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SMOOTHNESS AND THE ASYMPTOTIC–NORMING PROPERTIES OF BANACH SPACES

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We study some smoothness properties of a Banach space X that are related to the weak* asymptotic-norming properties of the dual space X^*. These properties imply that X is an Asplund space and are related to the duality mapping of X.

1.

Recently, Haydon [**6**] resolved a long standing conjecture in the negative by constructing an Asplund space that fails to admit an equivalent Frechét differentiable norm. The authors [**8**] introduced the weak*-asymptotic-norming properties in the dual Banach spaces and showed that there exists a Banach space X such that X^* has the Radon-Nikodym property but fails to have the weak*-asymptotic-norming property III. In this paper, we study some smoothness properties of X that are related to the weak*-asymptotic-norming properties in X^* and show that they imply that X is an Asplund space. We partially solve a question raised in [**1**] concerning the duality mapping of X.

For a Banach space X, let $S_X = \{x : x \in X, \|x\| = 1\}$ and $B_X = \{x : x \in X, \|x\| \leqslant 1\}$. A subset Φ of B_{X^*} is called a *norming set* of X if $\|x\| = \sup\{x^*(x) : x^* \in \Phi\}$ for all x in X. A sequence $\{x_n\}$ in S_X is said to be asymptotically normed by Φ [**9**] if for any $\varepsilon > 0$, there is x^* in Φ and N in $\mathbb{N}$ such that $x^*(x_n) > 1 - \varepsilon$ for all $n \geqslant N$.

For $\kappa = I, II$ or III, a sequence $\{x_n\}$ is said to have the property κ if

(I) $\{x_n\}$ is convergent;
(II) $\{x_n\}$ has a convergent subsequence;
(III) $\bigcap_{n=1}^{\infty} \overline{co}\{x_k : k \geqslant n\} \neq \phi$.

Let Φ be a norming set of X. Then X is said to have the *asymptotic-norming property* κ, $\kappa = I, II$, *or* III *with respect to* Φ (Φ-ANP-κ) if every sequence in S_X that is asymptotically normed by Φ has the property κ. X is said to have *the asymptotic-norming property* κ (ANP-κ) [**9**] if there is an equivalent norm $\|\cdot\|$ on X such that there is a norming set Φ with respect to $(X, \|\cdot\|)$ such that X has the Φ-ANP-κ,

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$\kappa = I, II$ or III. We say that a dual Banach space X^* has the *weak* asymptotic-norming property* κ (w*-ANP-κ) [8] if there is an equivalent norm $\|\cdot\|$ on X and a norming set Φ of $(X^*, \|\cdot\|)$ in B_X such that $(X^*, \|\cdot\|)$ has the Φ-ANP-κ, $\kappa = I, II$ or III.

For a Banach space X, let $X^\perp = \{x^\perp : x^\perp \in X^{***}, x^\perp(x) = 0$ for all x in $X\}$. A Banach space X is said to be *Hahn-Banach smooth* [11] if for all x^* in X^*, $\|x^* + x^\perp\| = \|x^*\| = 1$ implies that $x^\perp = 0$. In other words, x^* in X^{***} is the unique Hahn-Banach extension of $x^* |_X$. It is obvious that X is Hahn-Banach smooth if and only if $X^* = \{x^{***} : x^{***} \in X^{***}, \|x^{***}\| = \sup\{x^{***}(x) : x \in B_X\}\}$. Combining this with [8, Theorem 2.3 and Theorem 3.1], we have the following result.

Theorem 1. *Let $(X, \|\cdot\|)$ be a Banach space. The following are equivalent:*

(1) *$(X, \|\cdot\|)$ is Hahn-Banach smooth;*
(2) *X^* has the w*-ANP-III with respect to the norm $\|\cdot\|$;*
(3i) *there exists a norming set Φ of $(X, \|\cdot\|)$ in $B_{(X,\|\cdot\|)}$ such that $X^* = \{x^{***} : x^{***} \in X^{***}, \|x^{***}\| = \sup_{x\in\Phi} x^{***}(x)\}$;*
(3ii) *for any norming set Φ of X^* in $B_{(X,\|\cdot\|)}$, $X^* = \{x^{***} : x^{***} \in X^{***}, \|x^{***}\| = \sup_{x\in\Phi} x^{***}(x)\}$;*
(4) *the weak and weak* topologies coincide on $S_{(X^*,\|\cdot\|)}$.*

Corollary 2. [11, Theorem 6]. *If X is a Banach space such that $(S_{X^*}, w^*) = (S_{X^*}, \|\cdot\|)$, then X is Hahn-Banach smooth.*

Corollary 3. [1, Corollary 3.4]. *Every Hahn-Banach smooth space is Asplund.*

Proof: If X is Hahn-Banach smooth, then X^* has the w*-ANP-III. By [8], X^* has the Radon-Nikodym property. Hence X is Asplund. ∎

Remark. In [2, Lemma 6], it is proved that (1) and (4) in Theorem 1 are equivalent.

Example. Let $X = c_0(\omega_1)$ where ω_1 is the first uncountable ordinal. Then X is an Asplund space which admits an equivalent Frechét differentiable norm [12]. However, in [8], it is proved that X^* fails to have w*-ANP-III. Hence X is an Asplund space which is Frechét differentiable but fails to have an equivalent Hahn-Banach smooth norm. The spaces $C(K)$ and $C_0(L)$ constructed by Haydon in [6] are Asplund spaces that fail to admit an equivalent Frechét differentiable norm and they also fail to have an equivalent Hahn-Banach smooth norm. We don't know whether every Hahn-Banach smooth space admits an equivalent Frechét differentiable norm, even though Hahn-Banach smoothness is a property strictly stronger than the property that the space is Asplund.

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2.

The duality mapping D for a Banach space X is the set valued function from S_X to S_{X^*} defined by $D(x) = \{x^* : \|x^*\| = 1 = x^*(x)\}$, $x \in S_X$. X is said to be *very smooth* [11] if every element in S_X has a unique norming element in X^{***}. It is known that X is Frechét differentiable (respectively, very smooth) if and only if the duality mapping D is single-valued and is $(\|\cdot\| - \|\cdot\|)$ (respectively, $(\|\cdot\| - w)$) continuous.

DEFINITION: A Banach space X is said to be *quasi-Frechét differentiable* (respectively, *quasi-very smooth*) if, when $\{x_n\}$ is any convergent sequence in S_X, then for any $x_n^* \in D(x_n)$, $n \in \mathbb{N}$, the sequence $\{x_n^*\}$ has a norm-convergent (respectively, weakly convergent) subsequence.

It is clear that if X is smooth and quasi-Frechét differentiable (respectively, quasi-very smooth) then X is Frechét differentiable (respectively, very smooth). However, let c_0 be the usual sup norm; then $c_0^* = \ell_1$ has the w*-ANP-II [8]. By Theorem 4 below, c_0 is quasi-Frechét differentiable and Hahn-Banach smooth but is neither Frechét differentiable nor very smooth.

Let X and Y be topological spaces. A set valued function $D : X \longrightarrow Y$ is said to be *upper semi-continuous (u.s.c.) at* x, $x \in X$ if for any open set G in Y, $G \supset D(x)$, there exists a neighbourhood U of x in X such that $D(U) \subset G$. D is said to be *upper semi-continuous on* X if D is u.s.c. at every point of X. In the case that X is a normed space and $Y = X^*$, Giles, Gregory and Sims [1] introduced a restricted notion of upper semi-continuity for the duality mapping. The duality mapping D is said to be *GGS-u.s.c. (respectively GGS-w.u.s.c.) at* x if for every open set G of the form $D(x) + N$ where N is an open neighbourhood of 0 in $(X^*, \|\cdot\|)$ (respectively, (X^*, weak)), then there is a neighbourhood U of x such that $D(U) \subset G$. It is easy to see that if $D(x)$ is compact, then the two definitions of u.s.c. are the same. However, in general, $D(x)$ is not compact in either the norm or weak topology of X^*.

THEOREM 4. *Let $(X, \|\cdot\|)$ be a Banach space and let D be the duality mapping of X.*

(1) *If X^* has the w*-ANP-I with respect to the norm $\|\cdot\|$, then $(X, \|\cdot\|)$ is Frechét differentiable.*
(2) *If X^* has the w*-ANP-II with respect to the norm $\|\cdot\|$, then $(X, \|\cdot\|)$ is quasi-Frechét differentiable.*
(3) *If X^* has the w*-ANP-III with respect to the norm $\|\cdot\|$, then $(X, \|\cdot\|)$ is quasi-very smooth and so every Hahn-Banach smooth Banach space is quasi-very smooth.*
(4) *If X is quasi-Frechét differentiable then $D(x)$ is compact for all x in S_X and $D : (S_X, \|\cdot\|) \longrightarrow (S_{X^*}, \|\cdot\|)$ is u.s.c..*

(5) *If X is quasi-very smooth then $D(x)$ is weakly compact for all x in S_X and $D : (S_X, \|\cdot\|) \longrightarrow (S_{X^*}, w)$ is u.s.c.*

PROOF: (1) Let $\{x_n\}$ be a sequence in S_X such that $\lim_n \|x_n - x\| = 0$ for some $x \in X$. Then for any $x_n^* \in D(x_n)$, $n \in \mathbb{N}$, $x_n^*(x) \longrightarrow 1$. Hence $\{x_n^*\}$ is asymptotically normed by B_X. Since $(X^*, \|\cdot\|)$ has the w*-ANP-I, by [**8**, Corollary 3.2], it follows that $\{x_n^*\}$ is convergent in $(S_{X^*}, \|\cdot\|)$. Thus X is Frechét differentiable.

(2) and (3) are proved similarly to (1).

(4). It is clear that $D(x)$ is compact for all x in S_X when X is quasi-Frechét differentiable. To show that D is u.s.c., it suffices to show that if F is a norm closed subset in X^*, then the set $A = \{x : \|x\| = 1, D(x) \cap F \neq \phi\}$ is norm closed in S_X. Suppose $\{x_n\} \subset A$ and $\lim_n \|x_n - x\| = 0$ for some x in X. Choose $x_n^* \in D(x_n) \cap F$. Then $\lim_n x_n^*(x) = 1$. Since X is quasi-Frechét differentiable, there is a subsequence $\{x_{n_k}^*\}$ of $\{x_n^*\}$ and x^* in S_{X^*} such that $\lim_k \|x_{n_k}^* - x^*\| = 0$. It is clear that $x^* \in D(x) \cap F$. Therefore A is closed.

(5) is proved similarly to (4). □

REMARK. Since there exists a Frechét differentiable space X [**8**] that fails to admit an equivalent Hahn-Banach smooth norm, hence there exists a Frechét differentiable (respectively, quasi-Frechét differentiable) space X such that X^* does not have the w*-ANP-I (respectively, w*-ANP-II). We don't know if X is Frechét differentiable (respectively, quasi-Frechét differentiable) *and* Hahn-Banach smooth, whether or not X^* has the w*-ANP-I (respectively, w*-ANP-II).

3.

A Banach space X is said to be *weakly Hahn-Banach smooth* [**10**] if in X^{***}, for any $x^* \in X^*$, $x^\perp \in X^\perp$, $\|x^* + x^\perp\| = \|x^*\| = 1$ and $x^*(x) = \|x\| = 1$ for some x in X, then $x^\perp = 0$.

X is said to be *weakly very smooth* [**13**] if for all x in S_X, x_n^* in S_{X^*}, $\lim_n x_n^*(x) = 1$ implies that $\{x_n^*\}$ has a weakly convergent subsequence in X^*.

From [**1**, Theorem 3.1, Cororllary 3.2] and Theorem 4, we conclude that for a Banach space X, the weakly Hahn-Banach smoothness, weakly very smoothness and quasi-very smoothness are the same. From now on, we shall use the term weakly Hahn-Banach smooth only.

In the following, for simplicity, we say that the duality mapping is w.u.s.c. if $D : (S_X, \|\cdot\|) \longrightarrow (S_{X^*}, w)$ is u.s.c.

4.

In [**1**], a question was raised: if a Banach space X admits an equivalent norm for which the duality mapping D is GGS-w.u.s.c., must X be an Asplund space? We give necessary and sufficient conditions for the space X to be an Asplund space when the duality mapping D of X is GGS-w.u.s.c. and show that if the duality mapping D is w.u.s.c. then X is an Asplund space. In fact, we give several consequences of w.u.s.c. of D that imply X is Asplund.

For a subset A in X^*, an element x^* in A is called a *weak* strongly exposed* (respectively, *weak* denting*) *point* of A if A is strongly exposed at x^* by some element in X (respectively, for any $\varepsilon > 0$, there exists a slice of A determined by some element in X which contains x^* and has diameter less than ε). Also x^* is called a *weak*-weak point of continuity* of A if the identity mapping $Id : (A, w^*) \longrightarrow (A, w)$ is continuous at x^*. For a set A in X, let $[A]$ (respectively, $\overline{co}A$) denote the closed linear subspace in X spanned by A (respectively, the closed convex hull of A). If A is in X^*, $\overline{co}^*A$ denotes the closed convex hull of A under the weak* topology in X^*.

LEMMA 5. *If X is a separable Banach space such that the duality mapping D of X is GGS-weak upper semi-continuous then for any x in S_X, a weak*-weak point of continuity of $D(x)$ is a weak*-weak point of continuity of B_{X^*}.*

PROOF: Suppose $x^* \in D(x)$ and x^* is not a weak*-weak point of continuity of B_{X^*}. Since X is separable, there exist a sequence $\{x_n\}$ in X, x^{**} in X^{**} and $\varepsilon > 0$ such that $w^* - \lim x_n^* = x^*$ and $|\, x^{**}(x^* - x_n^*)\,| > 2\varepsilon$, $n \in \mathbb{N}$. Let $\{x_n\}$ be a dense sequence in $(S_X, \|\cdot\|)$. Since $w^* - \lim x_n^* = x^*$, choosing a subsequence if necessary, we may assume that $|\,(x^* - x_n^*)(x_m)\,| < 1/n$ for all $n \geqslant m$, $n, m \in \mathbb{N}$. Let $U_n = \{x^* : x^* \in X^*, |\, x^*(x_m)\,| < 1/n,\ m = 1, 2, \cdots, n \text{ and } |\, x^{**}(x^*)\,| < \varepsilon\}$, $n \in \mathbb{N}$. Since D is GGS-w.u.s.c. at x, by [**1**, Theorem 2.1], $D(x) + U_n$ contains a slice of B_{X^*} determined by x. Since $x^* \in D(x)$ and $w^* - \lim x_n^* = x^*$, there is a subsequence, say $\{y_n^*\}$, of $\{x_n^*\}$ such that $y_n^* \in D(x) + U_n$, $n \in \mathbb{N}$. Let $z_n^* \in D(x)$ and $z_n^* - y_n^* \in U_n$, $n \in \mathbb{N}$. It follows that $|\,(z_n^* - x^*)(y_m)\,| < 2/n$ for all $n \geqslant m$, $n, m \in \mathbb{N}$ and $|\, x^{**}(x^* - z_n^*)\,| > \varepsilon$, $n \in \mathbb{N}$. Hence x^* is not a weak*-weak point of continuity of $D(x)$. ∎

LEMMA 6. *For any Banach space X, the following are equivalent:*

(1) *For any subspace Y of X, $B_{Y^*} = \overline{co}\{$weak* strongly exposed points of $B_{Y^*}\}$.*
(2) *For any subspace Y of X, $B_{Y^*} = \overline{co}\{$weak* denting points of $B_{Y^*}\}$.*
(3) *For any subspace Y of X, $B_{Y^*} = \overline{co}\{$weak*-weak points of continuity of $B_{Y^*}\}$.*
(4) *For any separable subspace Y in X, $B_{Y^*} = \overline{co}\{$weak* strongly exposed points of $B_{Y^*}\}$.*
(5) *For any separable subspace Y in X, $B_{Y^*} = \overline{co}\{$weak* denting points of $B_{Y^*}\}$.*

(6) *For any separable subspace Y in X, $B_{Y^*} = \overline{co}\{$weak*-weak points of continuity of $B_{Y^*}\}$.*

Furthermore, each of them is a sufficient condition for X to be an Asplund space.

PROOF: It suffices to show that (6) $\Longrightarrow$ (1). We first show that (6) implies that X is Asplund. Let E be a separable subspace of X. Then there exists a separable norming subspace Φ of E in E^*. Since Φ is norming, $\overline{B}^*_{\Phi} = B_{E^*}$. Hence B_{Φ} contains all weak* to weak points of continuity of B_{E^*}. By (6), we have $B_{\Phi} = B_{E^*}$ and so E^* is separable. This implies that X is Asplund.

Let Y be any subspace of X, $A = \{y : \|y\| = 1, \|\cdot\|$ is Frechét differentiable at $y\}$ and $F = \{y^* : y^*$ is a weak* strongly exposed point of $B_{Y^*}\}$. Then $\overline{A} = S_Y$.

Suppose (1) is false. Then there exist $\varepsilon > 0$ and $y^* \in B_{Y^*}$ such that $d(y^*, coF) > \varepsilon$. Let A_1 be any countable subset of A. Then the set $\bigcup_{x \in A_1} D(x)$ is countable. Hence there exists a countable subset A_2 in A, $A_1 \subset A_2$, $S_{[A_1]} \subset \overline{A}_2$ and $\sup_{y \in A_2} (y^* - z^*)(y) > \varepsilon$ for all $z^* \in co\left(\bigcup_{x \in A_1} D(x)\right)$. Continue by induction; there is a sequence $\{A_n\}$ in A, $A_n \subset A_{n+1}$, $S_{[A_n]} \subset \overline{A}_{n+1}$ and $\sup_{y \in A_n} (y^* - z^*)(y) > \varepsilon$ for all $z^* \in co\left(\bigcup_{x \in A_n} D(x)\right)$. Let $Y_0 = [\bigcup_n A_n]$. Then Y_0 is separable; $S_{Y_0} = \overline{\bigcup_n A_n}$. Let D_0 be the duality mapping of Y_0. Then $B_{Y_0^*} = \overline{co}^*\{D_0(x) : x \in \bigcup_n A_n\}$. By (6), $B_{Y_0^*} = \overline{co}\{$weak*-weak points of continuity of $B_{Y_0^*}\}$. Hence $B_{Y^*} = \overline{co}\{D_0(x) : x \in \bigcup_n A_n\}$. However, $y^* |_{Y_0} \in B_{Y_0^*}$ and $\|y^* |_{Y_0} - z^*\|_{Y_0} > \varepsilon$ for all z^* in $co\{D_0(x) : x \in \bigcup_n A_n\}$ which is impossible. ∎

A Banach space X is called nicely smooth [2] if for all x^{**} in X^{**},

$$\bigcap_{x \in X} B_{X^{**}}(x, \|x^{**} - x\|) = \{x^{**}\}$$

where $B_{X^{**}}(x, r)$ is the closed ball in X^{**} with centre x and radius r. Equivalently [5, Lemma 2.4], X is nicely smooth if and only if X^* contains no proper closed norming subspace of X.

LEMMA 7. *Let X be a Banach space. Then the following are equivalent.*

(1) *Every subspace of X is nicely smooth.*

(2) *Every almost monotone basic sequence in X is shrinking.*

PROOF: (1) $\Longrightarrow$ (2). Let $\{x_n\}$ be an almost monotone basic sequence in X, and let $Y = [x_n]$. If $\{x_n^*\}$ is the coefficient functional of $\{x_n\}$ in Y^*, since $\{x_n\}$ is almost monotone, $[x_n^*]$ is a norming subspace of Y. Hence $[x_n^*] = Y^*$, that is, $\{x_n\}$ is shrinking.

(2) $\Longrightarrow$ (1). Suppose not. Without loss of generality, we assume that there exists a proper closed norming subspace F of X in X^*. Choose $x_0^* \in S_{X^*}$ and $x_0^{**} \in S_{X^{**}}$ such that $x_0^{**}(x_0^*) > 1/2$ and $x_0^{**}(F) = 0$. Let $0 < \varepsilon_n < 1$ and suppose $\prod_n (1-\varepsilon_n)$ converges. By the principle of local reflexivity, there exists $T_1 : [x_0^{**}] \longrightarrow X$, $\|T_1\| < 2$ and $x_0^*(T_1 x_0^{**}) = x_0^{**}(x_0^*)$. Let $x_1 = T_1(x_0^{**})$ and let F_1 be a finite subset of S_F such that F_1 $(1-\varepsilon_1)$-norms $[x_1]$. By the principle of local reflexivity again, there exists $T_2 : [x_0^{**}] \longrightarrow X$, $\|T_2\| < 2$ and $x_0^*(T_2 x_0^{**}) = x_0^{**}(x_0^*)$ for all x^* in $F_1 \cup \{x_0^*\}$. Let $x_2 = T_2 x_0^{**}$. Continue by induction; for each $n \in \mathbb{N}$, there exist $x_n \in X$, a finite subset F_n in S_F, F_n $(1-\varepsilon_n)$-norming set of $[x_1, \ldots, x_n]$ and $x^*(x_{n+1}) = x_0^{**}(x^*)$ for all $x^* \in F_n \cup \{x_0^*\}$. It follows that $\|x_n\| \leqslant 2$, $x_0^*(x_{n+1}) = x_0^{**}(x_0^*) > 1/2$ and $x_{n+1}(x^*) = x_0^{**}(x^*) = 0$ for all $x^* \in F_n$, $n \in \mathbb{N}$. Clearly $\{x_n\}$ is not shrinking. It remains to show that $\{x_n\}$ is an almost monotone basic sequence.

For any $x \in [x_1, \ldots, x_n]$, choose $x^* \in F_n$, $x^*(x) \geqslant (1-\varepsilon_n)\|x\|$. For any $\lambda \in \mathbb{R}$, $\|x + \lambda x_{n+1}\| \geqslant x^*(x + \lambda x_{n+1}) \geqslant (1-\varepsilon_n)\|x\|$. Since $\prod_n (1-\varepsilon_n)$ converges, it follows that $\{x_n\}$ is a basic sequence and if $\{P_n\}$ is the sequence of associated projections of $\{x_n\}$, then $\|P_n\| \leqslant 1/\left(\prod_{k\geqslant n}(1-\varepsilon_k)\right) \longrightarrow 1$. Thus $\{x_n\}$ is an almost monotone basic sequence. ∎

Let K be a bounded subset of X^*. A subset B of K is called a *boundary* [3] of K if for every x in X, there exists x^* in B such that $x^*(x) = \sup\{y^*(x) : y^* \in K\}$. Observe that if B is a boundary of K then B is also a boundary of $\overline{co}^* K$. We need the following fundamental fact.

THEOREM 8. [3, Theorem I.2]. *Let B be a boundary of a bounded closed convex set K in X^*. Suppose for any bounded convex set C in X and for any x^{**} in X^{**} which is in the closure of C for the topology σ_B of pointwise convergence on B, there exists a sequence $\{x_n\}$ in C such that $\sigma_B - \lim x_n = x^{**}$. Then K is weak* compact and $K = \overline{co}B$. In particular if B is a separable bounded set in X^* such that B is a boundary of itself, then $\overline{co}^* B = \overline{co} B$ and so $\overline{co}^* B$ is separable.*

Let us remark that Theorem 8 implies a result of Haydon [7]: If K is a weak* compact convex set in X^* such that the set of extreme points of K is norm separable, then K is separable in the norm topology.

THEOREM 9. *Let X be a Banach space such that the duality mapping D of X is GGS-weak upper semi-continuous. Then the following are equivalent:*

(1) *X is Asplund.*

(2) *For all x in S_X, $D(x)$ has the Radon-Nikodym property.*
(3) *For all separable subspace Y in X, $B_{Y^*} = \overline{co}\{$weak*-weak point of continuity of $B_{Y^*}\}$.*
(4) *Every subspace of X is nicely smooth.*

PROOF: (1) implies (2) is well-known.

$(2) \Longrightarrow (3)$. Let Y be a separable subspace of X and let D_0 be the duality mapping of Y. Since D is GGS-w.u.s.c. on X, D_0 is GGS-w.u.s.c. on Y. By (2) and Lemma 5, the set $w^* - wpcB_{Y^*}$ consisting of weak*-weak point of continuity of B_{Y^*} is non-empty and is a boundary of B_{Y^*}. Since Y is separable, $w^* - wpcB_{Y^*}$ satisfies the hypothesis of Theorem 8, and hence $\overline{co}(w^* - wpcB_{Y^*}) = B_{Y^*}$.

$(3) \Longrightarrow (4)$. Let Y be any subspace of X. By (3) and Lemma 6, $B_{Y^*} = \overline{co}\{w^*$-strongly exposed points of $B_{Y^*}\}$. Hence the set of weak* strongly exposed points of B_{Y^*} separates the point of X^{**}. It follows [**2**, Lemma 5] that Y is nicely smooth.

$(4) \Longrightarrow (1)$. Let Y be a separable subspace of X. Since Y is nicely smooth, Y^* contains no proper closed norming subspace of Y. Hence Y^* is separable and so X is Asplund. ∎

REMARK. The fact that the dual space of a separable nicely smooth space is separable has been proved in [**2**, Lemma 10]. The question of whether every Asplund space admits an equivalent nicely smooth norm has been raised in [**4**, Question E] and is still open.

THEOREM 10. *Let X be a Banach space and let D be the duality mapping of X. Consider the following statements.*

(1) *D is weakly upper semi-continuous.*
(2) *For any symmetric closed convex set F in X^*, the set $\{x : x \in S_X, D(x) \cap F \neq \phi\}$ is norm closed.*
(3) *For any separable subspace Y in X and for any dense sequence $\{y_n\}$ in S_Y, then for any $x_n^* \in D(y_n), n \in \mathbb{N}, B_{Y^*} = \overline{co}\{\pm x_n^* |_Y : n \in \mathbb{N}\}$.*
(4) *For any separable subspace Y in X, $B_{Y^*} = \overline{co}\{$weak* strongly exposed points of $B_{Y^*}\}$.*
(5) *Every subspace of X is nicely smooth.*
(6) *X is Asplund.*

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (5) \Longrightarrow (6)$.

PROOF: By definition of w.u.s.c. mapping, it is clear that $(1) \Longrightarrow (2)$.

$(2) \Longrightarrow (3)$. Let Y be a separable subspace of X and let D_0 be the duality mapping of Y. By (2), it is obvious that for any symmetric closed convex subset F in Y^*, $\{y : y \in S_Y, D_0(y) \cap F \neq \phi\}$ is norm closed in Y. Let $\{y_n\}$ be a dense sequence in S_Y and let $y_n^* \in D_0(y_n)$, $n \in \mathbb{N}$. Then $\{y_n^*\}$ is a norming set of Y and so $\overline{co}^*\{y_n^*\} = B_{Y^*}$. Let $F = \overline{co}\{\pm y_n^*\}$. Then F is a symmetric closed convex set in

Y^*. Hence the set $A \equiv \{y : y \in S_Y, D_0(y) \cap F \neq \phi\}$ is norm closed. Since $y_n \in A$, $n \in \mathbb{N}$, we conclude that $A = S_Y$. Thus for all y in S_Y, there exists $y^* \in F$ such that $y^*(y) = 1 = \sup_{z^* \in F} z^*(y)$. By Theorem 8, $F = \overline{F}^* = B_{Y^*}$.

$(3) \Longrightarrow (4)$. By (3), every separable subspace of X has a separable dual. Hence X is Asplund. Let Y be a separable subspace of X and let $\{y_n\}$ be a dense sequence of S_Y such that the norm is Frechét differentiable at y_n, $n \in \mathbb{N}$. Then $D_0(y_n)$ is a weak* strongly exposed point of B_{Y^*} and by (3), $B_{Y^*} = \overline{co}\{$weak* strongly exposed points of $B_{Y^*}\}$.

$(4) \Longrightarrow (5)$. Let Y be a subspace of X. If F is a close norming subspace of Y in Y^*, then $\overline{B}^*_F = B_{Y^*}$. Since $B_{Y^*} = \overline{co}\{$weak*-weak points of continuity of $B_{Y^*}\}$, $\overline{B}^*_F$ contains all weak* to weak points of continuity of B_{Y^*}. It follows that $B_F = \overline{B}^*_F = B_{Y^*}$. Hence Y^* contains no proper closed norming subspace of Y and so Y is nicely smooth.

$(5) \Longrightarrow (6)$. Let Y be a separable subspace of X. Since every separable space has a separable norming subspace in its dual, we conclude that Y^* is separable and so Y is Asplund. Thus X is Asplund. ∎

We conclude this section with the following characterisations of reflexive Banach spaces.

Theorem 11. *The following are equivalent for a Banach space X.*

(1) *X is reflexive*
(2) *For any equivalent norm $\|\cdot\|$ on X, $(X, \|\cdot\|)$ is Hahn-Banach smooth and $(X, \|\cdot\|)$ has the ANP-III.*
(3) *For any equivalent norm $\|\cdot\|$ on X, the duality mapping of $(X, \|\cdot\|)$ is w.u.s.c.*
(4) *For any equivalent norm $\|\cdot\|$ on X, every almost monotone basic sequence in $(X, \|\cdot\|)$ is shrinking.*
(5) *X admits an equivalent norm $\|\cdot\|$ such that $(X, \|\cdot\|)$ has the ANP-I and both $(X, \|\cdot\|)$ and $(X^*, \|\cdot\|)$ are locally uniformly rotund.*
(6) *X admits an equivalent norm $\|\cdot\|$ such that $(X, \|\cdot\|)$ has the ANP-III and the duality mapping of $(X, \|\cdot\|)$ is w.u.s.c.*

Proof: By the definition of ANP-III, it is clear that every reflexive space has the ANP-III and by Theorem 10, we conclude that $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4)$.

$(4) \Longrightarrow (1)$. Let $\{x_n\}$ be a basic sequence in X. Then there is an equivalent norm $\|\cdot\|$ on X such that $\{x_n\}$ is almost monotone in $(X, \|\cdot\|)$. Hence $\{x_n\}$ is a shrinking sequence. By the well-known result of Zippin [14], X is reflexive.

$(1) \Longrightarrow (5)$. Since every reflexive space admits an equivalent norm $\|\cdot\|$ such that $(X, \|\cdot\|)$ and $(X^*, \|\cdot\|)$ are locally uniformly rotund and every reflexive space has the

ANP-III, it follows that $(X, \|\cdot\|)$ has the ANP-I [8].

$(5) \Longrightarrow (6)$. Obvious.

$(6) \Longrightarrow (1)$. Without loss of generality, assume that X is separable. Let $\|\cdot\|$ be an equivalent norm of X such that $(X, \|\cdot\|)$ has the Φ-ANP-III for some norming set Φ. We may assume that Φ is closed convex. It follows that $\overline{\Phi}^* = B_{X^*}$. Since the duality mapping D of $(X, \|\cdot\|)$ is w.u.s.c., by Theorem 10, $B_{X^*} = \overline{co}\{\text{weak}^* \text{ strongly exposed points of } B_{X^*}\}$ and so $\Phi = B_{X^*}$. Hence $(X, \|\cdot\|)$ is B_{X^*}-ANP-III. By [8, Theorem 2.3], we conclude that $X = \{x^{**} : x^{**} \in X^{**}, \|x^{**}\| = \sup_{x^* \in B_{X^*}} x^{**}(x^*)\} = X^{**}$, that is, X is reflexive. □

5.

In Theorem 4, we have proved that if $(X^*, \|\cdot\|)$ has the weak* ANP-II, then $(X, \|\cdot\|)$ is quasi-Frechét differentiable and so the duality mapping D of $(X, \|\cdot\|)$ is u.s.c. and $D(x)$ is compact for all x in S_X. In the case that $D(x)$ is compact for all x in S_X, D is u.s.c. if and only if D is GGS-u.s.c. The next theorem, using the Hahn-Banach theorem only, extends [1, Theorem 2.1]. The conditions (3) – (7) show that in the case that D is GGS-u.s.c. at x, then $D(x)$ behaves like a "weak* strongly exposed set" of B_{X^*}.

Theorem 12. *Let X be a Banach space and let $x \in S_X$. Then the following are equivalent:*

(1) *The duality mapping D of X is GGS-u.s.c. at x.*
(2) *For any $\varepsilon > 0$, $D(x) + \varepsilon B_{X^*}$ contains a weak* slice of B_{X^*} determined by x.*
(3) *For any net $\{x_\alpha^*\}$ in B_{X^*}, if $x_\alpha^*(x) \longrightarrow 1$ then $d(x_\alpha^*, D(x)) \longrightarrow 0$ where $d(x_\alpha^*, D(x))$ is the distance from x_α^* to $D(x)$.*
(4) *For any sequence $\{x_n^*\}$ in B_{X^*}, if $x_n^*(x) \longrightarrow 1$, then $d(x_n^*, D(x)) \longrightarrow 0$.*
(5) *For any sequence $\{x_n\}$ in S_X, if $\lim_n \|x_n - x\| = 0$ for some x, then $d(x_n^*, D(x)) \longrightarrow 0$ for any $x_n^* \in D(x_n)$, $n \in \mathbb{N}$.*
(6) *For any sequence $\{x_n\}$ in S_X, if $\lim_n \|x_n - x\| = 0$ for some x then $d(D(x_n), D(x)) \longrightarrow 0$.*
(7) $\lim_{t\to 0} \sup_{\|y\|=1} \inf\{| 1/t(\|x+ty\| - \|x\|) - x^*(y) | : x^* \in D(x)\} = 0$.

Proof: $(2) \Longrightarrow (3) \Longrightarrow (4) \Longrightarrow (1) \Longrightarrow (5) \Longrightarrow (6)$ are obvious.

$(6) \Longrightarrow (7)$. Let $t \in (0,1)$, $y \in S_X$, $x^* \in D(x)$ and $y^* \in D(x + ty/\|x+ty\|)$. Then

$$x^*(y) = \big[x^*(x+ty) - x^*(x)\big]/t \leqslant \big(\|x+ty\| - \|x\|\big)/t$$
$$\leqslant \big[y^*(x+ty) - y^*(x)\big]/t = y^*(y).$$

Hence $|\ (\|x+ty\|-\|x\|)/2 - x^*(y)\ | \leqslant \|x^* - y^*\|$ for all $y \in S_X$. Thus $\inf\{|\ (\|x+ty\|-\|x\|)/t - x^*(y)\ |: x^* \in D(x)\} \leqslant d(D(x), D(x+ty)/\|x+ty\|)$. Therefore

$$\begin{aligned}\lim_{t\to 0}\sup_{\|y\|=1}\inf\Big\{|\ (\|x+ty\|-\|x\|)/t - x^*(y)\ |: x^* \in D(x)\Big\}\\ \leqslant \lim_{t\to 0}\sup_{\|y\|=1} d\Big(D(x), D\left((x+ty)/\|x+ty\|\right)\Big) = 0.\end{aligned}$$

(7) $\Longrightarrow$ (2). Assume (2) is false. Then there exist $\varepsilon > 0$, and $x_n^* \in B_{X^*}$, with $\lim_n x_n^*(x) = 1$ and $d(x_n^*, D(x)) > \varepsilon$. By the Hahn-Banach theorem, there exists $x_n \in S_X$ for each $n \in \mathbb{N}$ such that $(x_n^* - x^*)(x_n) > \varepsilon$ for all $x^* \in D(x)$. By (7), choose $\delta > 0$ such that for all $0 < |\ t\ | \leqslant \delta$,

$$\sup_{\|y\|=1}\inf\Big\{|\ \frac{1}{t}\Big(\|x+ty\|-\|x\|\Big) - x^*(y)\ |: x^* \in D(x)\Big\} < \frac{\varepsilon}{2}.$$

Let $y_n = \delta x_n, n \in \mathbb{N}$. Then for any $x^* \in D(x)$,

$$\begin{aligned}\delta\varepsilon < (x_n^* - x^*)(y_n) &= \left[x_n^*(x+y_n) - x^*(x) - x^*(y_n)\right]\\ &\quad - x_n^*(x) + x^*x \leqslant \Big(\|x+y_n\| - \|x\| - x^*(y_n)\Big) - x_n^*(x) + 1.\end{aligned}$$

Thus

$$\begin{aligned}\delta\varepsilon &\leqslant \inf\Big\{\|x+y_n\| - \|x\| - x^*(y_n) : x^* \in D(x)\Big\} - x_n^*(x) + 1\\ &\leqslant \delta\Big(\frac{\varepsilon}{2}\Big) - x_n^*(x) + 1 \longrightarrow \delta\Big(\frac{\varepsilon}{2}\Big)\end{aligned}$$

which is a contradiction. ∎

Remark. (1) $\Longleftrightarrow$ (2) were proved in [**1**].

Corollary 13. *Let X be a Banach space, $x \in S_X$, and let D be the duality mapping of X. Then the following are equivalent:*

(1) *D is u.s.c. at x and $D(x)$ is compact.*
(2) *For any $\varepsilon > 0$, $D(x) + \varepsilon B_{X^*}$ contains a weak* slice of B_{X^*} determined by x and $D(x)$ is compact.*
(3) *For any net $\{x_\alpha^*\}$ in B_{X^*}, if $x_\alpha^*(x) \longrightarrow 1$ then $\{x_\alpha^*\}$ has a norm convergent subnet.*
(4) *For any sequence $\{x_n^*\}$ in B_{X^*}, if $x_n^* \longrightarrow 1$, then $\{x_n^*\}$ has a norm convergent subsequence.*
(5) *If $\{x_n\}$ is a sequence in S_X such that $\lim_n \|x_n - x\| = 0$, then for any $x_n^* \in D(x_n)$, $\{x_n^*\}$ has a norm convergent subsequence.*

(6) *$D(x)$ is compact and for any sequence $\{x_n\}$ in S_X, if $\lim_n \|x_n - x\| = 0$ then $d(D(x_n), D(x)) \longrightarrow 0$.*

(7) *$D(x)$ is compact and* $\lim_{t\to 0} \sup_{\|y\|=1} \inf\Big\{ \big| 1/t\big(\|x+ty\| - \|x\|\big) - x^*(y) \big| : x^* \in D(x)\Big\} = 0$.

REMARK. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) are proved in [**1**, Theorem 3.2].

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