\mathcal{S}\gamma = \{A \in \mathcal{S}\gamma \mid \text{every homomorphic image of } A \text{ is in } \mathcal{S}\gamma\}.$$

(See [**1**].)

If the radical γ is *supernilpotent* (that is, γ is hereditary and contains all nilpotent rings) or *subidempotent* (that is, γ is hereditary and consists of idempotent rings), then its supplementing radical $\overline{\gamma}$ is subidempotent or supernilpotent, respectively, and thus hereditary.

γ is a *dual radical*, if $\overline{\overline{\gamma}} = \gamma$, and in this case $(\gamma, \overline{\gamma})$ is called a dual pair of radicals ($\overline{\gamma} = \overline{\overline{\overline{\gamma}}}$ is always true).

It is customary to talk about the lattice of all (or all hereditary) radicals, although they do not form a set. In the lattice of radicals the meet $\wedge$ is defined by

$$\gamma \wedge \delta = \gamma \cap \delta$$

and the join by

$$\gamma \vee \delta = \mathcal{L}(\gamma \cup \delta) = \mathcal{U}(\mathcal{S}\gamma \cap \mathcal{S}\delta).$$

In particular,

$$\gamma \vee \delta = \{\text{all rings}\} \text{ if and only if } \mathcal{S}\gamma \cap \mathcal{S}\delta = \{0\}.$$

PROPOSITION 1. ([5, Lemma 2.1]) *Let γ and δ be radicals. $\mathcal{E}\gamma = \mathcal{S}\delta$ if and only if $\mathcal{E}\delta = \mathcal{S}\gamma$. If either one of these conditions are satisfied, then γ and δ are hereditary.*

PROPOSITION 2. ([3, Lemma 0.10]) *If γ is a radical such that $\mathcal{E}\gamma$ is a semisimple class, then γ is either supernilpotent or subidempotent.*

THEOREM 3. *The essential cover $\mathcal{E}\gamma$ of a radical γ is a semisimple class if and only if γ is hereditary and has a complement in the lattice of all hereditary radicals.*

PROOF: Suppose that $\mathcal{E}\gamma$ is a semisimple class, the semisimple class $\mathcal{S}\delta$ of a radical δ. Then by Proposition 1, γ is hereditary and so is δ as well. Hence for every ring A we have

$$\gamma(A) \cap \delta(A) \in \gamma \cap \delta \subseteq \mathcal{E}\gamma \cap \delta = \mathcal{S}\delta \cap \delta = \{0\}.$$

Thus $\delta \subseteq \overline{\gamma}$ where $\overline{\gamma}$ denotes the radical supplementing γ, and so $\mathcal{S}\overline{\gamma} \subseteq \mathcal{S}\delta$.

Since γ is hereditary, we know that

$$\overline{\gamma} = h\mathcal{S}\gamma \subseteq \mathcal{S}\gamma.$$

Proposition 2 says that γ is either supernilpotent or subidempotent, so $\overline{\gamma}$ is subidempotent or supernilpotent, respectively. Thus $\overline{\gamma}$ is hereditary. If $A \in \gamma$, then

$$\overline{\gamma}(A) = A \cap \overline{\gamma}(A) = \gamma(A) \cap \overline{\gamma}(A) = 0,$$

that is, $A \in \mathcal{S}\overline{\gamma}$ which implies $\gamma \subseteq \mathcal{S}\overline{\gamma}$. Since $\overline{\gamma}$ is hereditary, $\mathcal{S}\overline{\gamma}$ is closed under essential extensions, and so we get

$$\mathcal{S}\delta = \mathcal{E}\gamma \subseteq \mathcal{E}\mathcal{S}\overline{\gamma} = \mathcal{S}\overline{\gamma}$$

yielding

$$\mathcal{E}\gamma = \mathcal{S}\delta = \mathcal{S}\overline{\gamma} \text{ and also } \delta = \overline{\gamma}.$$

Thus $\delta = \overline{\gamma}$ is the radical supplementing γ, and $\mathcal{E}\gamma = \mathcal{S}\overline{\gamma}$.

Next, let us consider the intersection $\mathcal{S}\gamma \cap \mathcal{S}\overline{\gamma}$ of the semisimple classes of γ and $\overline{\gamma}$, and let us take an arbitrary ring A from $\mathcal{S}\gamma \cap \mathcal{S}\overline{\gamma}$. Then $A \in \mathcal{S}\overline{\gamma} = \mathcal{E}\gamma$, and so A has an essential ideal $B \in \gamma$. On the other hand, $A \in \mathcal{S}\gamma$ and therefore the only ideal of A which is in γ, is the ideal 0. Hence $B = 0$, and by $B \triangleleft \cdot A$ it follows $A = 0$. Thus $\mathcal{S}\gamma \cap \mathcal{S}\overline{\gamma} = \{0\}$, and so $\gamma \vee \overline{\gamma} = \{\text{ all rings}\}$.

Since $\overline{\gamma}$ is supplementing the radical γ, also

$$\gamma \wedge \overline{\gamma} = \{0\}$$

holds, whence $\overline{\gamma}$ is the complement of γ in the lattice of hereditary radicals.

Conversely, assume that γ has a complement γ' in the lattice of hereditary radicals. Then $\gamma \wedge \gamma' = \{0\}$ and by the hereditariness of γ and γ' this is equivalent to

$$\gamma(A) \cap \gamma'(A) = 0 \text{ for all rings } A.$$

Since the radical $\overline{\gamma}$ supplementing γ is the largest radical with the latter property, it follows that $\gamma' \subseteq \overline{\gamma}$. Thus

$$\gamma \vee \overline{\gamma} \supseteq \gamma \vee \gamma' = \{\text{all rings}\}$$

is valid, whence also $\overline{\gamma}$ is a complement of γ. Since the complement (if it exists) is unique in the lattice of hereditary radicals, we conclude that $\gamma' = \overline{\gamma}$. Thus we have also

$$\mathcal{S}\gamma \cap \mathcal{S}\overline{\gamma} = \{0\}.$$

Our aim is to show that $\mathcal{E}\gamma$ is a semisimple class. Suppose that $\mathcal{E}\gamma$ is not a semisimple class. By definition of the supplementing radical we have $\gamma \subseteq \overline{\overline{\gamma}}$ where $\overline{\overline{\gamma}}$ is the radical supplementing $\overline{\gamma}$. Since $\overline{\gamma} = \gamma'$ is hereditary, $\overline{\overline{\gamma}}$ is given as $h\mathcal{S}\overline{\gamma}$, and so

$$\gamma \subseteq \overline{\overline{\gamma}} = h\mathcal{S}\overline{\gamma} \subseteq \mathcal{S}\overline{\gamma}$$

holds. By the hereditariness of $\overline{\gamma}$ the semisimple class $\mathcal{S}\overline{\gamma}$ is closed under essential extensions, and so by $\gamma \subseteq \mathcal{S}\overline{\gamma}$ it follows that $\mathcal{E}\gamma \subseteq \mathcal{S}\overline{\gamma}$. Moreover, by the assumption $\mathcal{E}\gamma \neq \mathcal{S}\overline{\gamma}$, there exists a ring A in the class $\mathcal{S}\overline{\gamma} \backslash \mathcal{E}\gamma$. Hence $\gamma(A)$ is not an essential ideal of A, and so A possesses a nonzero ideal B such that $\gamma(A) \cap B = 0$. From $B \triangleleft A \in \mathcal{S}\overline{\gamma}$ we conclude that $B \in \mathcal{S}\overline{\gamma}$. Furthermore, the ADS-Theorem yields $\gamma(B) \triangleleft A$, and so

$$\gamma(B) \subseteq \gamma(A) \cap B = 0.$$

Hence $B \in \mathcal{S}\gamma$. Thus $0 \neq B \in \mathcal{S}\gamma \cap \mathcal{S}\overline{\gamma}$ holds, contradicting the already proven relation $\mathcal{S}\gamma \cap \mathcal{S}\overline{\gamma} = \{0\}$. Thus $\mathcal{E}\gamma$ must be a semisimple class. ∎

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Supplement to Theorem 3. *If $\mathcal{E}\gamma$ is a semisimple class, then $\mathcal{E}\gamma$ is the semisimple class of the radical $\overline{\gamma}$ supplementing γ. If γ has a complement γ' in the lattice of hereditary radicals, then γ' is the radical $\overline{\gamma}$ supplementing γ. If $\mathcal{E}\gamma$ is a semisimple class, then $(\gamma, \overline{\gamma})$ forms a dual pair of a supernilpotent radical and a subidempotent radical.*

The proof of the first and second statement is included in that of Theorem 3. The last assertion is obvious by $\gamma = \gamma'' = \overline{\overline{\gamma}}$ and Proposition 1.

Combining Theorem 3 with Snider's [**9**] description of complemented hereditary radicals, and taking into account the supplement and Andrunakievich's Theorem 10 in [**1**] to determine subidempotent dual radicals, we get

Corollary 4. *The essential cover $\mathcal{E}\gamma$ of a radical γ is a semisimple class if and only if there exists a finite set*

$$\varphi = \{M_{n_1}(F_1), \dots, M_{n_k}(F_k)\}$$

of matrix rings over finite fields $F_1, \dots, F_k$ such that either γ is the upper radical $\mathcal{U}\varphi$ or γ is the upper radical $\mathcal{U}\psi$ where

$$\psi = \{A \text{ is subdirectly irreducible } \mid A \notin \varphi\}.$$

The connection established in Theorem 3 enables us to complete Snider's result on complemented subidempotent radicals by using the description given in [**3**]. Given a simple ring M, a ring A will be called an *M-ring*, if A is semiprime and every prime homomorphic image of A is isomorphic to M.

Corollary 5. *A subidempotent radical γ is complemented in the lattice of all hereditary radicals if and only if there exist finitely many matrix rings $M_{n_1}(F_1), \dots,$ $M_{n_k}(F_k)$ over finite fields $F_1, \dots, F_k$ such that γ is the lower radical generated by all the $M_{n_i}(F_i)$-rings for $i = 1, 2, \dots, k$.*

Recently, it has come to our attention that Wu Tongsuo [**12**] has also characterised the semisimplicity of the essential cover of a supernilpotent radical ρ by showing that $\mathcal{E}\rho$ is semisimple if and only if $\rho(R/\overline{\rho}(R))$ is essential in $R/\overline{\rho}(R)$ for any ring R.

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