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ON NORMAL COVERS OF LOCALLY COMPACT SPACES

YUKINOBU YAJIMA

Dedicated to Professor Ryosuke Nakagawa on his 60th birthday

In this paper, we deal with the following question: What kind of open covers are normal if they have cushioned open refinements? For this, we prove that an open cover consisting of members with compact closure is a desired one.

1. INTRODUCTION

Originally, normal covers of topological spaces were characterised by many forms (for example, see [**5**, Theorem 1.2]). However, all of these seem to be closely related to local finiteness. So it has raised the following question:

(*) In what kinds of spaces are normal covers characterised by more general properties such as closure-preserving or cushioned ones?

In normal spaces, the author [**7, 8**] characterised normal covers by σ-cushioned-like properties. Unfortunately, these are not exactly σ-cushioned property. In this paper, we shall give an answer to the above question. Moreover, we consider when an open cover has an open star-refinement if it has a cushioned open refinement.

Gruenhage [**2**] showed that a locally compact space X is metacompact if and only if every directed open cover of X has a cushioned refinement. Jiang [**3**] showed that an orthocompact space X is metacompact if and only if every directed open cover of X has a cushioned refinement. These give a good line for our questions.

Throughout this paper, all spaces are assumed to be Hausdorff.

2. DEFINITIONS

Let X be a space and $\mathcal{U}$ a cover of X. A cover $\mathcal{V}$ of X is a *refinement* of $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in some member of $\mathcal{U}$. A cover $\mathcal{V}$ of X is a *star-refinement* (*point-star refinement*) of $\mathcal{U}$ if for each $V \in \mathcal{V}$, $\mathrm{St}(V, \mathcal{V}) = \cup\{V' \in \mathcal{V} : V' \cap V \neq \emptyset\}$ (for each $x \in X$, $\mathrm{St}(x, \mathcal{V}) = \cup\{V \in \mathcal{V} : x \in V\}$) is contained in some member of $\mathcal{U}$.

An open cover $\mathcal{U}$ of a space X is said to be *normal* if there is a sequence $\{\mathcal{V}_n\}$ of open covers of X such that $\mathcal{V}_0 = \mathcal{U}$ and $\mathcal{V}_{n+1}$ is a star-refinement (or point-star refinement) of $\mathcal{V}_n$ for each $n \in \omega$.

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A subset U of a space X is a *cozero set* if there is a continuous function f from X into the closed interval [0, 1] such that $U = \{x \in X : f^{-1}(x) > 0\}$. It is well-known that an open cover $\mathcal{U}$ of a space X is normal if and only if it has a σ-locally finite cozero refinement (see [**6**, Theorem 1.2]).

Let $\mathcal{U}$ and $\mathcal{V}$ be collections of subsets of a space X. We say that $\mathcal{V}$ is *cushioned in* $\mathcal{U}$ [4] if for each $V \in \mathcal{V}$ one can assign a $U_V \in \mathcal{U}$ such that for every $\mathcal{V}' \subset \mathcal{V}$,

$$\mathrm{Cl}(\cup\{V : V \in \mathcal{V}'\}) \subset \cup\{U_V : V \in \mathcal{V}'\}.$$

For two covers $\mathcal{U}$ and $\mathcal{V}$ of a space X, $\mathcal{V}$ is a *cushioned refinement* of $\mathcal{U}$ if $\mathcal{V}$ is cushioned in $\mathcal{U}$.

3. Results

The following is known and the proof is routine. For example, it is found in the proof of (iv) $\to$ (i) of [**1**, Theorem 2.6].

LEMMA 1. *Let X be a space and $\mathcal{U}$ an open cover of X. If $\mathcal{U}$ has an open star-refinement, then it has a cushioned open refinement.*

Gruenhage [**2**, Theorem 2] showed:

THEOREM 2. *Let X be a locally compact space and $\mathcal{U}$ an open cover of X by sets with compact closure. Then $\mathcal{U}$ has a point-finite open refinement if and only if $\mathcal{U}^F$ has a cushioned refinement, where $\mathcal{U}^F$ denotes the cover consisting of all finite unions of members of $\mathcal{U}$.*

We prove the following analogue of Theorem 2. The main idea of the proof of it is due to Gruenhage.

THEOREM 3. *Let X be a locally compact space and $\mathcal{U}$ an open cover of X by sets with compact closure. Then $\mathcal{U}$ is normal if and only if it has a cushioned open refinement.*

PROOF: The "only if" part is obvious from Lemma 1. We show the "if" part.

Let $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ such that $\mathrm{Cl}\, U_\alpha$ is compact for each $\alpha < \kappa$, where κ is some cardinal. Let $\mathcal{V}$ be a cushioned open refinement of $\mathcal{U}$. We may assume that $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ and that $\mathrm{Cl}\left(\bigcup_{\alpha\in\Gamma} V_\alpha\right) \subset \bigcup_{\alpha\in\Gamma} U_\alpha$ for each $\Gamma \subset [0, \kappa)$ (see [**4**, Proposition 2.1]). For each $x \in X$, we take an $\alpha(x) < \kappa$ with $x \in V_{\alpha(x)}$.

For each $\alpha < \kappa$, we can construct two collections $\mathcal{U}_\alpha$ and $\{W(x) : x \in S_\alpha\}$ of open sets in X, satisfying the following conditions:

(1) $\mathcal{U}_\alpha$ is a countable subcollection of $\mathcal{U}$,

(2) $\{U_\beta : \beta < \alpha\} \subset \bigcup_{\beta\leqslant\alpha} \mathcal{U}_\beta$,

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(3) S_α is a countable subset of $X \setminus \bigcup_{\beta<\alpha} (\cup\mathcal{U}_\beta)$,

(4) for each $x \in S_\alpha$, $W(x)$ is a cozero set in X such that

$$x \in W(x) \subset V_{\alpha(x)} \setminus \operatorname{Cl}(\cup\{V_\gamma \colon U_\gamma \in \mathcal{U}_\beta \text{ and } \beta < \alpha\}),$$

(5) $(\cup\mathcal{U}_\alpha) \setminus \bigcup_{\beta<\alpha}(\cup\mathcal{U}_\beta) \subset \cup\{W(x) \colon x \in S_\alpha\}$,

(6) $\{U_{\alpha(x)} \colon x \in S_\alpha\} \subset \mathcal{U}_\alpha$.

This construction is similar to Gruenhage's in the proof of [**2**, Theorem 2]. The details are left to the reader.

Let $\mathcal{W} = \{W(x) \colon x \in S_\alpha \text{ and } \alpha < \kappa\}$. It is easily seen from (2), (4) and (5) that $\mathcal{W}$ is a cozero refinement of $\mathcal{U}$. We show that $\mathcal{W}$ is σ-locally finite in X. For each $\alpha < \kappa$, let $G_\alpha = \cup\{W(x) \colon x \in S_\alpha\}$. Since each S_α is countable, it suffices to show that

(*) $\{G_\alpha \colon \alpha < \kappa\}$ is locally finite in X.

Assuming the contrary, so say $\{G_\alpha \colon \alpha < \kappa\}$ is not locally finite at some $p \in X$. Since $U_\alpha(p)$ meets infinitely many G_α's, there is a sequence $\{\alpha_n\}$ such that $\alpha_0 < \alpha_1 < \cdots < \kappa$ and $U_{\alpha(p)}$ meets all G_{α_n}'s. For each $n \in \omega$, pick an $x_n \in S_{\alpha_n}$ such that $U_{\alpha(p)}$ meets $W(x_n)$. Since $\operatorname{Cl} U_{\alpha(p)}$ is compact, it follows that

(**) $\{W(x_n) \colon n \in \omega\}$ is not locally finite in X.

Pick any $x \in \operatorname{Cl}\left(\bigcup_{n\in\omega} W(x_n)\right)$. By (4), we have

$$x \in \operatorname{Cl}\left(\bigcup_{n\in\omega} V_{\alpha(x_n)}\right) \subset \bigcup_{n\in\omega} U_{\alpha(x_n)}.$$

Take a $k \in \omega$ with $x \in U_{\alpha(x_k)}$. Since $\bigcup_{\alpha<\kappa} \mathcal{U}_\alpha$ covers X, take some $\delta < \kappa$ such that $x \in (\cup\mathcal{U}_\delta) \setminus \bigcup_{\beta<\delta} (\cup\mathcal{U}_\beta)$. By (6), we have $U_{\alpha(x_k)} \in \mathcal{U}_{\alpha_k}$. So we get $\delta \leqslant \alpha_k$. By (5), take some $y \in S_\delta$ with $x \in W(y)$. It follows from (4) and (6) that

$$x \in W(y) \subset V_{\alpha(y)} \subset U_{\alpha(y)} \in \mathcal{U}_\delta.$$

Take any $n > k$. Since $\alpha_n > \alpha_k \geqslant \delta$, it follows from (4) that

$$W(x_n) \cap V_{\alpha(y)} \subset W(x_n) \cap (\cup\{V_\gamma : U_\gamma \in \mathcal{U}_\beta \text{ and } \beta < \alpha_n\}) = \emptyset.$$

Hence $x \notin \operatorname{Cl}(\cup\{W(x_n) \colon n > k\})$. This contradicts (**), so that we have shown (*). Thus $\mathcal{W}$ is a σ-locally finite cozero refinement of $\mathcal{U}$, and so $\mathcal{U}$ is normal. ∎

QUESTION A: In Theorem 3, can the "cushioned" be replaced by "σ-cushioned"?

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An open cover $\mathcal{U}$ of a space X is *interior-preserving* if $\cap\mathcal{U}'$ is open in X for each $\mathcal{U}' \subset \mathcal{U}$. A space X is said to be *orthocompact* if every open cover of X has an interior-preserving open refinement.

Jiang [**3**, Theorem 2.1] showed

THEOREM 4. *Let X be a space and $\mathcal{U}$ an interior-preserving open cover of X. Then $\mathcal{U}$ has an open point-star refinement if and only if it has a cushioned refinement.*

We can get an analogue of Theorem 4.

THEOREM 5. *Let X be a space and $\mathcal{U}$ an interior-preserving open cover of X. Then $\mathcal{U}$ has an open star-refinement if and only if it has a cushioned open refinement.*

PROOF: The "only if" part follows immediately from Lemma 1.

Let $\mathcal{V}$ be a cushioned open refinement of $\mathcal{U}$. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$. We may assume that $\mathcal{V} = \{V_\alpha : \alpha \in A\}$ and that $\mathrm{Cl}\left(\bigcup_{\alpha\in B} V_\alpha\right) \subset \bigcup_{\alpha\in B} U_\alpha$ for each $B \subset A$.

For each $x \in X$, choose an $\alpha(x) \in A$ with $x \in V_{\alpha(x)}$, and let

$$W(x) = \left(\cap\{U_\alpha : x \in U_\alpha\} \cap V_{\alpha(x)}\right) \setminus \mathrm{Cl}(\cup\{V_\alpha : x \notin U_\alpha\}).$$

Let $\mathcal{W} = \{W(x) : x \in X\}$. Since each $W(x)$ is an open neighbourhood of x in X, $\mathcal{W}$ is an open cover of X. Pick any $x \in X$. It suffices to show that $\mathrm{St}(W(x), \mathcal{W}) \subset U_{\alpha(x)}$. Suppose that $W(x) \cap W(y) \neq \emptyset$. If $y \notin U_{\alpha(x)}$, we have $W(y) \cap V_{\alpha(x)} = \emptyset$. Since $W(x) \subset V_{\alpha(x)}$, this is a contradiction. So $y \in U_{\alpha(x)} \in \mathcal{U}$. Therefore it follows that $W(y) \subset \cap\{U_\alpha : y \in U_\alpha\} \subset U_{\alpha(x)}$. ∎

QUESTION B: Let X be an orthocompact space and $\mathcal{U}$ an open cover of X. If $\mathcal{U}$ has a cushioned open refinement, does it have an open star-refinement?

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Department of Mathematics
Kanagawa University
3-27-1 Rokkakubashi
Yokahama 221
Japan