

PLÜCKER FORMULAE FOR THE ORTHOGONAL GROUP

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Plücker formulae for horizontal curves in $SO(m)$ -flag manifolds are derived. These formulae are seen to generalise the usual Plücker formulae for projective space curves. They also have applications in the theory of minimal surfaces in Euclidean sphere and the complex hyperquadric.

INTRODUCTION

Plücker formulae for projective space curves give relationships between an intrinsic invariant (the genus) and a set of extrinsic invariants (associated degrees and ramification indices). On the other hand the method of moving frames allows us to think of projective curves (whether compact or not) as horizontal curves in the complex flag manifold $SU(n+1)/T$. This gives a new interpretation of the Plücker formulae and indeed one can derive the formulae for horizontal curves in any G/T , where G is a connected compact simple Lie group, and T a maximal torus in G . (G/T is called the G -flag manifold.) We note that the resulting formulae have interesting applications to the theory of harmonic maps of real surfaces into Riemannian homogeneous spaces, especially type 1 inner symmetric spaces (see [1, Section 4]).

In this paper we derive explicitly the formulae for horizontal curves in $SO(m)/T$. We do this for compact curves so as to obtain directly the integrated form of the formulae. The cases of $G = SU(n)$, $Sp(n)$ have been dealt with in [4, 5]. It would also be interesting to write down the exceptional Plücker formulae.

1. HORIZONTAL CURVES IN G -FLAG MANIFOLDS

In this section we give an overview of the theory of horizontal curves in G -flag manifolds and indicate how the Plücker formulae are derived in general. It should be pointed out that the derivation depends essentially only upon the Maurer–Cartan structure equations of G . Horizontal curves form a small subset of a large class of maps called pseudocomplex maps (they are maps from an almost complex manifold to a homogeneous space of the form G/H , where G is a compact semisimple Lie group and H is a closed subgroup of maximal rank) and a detailed account of them is given in [4]. In particular the proofs of results collected in this section can be found there.

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As a motivational example we give a brief discussion of projective curves (see [3, Chapter II]). Consider a projective algebraic curve, that is, a holomorphic map

$$f = f_0: M \rightarrow CP^n = SU(n + 1)/S(U(1) \times U(n)),$$

where $M = M_g$ is a compact Riemann surface of genus g . There are the associated curves

$$f_i: M \rightarrow CG_{n+1, i+1} \subset CP^N,$$

where $N = \binom{N+1}{i+1} - 1, 0 \leq i \leq n - 1$. Let d_i denote the degree of $f_i(M) \subset CP^N$ as an algebraic curve and also let $\#_i$ denote the total ramification index of f_i . We then have

$$2g - 2 - \#_i = d_{i-1} - 2d_i + d_{i+1}, 0 \leq i \leq n - 1,$$

where $d_{-1} = d_n = 0$. Let ds_N^2 denote the normalised Fubini–Study metric on CP^N . Assume that f is nondegenerate, that is $f(M)$ does not lie in a lower dimensional projective subspace. Then each $f_i^* ds_N^2$ is a singular metric (also called a pseudo-metric, for a definition see [4, p.81]) on M and can be written as $\varphi_i \bar{\varphi}_i$ for some local type $(1, 0)$ form φ_i on M . The zeros of φ_i (which are globally defined) define an integral divisor on M and its degree equals $\#_i$.

Now consider $SU(n + 1)/T$, where $T = S(U(1)^{n+1})$ is a maximal torus. Fix an invariant complex structure on $SU(n + 1)/T$. This amounts to fixing a system of positive roots of $SU(n + 1)$ relative to T . There are exactly n simple roots of $SU(n + 1)$ and the corresponding root spaces generate a rank n holomorphic subbundle, called the horizontal bundle, of the holomorphic tangent bundle of $SU(n + 1)/T$. The inclusion $T \subseteq H = (SU(1) \times U(n))$ induces the projection $\pi: SU(n + 1)/T \rightarrow CP^n$. On CP^n we use the complex structure coming from the one on $SU(n + 1)/T$ so that π is holomorphic. We then have

THEOREM. *Let $f: M \rightarrow CP^n$ be a nondegenerate holomorphic curve. Then its Frenet lifting gives rise to a holomorphic horizontal (tangential to the horizontal bundle) curve $\Phi_f: M \rightarrow SU(n + 1)/T$ which is nondegenerate in the sense that the image does not lie in any $G'/T \cap G'$, with G' a closed subgroup of lower rank. Conversely, given a holomorphic horizontal nondegenerate curve $F: M \rightarrow SU(n + 1)/T$ the composition $\pi \circ F$ is a nondegenerate holomorphic curve in CP^n .*

PROOF: See [4, p.84–85] for the proof. □

Hereafter, G will denote a compact connected simple Lie group of rank n and, T a maximal torus in G . By a horizontal curve in G/T we will mean a holomorphic horizontal map $f: M \rightarrow G/T$, where $M = M_g$ is a compact Riemann surface of genus

g . We say f is nondegenerate if $f(M)$ does not lie in any $G'/T \cap G'$ for G' , a closed subgroup of lower rank.

Consider a nondegenerate horizontal curve $f: M \rightarrow G/T$. Let $\mathcal{H} \subseteq T^{1,0}(G/T) \rightarrow G/T$ denote the horizontal bundle over G/T . \mathcal{H} decomposes as

$$\mathcal{H} = L_1 \oplus \dots \oplus L_n,$$

where $L_i \rightarrow G/T$ is the holomorphic line bundle generated by the root space corresponding to the i -th simple root (Order the set of simple roots.) On G/T we fix an invariant metric $ds^2_{G/T}$. Let

$$\langle \cdot, \cdot \rangle_i = \text{the restriction of } ds^2_{G/T} \text{ to } L_i.$$

The nondegeneracy assumption of f implies that $f_i^* \langle \cdot, \cdot \rangle_i$ is a singular metric on M for every i . Put $f_i^* \langle \cdot, \cdot \rangle_i = \varphi_i \bar{\varphi}_i$, for some local type $(1, 0)$ form φ_i on M . Let D_i denote the zero divisor of φ_i and put $\#_i = \deg D_i$. Also let Λ_i denote the Kähler form of $(M, \varphi_i \bar{\varphi}_i)$. Λ_i defined on the support of $\varphi_i \bar{\varphi}_i$, hence on all of M except for a finite set. Put $d_i = \int_M \Lambda_i$, $1 \leq i \leq n$. So d_i is the area of $(M, \varphi_i \bar{\varphi}_i)$. Observe that if $G = SU(n + 1)$ then d_i is the degree of the i -th associated curve of f by Wirtinger's theorem.

We now indicate how the general derivation of Plücker formulae goes. Choose a G -frame $e: U \subset M \rightarrow G$, $\pi \circ e = f$, $\pi: G \rightarrow G/T$. By its very construction we can write φ_i as $e^* \Omega_i$, where Ω_i is a left-invariant form on G . Now

$$d\varphi_i = -\omega_i \wedge \varphi_i,$$

where ω_i is the complex connection form. On the other hand

$$d\varphi_i = d e^* \Omega_i = e^* d\Omega_i$$

which we can compute using the Maurer–Cartan structure equations of G . This allows us to write ω_i as the pullback of left-invariant forms. Exterior differentiation of both sides of the equations thus obtained gives relations amongst the curvature forms, the Gaussian curvatures, and the Kähler forms, which give the infinitesimal form of the Plücker formulae. Integration over M finally yields the formulae.

Remarks. Combining the above theorem on projective curves with the well-known result that any compact Riemann surface can be holomorphically embedded in CP^3 we can show that any compact Riemann surface can be holomorphically and horizontally embedded in $SU(4)/T$. In light of this it would be interesting to establish the following conjecture: any compact Riemann surface can be holomorphically and horizontally embedded in G/T , where G is any connected compact simple Lie group of rank 3. Another interesting problem is to describe the structure of the moduli of horizontal curves. It seems that the set of compact horizontal curves in any G/T forms at least a complex analytic space.

2. PLÜCKER FORMULAE FOR $SO(2n)$

Let $G = SO(2n) = \{X \in GL(2n; \mathbb{R}) \mid {}^tXX = 1, \det X = 1\}$ and also let $T = \{\text{diag}(D_1, \dots, D_n) \mid D_i = \begin{pmatrix} c_i & -s_i \\ s_i & c_i \end{pmatrix}, \text{ where } c = \cos 2\pi x_i, s = \sin 2\pi x_i\} = SO(2)^n$. T is a maximal torus in G and

$$G/T = \tilde{F}_{2,4,\dots,2n}(\mathbb{R}^{2n}),$$

the full oriented even flags in \mathbb{R}^{2n} .

Notation. We will identify $SO(2)$ with $U(1)$ and write e^{ix} (or $\exp(ix)$) instead of $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$ from now on.

We note that the standard injection $U(n) \rightarrow SO(2n)$ projects down to a totally geodesic embedding $U(n)/U(1)^n \rightarrow G/T$. For this reason one should consider the study of horizontal maps in $SO(2n)/SO(2)^n$ to be more general in scope than that in the complex flag manifold.

There is the Lie algebra decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$, where \mathfrak{m} is the orthogonal vector space complement of \mathfrak{t} (the Lie algebra of T) with respect to the Killing form.

Let $E_{ij} = [e_{mn}]$, $F_{ij} = [f_{mn}]$, $E'_{ij} = [e'_{mn}]$ and $F'_{ij} = [f'_{mn}]$ be $2n \times 2n$ matrices with all entries zero except for

$$\begin{aligned} e_{2i-1, 2j-1} &= e_{2i, 2j} = -e_{2j-1, 2i-1} = -e_{2j, 2i} = 1, \\ -f_{2i-1, 2j} &= f_{2i, 2j-1} = -f_{2j-1, 2i} = f_{2j, 2i-1} = 1, \\ e'_{2i-1, 2j-1} &= -e'_{2i, 2j} = -e'_{2j-1, 2i-1} = e'_{2j, 2i} = 1, \\ f'_{2i-1, 2j} &= f'_{2i, 2j-1} = -f'_{2j-1, 2i} = -f'_{2j, 2i-1} = 1. \end{aligned}$$

Then

$$\mathfrak{m} = \sum V_{ij} \oplus V'_{ij} (1 \leq i < j \leq n),$$

where $V_{ij} = \mathbb{R} - \text{span}\{E_{ij}, F_{ij}\}$ and $V'_{ij} = \mathbb{R} - \text{span}\{E'_{ij}, F'_{ij}\}$.

For $t = \text{diag}(D_1, \dots, D_n) \in T$, $v = xE_{ij} + yF_{ij} \in V_{ij}$ and $v' = xE'_{ij} + yF'_{ij} \in V'_{ij}$ we compute that

$$\begin{aligned} Ad_t: v &\leftrightarrow x + iy \mapsto \exp(2\pi i(x_i - x_j)) \cdot (x + iy), \\ v' &\leftrightarrow x + iy \mapsto \exp(2\pi i(x_i + x_j)) \cdot (x + iy), \end{aligned}$$

where we use the complex notation to write v, v' relative to their respective bases. It follows that the root spaces of G are V_{ij}, V'_{ij} and the corresponding roots are $\Delta = \{\pm(x_i - x_j), \pm(x_i + x_j) \mid 1 \leq i < j \leq n\}$.

Put $\theta_{ij} = x_i - x_j, \theta'_{ij} = x_i + x_j$.

Write $\Delta = \Delta_+ \cup \Delta_-$, where $\Delta_+ = \{\theta_{ij}, \theta'_{ij} : 1 \leq i < j \leq n\}$. Δ_+ forms a system of positive roots and the resulting simple roots are

$$\Delta_s = \{\theta_{12}, \theta_{23}, \dots, \theta_{n-1,n}, \theta'_{n-1,n}\}.$$

$\Omega = (\Omega_\beta^\alpha)$, $1 \leq \alpha, \beta \leq 2n$, denotes the $\mathfrak{o}(2n)$ -valued Maurer–Cartan form of G . The Maurer–Cartan structure equations are

$$d\Omega_\beta^\alpha = -\Omega_\gamma^\alpha \wedge \Omega_\beta^\gamma.$$

Let J denote the invariant complex structure on G/T arising from the choice, Δ_+ , of positive roots. For $1 \leq i < j \leq n$, put

$$\begin{aligned} \Theta^{ij} &= \frac{1}{2}[(\Omega_{2j-1}^{2i-1} + \Omega_{2j}^{2i}) + i(\Omega_{2j-1}^{2i} - \Omega_{2j}^{2i-1})], \\ \Theta^{iij} &= \frac{1}{2}[(\Omega_{2j-1}^{2i-1} - \Omega_{2j}^{2i}) + i(\Omega_{2j-1}^{2i} + \Omega_{2j}^{2i-1})]. \end{aligned}$$

The type $(1, 0)$ forms on G/T are given by \mathbb{C} -linear combinations of the pullbacks of Θ^{ij} , Θ^{iij} .

For an invariant metric on G/T we take the pullback of

$$\frac{1}{2} \sum (\Omega_{2j-1}^{2i-1})^2 + (\Omega_{2j}^{2i})^2 + (\Omega_{2j-1}^{2i})^2 + (\Omega_{2j}^{2i-1})^2, \quad 1 \leq i < j \leq n.$$

$ds_{G/T}^2$ will denote the above metric.

The horizontal bundle $\mathcal{H} \subset T^{1,0}(G/T)$ is given by the following Pfaffian system on G/T :

$$\Sigma = \{\theta^{ij} = 0 \text{ unless } j = i + 1; \theta^{iij} = 0 \text{ unless } (i, j) = (n - 1, n)\},$$

where $\theta^{ij} = e^* \Theta^{ij}$ and e is a local section of $G \rightarrow G/T$, etcetera. Write

$$\mathcal{H} = L_1 \oplus \dots \oplus L_n,$$

where for $1 \leq i \leq n - 1$ L_i is the holomorphic line bundle over G/T generated by the root space $V_{i,i+1}$ and L_n is the line bundle generated by $V'_{n-1,n}$.

Consider a horizontal curve (that is, horizontal holomorphic map from a compact Riemann surface of genus g) $f: M \rightarrow G/T$. Let $e: U \subset M \rightarrow G$ be a local lifting of f . The holomorphy of f is reflected by the fact that the forms $e^* \Theta^{ij}$, $e^* \Theta^{iij}$ are all of type $(1, 0)$ on M .

Notation. $e^* \Theta^{ij} = \theta^{ij}$, $e^* \Omega_\beta^\alpha = \omega_\beta^\alpha$.

The horizontality of f implies that

$$(1) \quad \{\theta^{ij} = 0 \text{ for } j \neq i + 1; \theta^{ij} = 0 \text{ for } (i, j) \neq (n - 1, n)\}.$$

So the surviving type $(1, 0)$ forms on M are

$$(2) \quad \varphi_i = \theta^{i,i+1}, 1 \leq i \leq n - 1, \text{ and } \varphi_n = \theta^{n-1,n}.$$

As in Section 1 let \langle, \rangle_i denote $ds_{G/T}^2$ restricted to the line bundle L_i . Then we have

$$f^* \langle, \rangle_i = \varphi_i \bar{\varphi}_i, 1 \leq i \leq n.$$

Each $\varphi_i \bar{\varphi}_i$ defines a singular metric on M . In particular, φ_i has only isolated zeros of finite multiplicity. (f is nondegenerate if and only if none of the φ_i 's is identically zero. See [4, p.64 and p.76]). Put

$$D_i = \text{the zero divisor of } \varphi_i \text{ on } M, \\ \#_i = \text{deg } D_i.$$

Away from the zeros of φ_i we have

$$(3) \quad d\varphi_i = -\omega_i \wedge \varphi_i,$$

$$(4) \quad d\omega_i = \frac{1}{2} K_i \varphi_i \wedge \bar{\varphi}_i = (-iK_i) \cdot \Lambda_i,$$

where ω_i is the complex connection form, $d\omega_i$ is the curvature form, K_i is the Gaussian curvature, and Λ_i denotes the Kähler form of $(M, \varphi_i \bar{\varphi}_i)$.

Differentiating both sides of the equations in (1) we obtain

$$(5) \quad \omega_{2j-1}^{2i-1} = \omega_{2j}^{2i} = \omega_{2j-1}^{2i} = \omega_{2j}^{2i-1} = 0, j \neq i + 1, 1 \leq i \leq n - 2, \\ \omega_{2i+1}^{2i-1} = \omega_{2i+2}^{2i}, \omega_{2i+1}^{2i} = -\omega_{2i+2}^{2i-1}, 1 \leq i \leq n - 2.$$

Using (5) we can rewrite the equations in (2) as follows.

$$(6) \quad \varphi_i = \omega_{2i+1}^{2i-1} + i\omega_{2i+1}^{2i}, 1 \leq i \leq n - 2, \\ \varphi_{n-1} = \frac{1}{2} [(\omega_{2n-1}^{2n-3} + \omega_{2n}^{2n-2}) + i(\omega_{2n-1}^{2n-2} - \omega_{2n}^{2n-3})], \\ \varphi_n = \frac{1}{2} [(\omega_{2n-1}^{2n-3} - \omega_{2n}^{2n-2}) + i(\omega_{2n-1}^{2n-2} + \omega_{2n}^{2n-3})].$$

Using (3), (6) and the Maurer–Cartan structure equations we differentiate both sides of the equations in (2) and obtain

$$(7) \quad \begin{aligned} \omega_i &= -i(\omega_{2i}^{2i-1} - \omega_{2i+2}^{2i+1}), \quad 1 \leq i \leq n-1, \\ \omega_n &= -i(\omega_{2n-2}^{2n-3} + \omega_{2n}^{2n-1}). \end{aligned}$$

Using the Maurer–Cartan structure equations, (2), and the definition of the Λ_i 's we also obtain

$$(8) \quad \begin{aligned} d\omega_{2i}^{2i-1} &= -2\Lambda_{i-1} + 2\Lambda_i, \quad 1 \leq i \leq n-2, \quad \Lambda_0 = 0, \\ d\omega_{2n-2}^{2n-3} &= -2\Lambda_{n-2} + 2(\Lambda_{n-1} + \Lambda_n), \\ d\omega_{2n}^{2n-1} &= -2\Lambda_{n-1} + 2\Lambda_n. \end{aligned}$$

Exterior differentiation of the equations in (7), using (8) together with (4), now gives

$$(9) \quad \begin{aligned} K_i \Lambda_i &= -2\Lambda_{i-1} + 4\Lambda_i - 2\Lambda_{i+1}, \quad 1 \leq i \leq n-3, \quad \Lambda_0 = 0, \\ K_{n-2} \Lambda_{n-2} &= -2\Lambda_{n-3} + 4\Lambda_{n-2} - 2\Lambda_{n-1} - 2\Lambda_n, \\ K_{n-1} \Lambda_{n-1} &= -2\Lambda_{n-2} + 4\Lambda_{n-1}, \\ K_n \Lambda_n &= -2\Lambda_{n-2} + 4\Lambda_n, \end{aligned}$$

where we assume $G = SO(2n)$ with $n \geq 5$.

For $n = 4$ we get

$$\begin{aligned} K_1 \Lambda_1 &= 4\Lambda_1 - 2\Lambda_2, \\ K_2 \Lambda_2 &= -2\Lambda_1 + 4\Lambda_2 - 2\Lambda_3 - 2\Lambda_4, \\ K_3 \Lambda_3 &= -2\Lambda_2 + 4\Lambda_3, \\ K_4 \Lambda_4 &= -2\Lambda_2 + 4\Lambda_4. \end{aligned}$$

For $n = 3$ we get

$$\begin{aligned} K_1 \Lambda_1 &= 4\Lambda_1 - 2\Lambda_2 - 2\Lambda_3, \\ K_2 \Lambda_2 &= -2\Lambda_1 + 4\Lambda_2, \\ K_3 \Lambda_3 &= -2\Lambda_1 + 4\Lambda_3. \end{aligned}$$

Finally for $n = 2$,

$$\begin{aligned} K_1 \Lambda_1 &= 4\Lambda_1, \\ K_2 \Lambda_2 &= 4\Lambda_2. \end{aligned}$$

The generalised Gauss–Bonnet–Chern theorem ([3, p.144] or [4, p.81]) states that

$$\int_M d\omega_i = 2\pi i \cdot (2g - 2 - \#i),$$

where g is the genus of M . This allows us to integrate the left hand sides of the equations in (9). Also put

$$d_i = \frac{1}{\pi} \int_M \Lambda_i = \text{the area of } (M, \varphi_i \bar{\varphi}_i).$$

Upon integration over M the equations in (9) yields

$$\begin{aligned} 2g - 2 - \#i &= d_{i-1} - 2d_i + 2d_{i+1}, \quad 1 \leq i \leq n - 3, \quad d_0 = 0, \\ 2g - 2 - \#_{n-2} &= d_{n-3} - 2d_{n-2} + d_{n-1} + d_n, \\ 2g - 2 - \#_{n-1} &= d_{n-2} - 2d_{n-1}, \\ 2g - 2 - \#_n &= d_{n-2} - 2d_n. \end{aligned}$$

3. PLÜCKER FORMULAE FOR $SO(2n + 1)$

For the rest of the paper we let $G = SO(2n + 1)$. Include $SO(2n)$ in $SO(2n + 1)$ via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. For a maximal torus in G we use T in Section 2 included in G which we again denote by T . Generally speaking we will identify an object in $SO(2n)$ (or in $\mathfrak{o}(2n)$) with its counterpart in $SO(2n + 1)$ (or in $\mathfrak{o}(2n + 1)$).

As the root structure of $SO(2n + 1)$ is different from that of $SO(2n)$ it is necessary that we treat this case separately. Nevertheless the computations involved are similar and will mostly be suppressed.

As before we have the Lie algebra decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$, $\mathfrak{m} = \mathfrak{t}^\perp$. Let E_{ij} , F_{ij} , E'_{ij} , F'_{ij} be as in Section 2 except that they are now interpreted to be vectors in $\mathfrak{g} = \mathfrak{o}(2n + 1)$. Let $E_\gamma = [e_{ik}]$ be the $(2n + 1) \times (2n + 1)$ matrix with all entries zero except for $e_{\gamma, 2n+1} = -e_{2n+1, \gamma} = 1$. Then

$$\mathfrak{m} = \sum V_{ij} \oplus V'_{ij} \oplus V_i = \mathfrak{o}(2n) \oplus \sum V_i (1 \leq i < j \leq n, 1 \leq i \leq n),$$

where $V_i = \mathbb{R}\text{-span} \{E_{2i-1}, E_{2i}\}$ and V_{ij}, V'_{ij} as before.

For $t = \text{diag} (D_1, \dots, D_n, 1) \in T$ and $v = xE_{2i-1} + yE_{2i} = \begin{pmatrix} x \\ y \end{pmatrix}$ we compute that

$$Ad_t: v \leftrightarrow x + iy \rightarrow \exp(2\pi i x_i) \cdot (x + iy).$$

This together with the earlier computation shows that V_{ij}, V'_{ij}, V_i are the root spaces of G relative to T . The roots are

$$\Delta = \{\pm(x_i - x_j), \pm(x_i + x_j), \pm x_i : 1 \leq i < j \leq n\}.$$

Put $\theta_{ij} = x_i - x_j, \theta'_{ij} = x_i + x_j, \theta_i = x_i$.

For positive roots we take $\Delta_+ = \{\theta_{ij}, \theta'_{ij}, \theta_i : 1 \leq i < j \leq n\}$. The resulting simple roots are $\Delta_s = \{\theta_{12}, \theta_{23}, \dots, \theta_{n-1,n}, \theta_n\}$.

$\Omega = (\Omega_{\alpha}^{\beta})$, $1 \leq \alpha, \beta \leq 2n + 1$, denotes the $\mathfrak{o}(2n + 1)$ -valued Maurer–Cartan form of G . For $1 \leq i < j \leq n$ put

$$\begin{aligned} \Theta^{ij} &= \frac{1}{2}[(\Omega_{2j-1}^{2i-1} + \Omega_{2j}^{2i}) + i(\Omega_{2j-1}^{2i} - \Omega_{2j}^{2i-1})], \\ \Theta'^{ij} &= \frac{1}{2}[(\Omega_{2j-1}^{2i-1} - \Omega_{2j}^{2i}) + i(\Omega_{2j-1}^{2i} + \Omega_{2j}^{2i-1})], \\ \Theta^i &= \Omega_{2n+1}^{2i-1} + i\Omega_{2n+1}^{2i}. \end{aligned}$$

The pullbacks of $\Theta^{ij}, \Theta'^{ij}, \Theta^i$ to G/T are all of type $(1, 0)$ with respect to the invariant complex structure J determined by Δ_+ .

For an invariant metric on G/T we take the pullback of

$$\frac{1}{2} \sum [(\Omega_{2j+1}^{2i-1})^2 + (\Omega_{2j}^{2i})^2 + (\Omega_{2j-1}^{2i})^2 + (\Omega_{2j}^{2i-1})^2 + 2(\Omega_{2n+1}^{2i-1})^2 + 2(\Omega_{2n+1}^{2i})^2].$$

We consider a horizontal curve $f: M \rightarrow G/T$. Letting $e: U \subseteq M \rightarrow G$ be a local lifting of f we write $e^*\Omega = \omega, e^*\Theta = \theta$. Then the horizontality of f implies that

$$(1') \quad \{\theta^{ij} = 0 \text{ unless } j = i + 1; \theta'^{ij} \text{ for } 1 \leq i < j \leq n; \theta^i = 0 \text{ unless } i = n\}.$$

The surviving type $(1, 0)$ forms on M are

$$(2') \quad \varphi_i = \theta^{i,i+1}, 1 \leq i \leq n - 1; \varphi_n = \theta^n = \omega_{2n+1}^{2n-1} + i\omega_{2n+1}^{2n}.$$

As before put

$\#_i =$ the degree of the zero divisor of φ_i ,

$$\begin{aligned} d_i &= \frac{1}{\pi} \int_M \Lambda_i \text{ for } 1 \leq i \leq n - 1, \\ d_n &= \frac{1}{2\pi} \int_M \Lambda_n, \end{aligned}$$

where $\Lambda_i(1 \leq i \leq n)$ denotes the Kähler form of $(M, \varphi_i; \bar{\varphi}_i)$. Going through a derivation similar to the one already given in Section 2 we obtain

$$(10') \quad 2g - 2 - \#_i = d_{i-1} - 2d_i + d_{i+1}, 1 \leq i \leq n,$$

where $d_0 = 0, d_{n+1} = d_n$.

Remark. Let $H = SO(2) \times SO(2n - 2)$, $G/H = Q_{2n-2} \subset CP^{2n-1}$, the hyperquadric. An inclusion $T \subset H$ induces a projection $G/T \rightarrow G/H$. Then a horizontal curve in G/T projects down to a harmonic map into G/H by Bryant's theorem. Also if $G = SO(2n + 1)$ then we can take $H = SO(2n)$ and produce harmonic maps into $S^{2n} = G/H$ from horizontal curves. It is not difficult to show that every minimal two-sphere in S^{2n} arises as a horizontal curve. Thus our formulae in this section represent a generalisation of the similar formulae given by Chern in [2].

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