QUASIPRIMITIVITY AND QUASIGROUPS

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It is well known that $Q$ is a simple quasigroup if and only if $\text{Mlt } Q$ acts primitively on $Q$. Here we show that $Q$ is a simple quasigroup if and only if $\text{Mlt } Q$ acts quasiprimitively on $Q$, and that $Q$ is a simple right quasigroup if and only if $\text{RMlt } Q$ acts quasiprimitively on $Q$.

A quasigroup is set with a single binary operation, denoted by juxtaposition, such that in $xy = z$, knowledge of any two of $x, y$ and $z$ specifies the third uniquely. A right quasigroup is a set with a single binary operation whose right translations biject.

The multiplication group, $\text{Mlt } Q$, of a quasigroup $Q$ is the subgroup of the group of all bijections on $Q$ generated by right and left translations, that is $\text{Mlt } Q := \langle R(x), L(x) : x \in Q \rangle$, where $R(x)$ (respectively, $L(x)$) is right (respectively, left) translation by $x$. The right multiplication group, $\text{RMlt } Q$, of a right quasigroup $Q$ is the subgroup of the group of all bijections on $Q$ generated by right translations, that is $\text{RMlt } Q := \langle R(x) : x \in Q \rangle$. A quasigroup $Q$ is called type 1 if $\text{RMlt } Q = \text{Mlt } Q$. For example, commutative quasigroups are type 1; so too are finite simple Moufang loops [4].

A permutation group $G$ on a set $Q$ acts primitively on $Q$ if the only $G$-invariant partitions of $Q$ are the two trivial partitions $\{Q\}$ and $\{\{x\} : x \in Q\}$. Of course, if $G$ acts primitively on $Q$ then each nontrivial normal subgroup of $G$ is transitive on $Q$. In [5], Praeger used this fact to generalise the notion of primitivity: a permutation group $G$ on a set $Q$ acts quasiprimitively on $Q$ if each non-trivial normal subgroup of $G$ is transitive on $Q$. This definition is useful because, as Praeger proved [5], there is an O'Nan-Scott type theorem classifying all quasiprimitive permutation groups as one of eight types.

Given a quasigroup $Q$, there exist two binary operations $/\,$, $\setminus$ on $Q$ such that $(xy)/y = x$, $(x/y)y = x$, $x\setminus(xy) = y$, and $x(x\setminus y) = y$. Conversely, an algebra with three binary operations satisfying these four identities is a quasigroup — as defined at the beginning of this paper — under any one of these operations [2]. Similarly, right quasigroups are axiomatised by the first two of the four quasigroup identities above. A
congruence on a (right) quasigroup $Q$ is an equivalence relation $V$ on $Q$ such that if $x_1 V y_1$ and $x_2 V y_2$, then $x_1 x_2 f V y_1 y_2 f$, where $f$ is any one of the (two) three binary operations on $Q$. $Q$ is simple if its only congruences are the trivial congruence and the improper congruence. We record the following well known fact [2]:

**Proposition 1.** A quasigroup $Q$ is simple if and only if $\text{Mlt} \, Q$ acts primitively on $Q$.

**Theorem 2.** A quasigroup $Q$ is simple if and only if $\text{Mlt} \, Q$ acts quasiprimitively on $Q$.

**Proof:** ($\implies$) Quasiprimitivity is a generalisation of primivity.

($\impliedby$) Let $V$ be a nontrivial congruence on $Q$. There is a corresponding normal subgroup $\text{Nor}(V) := \{g \in \text{Mlt} \, Q : \forall q \in Q, (q, qg) \in V\}$ of $\text{Mlt} \, Q$. Now pick $x \in Q$. The subset $x \text{Nor}(V)$ of $Q$ is contained in the congruence class of $x$. But since $\text{Mlt} \, Q$ acts quasiprimitively on $Q$, $x \text{Nor}(V) = Q$. Thus $V$ is improper and $Q$ is simple. □

An important research program from the theory of quasigroups is to determine which permutation group actions are the actions of a multiplication group of a quasigroup, and which permutation groups are not [2, 4]. The following corollary to Theorem 2 advances this program.

**Corollary 3.** An imprimitive, quasiprimitive action is not a multiplication group action.

For examples of imprimitive, quasiprimitive actions see [5]. While every symmetric group, under its natural action, can be realised as a multiplication group of some quasigroup, [3, Proposition 4.2] shows that the actions of symmetric groups of odd prime power degree bigger than three on unordered pairs cannot be multiplication group actions. Corollary 3 above expands on this theme: it is the first example of a general class of permutation actions — that is, the quasiprimitive, imprimitive actions — that cannot be multiplication group actions. The following result gives a unilateral version of Theorem 2.

**Theorem 4.** $Q$ is a simple right quasigroup if and only if $\text{RMlt} \, Q$ acts quasiprimitively on $Q$.

**Proof:** ($\implies$) Let $N$ be a non-trivial normal subgroup of $\text{RMlt} \, Q$. The sets $xN, x \in Q$, partition $Q$. Let $Q/N$ denote the set of equivalence classes. Define a binary operation on $Q/N$ by $(xN)(yN) = (xyN)$. It is easy to check that this operation is well defined and that under this operation $Q/N$ is a right quasigroup [1]. It is also clearly a proper homomorphic image of $Q$. But $Q$ is simple, so this image must be trivial. That is, there is only one equivalence class, and $N$ is a transitive on $Q$.

($\impliedby$) Given a right quasigroup epimorphism $f : Q \to M$, there is a group epimorphism $F : \text{RMlt} \, Q \to \text{RMlt} \, M$; $R(x) \to R(xf)$. Pick $x \in Q$. The subset $x \text{Ker} \, F$...
is in the congruence (ker \( f \)) class of \( x \). But since \( R\text{Mlt}Q \) acts quasiprimatively on \( Q, x \text{Ker } F = Q \). Hence, ker \( f \) is improper, and \( Q \) is simple. □

**Corollary 5.** If \( Q \) is a type 1 simple quasigroup, then \( Q \) is a simple right quasigroup.

**References**


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