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THE ALGEBRA OF STABLE OPERATIONS FOR p-LOCAL COMPLEX K-THEORY

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ABSTRACT. The multiplicative structure of the algebra of stable operations for p-local complex K-theory is studied, and the units and zero divisors are identified.

This paper is devoted to studying the structure of the algebra of stable degree 0 operations of complex K-theory localized at a prime p. This algebra is large, in fact uncountable, as was shown in [2], and its additive structure is that of the dual of a free $Z_{(p)}$ module. We will be concerned with its multiplicative structure.

Our results are:

THEOREM 1.

(i) *The stable degree 0 operation* α *is a unit iff the homomorphism*

$$\alpha_*:\pi_*(K) \to \pi_*(K)$$

is an isomorphism.

(ii) *The only zero divisors in the algebra of stable operations of degree 0 are those arising from the Adams splitting of K if p is odd or from the relation*

$$(\Psi^1 + \Psi^{-1})(\Psi^1 - \Psi^{-1}) = 0 \text{ if } p = 2.$$

Here Ψ^i denotes the i-th Adams operation. It follows from the second part of this theorem that in the corresponding algebra for unlocalized K-theory the only zero divisors are those described above for the case $p = 2$, a result due to Geoff Mess [5].

COROLLARY 2. *For p an odd prime the algebra of stable operations of degree 0 of one of the Adams summands of K is an integral domain and a local ring with residue field Z/pZ.*

The proof of this theorem is contained in section 2. Section 1 is devoted to recalling some known results about complex K-theory which are required in the proof.

§1. Let us denote by K the spectrum representing complex K-theory localized at a prime p. The main result in [2] was established by showing that K^0K is the Hopf algebra

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dual to K_0K (i.e. $Hom(K_0K, Z_{(p)})$). This later Hopf algebra had been investigated previously in [1]. There, the unlocalized analog of K_0K was described as the Hopf algebra of Laurent polynomials in one variable over Q satisfying a certain integrality condition. Since that description is the starting point for our results we begin by recalling its p-local version and fixing some notation.

THEOREM 3. *The natural map*

$$K_0K \to K_0K \otimes Q \cong Q[w, w^{-1}]$$

is injective, and its image consists of the set of Laurent polynomials $f(w)$ satisfying the condition that for any integer k prime to p, $f(k) \in Z_{(p)}$

NOTATION 4. *Let us denote by C this Hopf algebra of Laurent polynomials and by B the intersection of C with $Q[w]$ (the coproduct for these Hopf algebras is determined by the formula $\psi(w) = w \otimes w$). Also, let us denote by A the Hopf algebra consisting of those polynomials $f(w) \in Q[w]$ with the property that $f(k) \in Z_{(p)}$ for all $k \in Z$.*

We may, therefore, represent a stable degree 0 cohomology operation by a homomorphism $\alpha \in Hom(C, Z_{(p)}) = C^*$. In [4], theorem 1, it was shown that for such an α the numbers $\alpha(w^n)$, $n \in Z$, (which correspond to the action of α on $\pi_{2n}(K)$) satisfy certain congruences. Furthermore a set of congruences was constructed which is complete in the sense that a squence $\{\lambda_n | n \in Z\}$ satisfying these congruences equals $\{\alpha(w^n) | n \in Z\}$ for a unique operation α. This allows us to construct stable operations by specifying the numbers λ_n. We will exploit this to identify the units in K^0K.

To investigate the zero divisors in K^0K we will also need to know how the Adams splitting of K is reflected in the algebraic description of K_0K above.

NOTATION 5. *Let us denote $C \cap Q[w^{p-1}, w^{-(p-1)}]$ by C_0 and $B \cap Q[w^{p-1}]$* by B_0.

PROPOSITION 6. ([3], Lemma 1.1).

(i) $B \cong \bigoplus_{j=1}^{p-2} w^j \cdot B_0$

(ii) $C \cong \bigoplus_{j=1}^{p-2} w^j \cdot C_0$

§**2**. The first part of the theorem 1 will follow from:

THEOREM 7. $\alpha \in C^*$ *is a unit iff* $\alpha(w^n) \in Z^*_{(p)}$ *for all n.*

PROOF. We first note that the product in C^* is given by $(\alpha_1 \cdot \alpha_2)(w^n) = \alpha_1(w^n) \cdot \alpha_2(w^n)$. Thus we will be done if we can show that the numbers

$$\{1/\alpha(w^n) | n = \dots, -2, -1, 0, 1, 2, \dots\}$$

satisfy the appropriate congruences of theorem 1 of [4].

Since $\alpha(1) \in Z^*_{(p)}$ we may assume, by dividing if necessary, that $\alpha(1) = 1$. Also, by replacing α by α^{p-1} if necessary, we may assume that $\alpha(w^n) \equiv 1 \pmod p$ for all n. Thus we may write $\alpha = 1 + \bar{\alpha}$ with $\bar{\alpha}(w^n) \equiv 0 \pmod p$.

Let us define $\beta_n = \Sigma_{i=0}^{n} (-1)^i \bar{\alpha}^i$. Then $\beta_n \alpha = 1 \pm \bar{\alpha}^{n+1}$ and so we have:

$$\beta_n \alpha(w^n) \equiv 1 \pmod{p^{n+1}}$$

Since β_n is a well defined element of C^* the numbers $\lambda_m = \beta_n(w^m)$ satisfy the congruences of theorem 1 of [4], and also $\lambda_m \equiv 1/\alpha(w^m) \pmod{p^{m+1}}$. Choosing $n > \gamma_p(k!)$ we see that the numbers $1/\alpha(w^m)$ satisfy the first k of these congruences. Since we are free to choose n arbitrarily large, the result follows.

Based on this theorem we see that C_0^* is a local ring. Indeed, writing $x = w^{p-1}$, the set of non-units consists of those α for which $\alpha(x^m)$ is p-divisible in $Z_{(p)}$ for some m, and so for all m since $(x - 1)/p \in C_0$. This clearly forms an ideal.

To describe the zero divisors among the stable operations it suffices, by proposition 6, to describe those in B_0^* and C_0^*. This is accomplished by:

THEOREM 8. *If $\alpha, \beta \in B_0^*$ or C_0^* are such that $\alpha \cdot \beta = 0$ then:*
(i) *if p is odd, then either $\alpha = 0$ or $\beta = 0$ or both.*
(ii) *if $p = 2$ then either $\alpha = 0$ or $\beta = 0$ or*

$$2\alpha = \bar{\alpha}(\Psi^1 \pm \Psi^{-1})$$
$$\text{and } 2\beta = \bar{\beta}(\Psi^{-1} \pm \Psi^1)$$

The proof of this theorem will occupy the remainder of the paper and rests on the following proposition concerning the possibilities for the kernel of elements of C_0^* and B_0^*:

PROPOSITION 9. *If $\alpha \in B_0^*$ or C_0^* is such that $\alpha(x^i) = 0$ if $n|i$ then*
(i) *if p is odd or $p = 2$ and n is odd, then $\alpha = 0$.*
(ii) *if p, n are both even, then $\alpha(x^{2i}) = 0$ for all i.*

PROOF. We will establish the result for $\alpha \in B_0^*$ first, and deduce the general result from this. Let $n = p^a b$, $(b, p) = 1$. Since $(x^{p^m} - 1)/p^{m+1} \in B$ for any m, we have, for all i:

$$\alpha(x^i) \equiv \alpha(x^{i+p^m}) \pmod{p^{m+1}}$$

Thus, given i such that $p^a|i$ we may find k, l such that $kn - lp^m = i$, $k, l > 0$ and so have:

$$\begin{aligned} \alpha(x^i) &= \alpha(x^{i+lp^m}) \\ &\equiv \alpha(x^{kn}) \\ &= 0 \pmod{p^{m+1}} \end{aligned}$$

Since this is true for any m, we see that $\alpha(x^i) = 0$ if $p^a|i$ and so we may assume with out loss of generality that $b = 1$. In particular, if n is relatively prime to p we are finished.

REMARK 10. The preceding argument would be sufficient to prove the analogous theorem concerning unlocalized K-theory. We would simply choose p so that $p(p - 1)$ is relatively prime to n.

NOTATION 11. *Let $B_a = B_0 \cap Q[x^{p^a}]$ and $C_a = C_0 \cap Q[x_{\pm}^{p^a}]$.*

It will be sufficient for us to show that for all a

$$Hom(B_0/B_a, Z_{(p)}) = 0$$

if p is odd and

$$Hom(B_1/B_a, Z_{(p)}) = 0$$

if $p = 2$, and the corresponding result for C.

First, let us suppose that p is odd. Since any polynomial in B_0 can be expressed in the form $g((x-1)/p)$ with $g \in A$, the polynomial $(x-1)/p$ plays a distinguished role in our proof. We will begin by showing that it can be approximated modulo p^m in each B_a.

LEMMA 12. *For any positive integer m there exists $f_m(x) \in B_a$ such that for any integer k,*

$$f_m(1 + kp) \equiv k \pmod{p^m}$$

PROOF. We construct the polynomials f_m inductively. To begin, let $f_1(x) = (x^{p^m} - 1)/p^{m+1}$. We then have, for any integer k:

$$f_1(1 + kp) = k + p \cdot g_1(k)$$

where $g_1 \in Z_{(p)}[x]$, and so $g_1(k + lp) \equiv g_1(k) \pmod{p}$.

Suppose now that we have constructed $f_m(x)$ in such a way that for any integer k:

$$f_m(1 + kp) = k + p^m \cdot g_m(k)$$

with $g_m(x) \in Z_{(p)}[x]$. Let us define:

$$f_{m+1}(x) = f_m(x) - p^m \cdot g_m(f_1(x))$$

Certainly $f_{m+1} \in B_a$ and $g_n(f_1(1 + kp)) \equiv g_n(k) \pmod{p}$ so we have:

$$\begin{aligned} f_{m+1}(1 + kp) &= (k + p^m g_m(k)) - p^m g_m(f_1(1 + kp)) \\ &= k \pmod{p^{m+1}} \end{aligned}$$

If we let $g_{m+1}(x) = (g_m(x) - g_m(x + pg_1(x)))/p$ then

$$f_{m+1}(1 + kp) \equiv k + p^{m+1} \cdot g_{m+1}(k)$$

and it is easily checked, using the binomial theorem, that $g_{m+1} \in Z_{(p)}[x]$.

Using this lemma, we will now show that B_0/B_a is p-divisible. This will imply that the first Hom group mentioned above is zero. Suppose that $f \in B_0$, that $g \in A$ is such that $f(x) = g(x - 1/p)$ and that m is an integer large enough that if $k \equiv k' \pmod{p^m}$ then $g(k) \equiv g(k') \pmod{p}$. Consider

$$f(x) - g(f_m(x)) = g(x - 1/p) - g(f_m(x))$$

From the way we chose m, it follows that for any integer k, $f(1 + kp) \equiv g(f_m(1 + kp)) \pmod{p}$ and so that $f(x) - g(f_m(x))$ is divisible by p in B_0. It is also clear that $g(f_m(x)) \in B_a$, since $f_m(x)$ is. Thus B_0/B_a is p-divisible.

For the case $p = 2$ the polynomial $(x - 1)/p$ must be replaced by $(x^2 - 1)/2^3$. This is because:

LEMMA 13. *If* $f(x) \in B_1$ *then* f *can be expressed in the form* $f(x) = g((x^2 - 1)/2^3)$ *with* $g(x) \in A$.

PROOF. Since f is even, we can certainly express it in the form above for some polynomial $g(x)$. The question is whether $g(x) \in A$. If $x = 1 + 2k$, then $(x^2 - 1)/2^3 = k(k + 1)/2$ and so $g(k(k + 1)/2) \in Z(2)$ for any integer k. To see that this implies that $g(x) \in A$, choose n large enough that $2^n g(x) \in Z_{(2)}[x]$. If $k \equiv k' \pmod{2^n}$ and $g(k) \in Z(2)$ then $g(k') \in Z_{(2)}$ and so it will suffice to show that the congruence

$$x(x + 1) \equiv 2k \pmod{2^n}$$

is solvable for any k. Using the quadratic formula we see that this is equivalent to showing that $1 + 8k$ is a quadratic residue $\bmod 2^n$. This is well known.

LEMMA 14. *For any integer m there exists a polynomial* $f_m(x) \in B_a$ *such that for any integer* k,

$$f_m(1 + 2k) \equiv k(k + 1)/2 \pmod{2^m}$$

PROOF. As in the case of p odd, we construct the polynomials $f_m(x)$ inductively, starting with:

$$f_1(x) = (x^{2^a} - 1)/2^{a+2}$$

which we check has the property that

$$f_1(1 + 2k) = k(k + 1)/2 + 2g_1(k(k + 1)/2)$$

with $g_1(x) \in Z_{(2)}[x]$. We then suppose that we have constructed $f_m(x)$ such that

$$f_m(1 + 2k) = k(k + 1)/2 + 2^m g_m(k(k + 1)/2)$$

with $g_m(x) \in Z_{(2)}[x]$ and define

$$f_{m+1}(x) = f_m(x) - 2^m g_m(f_1(x))$$

Certainly $f_{m+1}(x) \in B_a$, and we can check that $f_{m+1}(1 + 2k)$ has the required form in the same way as for odd p.

We may now show that B_1/B_a is infinitely 2-divisible. If $f(x) \in B_1$ with $f(x) = g((x^2 - 1)/2^3)$ as in lemma 13, then $f(x) - g(f_m(x))$ is 2-divisible, if we choose m large enough, and $g(f_m(x)) \in B_a$.

We have now established proposition 9 for $\alpha \in B_0^*$. Suppose next that $\alpha \in C_0^*$ is such that $\alpha(x^i) = 0$ if $n|i$ and that $g(x) \in C_0$ ($\in C_1$ if $p = 2$) is such that $\alpha(g) \neq 0$. If we define $\alpha^-(f) = \alpha(x^{-p^m n} f)$ with m chosen large enough that $x^{p^m n} g(x) \in B_0$, then $\alpha^- \in B_0^*$, $\alpha^-(x^i) = 0$ if $n|i$ and $\alpha^-(g) \neq 0$, a contradiction.

Finally, we return to the proof of theorem 8. First suppose that p is odd, and that $\alpha\beta = 0$. Fix a positive integer m, and suppose that there exists an integer k with $1 < k < p^m - 1$ and $\beta(x^{k+jp^m}) \neq 0$ for all j. We claim that this implies that $\alpha = 0$.

To see this note that we have $\alpha(x^{k+jp^m}) = 0$ for all j. If we define $\alpha' \in B_0$ or C_0 by $\alpha'(x^r) = \alpha(x^{r+k})$ then α' satisfies the hypothesis of proposition 9(i) with $n = p^m$ and so $\alpha' = 0$. In the case $\alpha \in C_0$ this shows that $\alpha(x^r) = 0$ for all integers r, and so that $\alpha = 0$. In the case $\alpha \in B_0$ we have $\alpha(x^r) = 0$ if $r > k$. Since $(x^{p^s} - 1)/p^{s+1} \in B_0$ for any s, we have $\alpha(1) \equiv 0 \pmod{p^{s+1}}$ for any s, and so $\alpha(1) = 0$. An application of proposition 9(i) with $n = 2k$ now shows that $\alpha = 0$ in this case also.

We may, therefore, assume that, given k as above, there exists k' such that $k \equiv k' \pmod{p^m}$ and $\beta(x^{k'}) = 0$. Since $(x^{p^m} - 1)/p^{m+1} \in B_0$, $\beta(x^k) \equiv \beta(x^{k'}) \pmod{p^{m+1}}$ and so $\beta(x^k) \equiv 0 \pmod{p^{m+1}}$. Since this is true for all m, $\beta = 0$.

Next suppose that $p = 2$, that $\alpha\beta = 0$, and that $\alpha \neq 0$. As before choose a positive integer m, and let $k = 0, 1, 2, \ldots, 2^m - 1$. Using proposition 9 we can show that we can find j such that $\beta(x^{k+2^m j}) = 0$

Case 1. for all n, k.

Case 2. for all n, for all even k.

Case 3. for all n, for all odd k.

In case 1 we can show as for p odd that $\beta = 0$. In case 2 we can show that $\beta(x^{2i}) = 0$ and $\alpha(x^{2i+1}) = 0$ for all i. Case 3 is the reverse of case 2.

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