Can. J. Math., Vol. XL, No. 3, 1988, pp. 750-768

CONVERGENCE OF INTERPOLATION TO TRANSFORMS OF TOTALLY POSITIVE KERNELS

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1. Introduction. Convergence of exponential sums

$$\sum_{j=1}^{n} a_j e^{\lambda_j x}$$

that interpolate to Laplace transforms

$$(1.1) \quad f(x) := \int_0^\infty e^{-xt} d\mu(t),$$

have been studied by several authors [**3**, **6**, **8**, **15**]. For rational functions that interpolate to Markov functions (also called Hamburger or Stieltjes Series or Hilbert Transforms)

$$(1.2) \quad f(x) := \int_{-1}^{1} 1/(1 - xt) d\mu(t),$$

far more detailed convergence results are available (see [**10**, **11**, **16**] and references therein). Both (1.1) and (1.2) are special cases of the transform

$$(1.3) \quad f(x) := \int K(x, t) d\mu(t),$$

where $K(x, t)$ is a strictly totally positive kernel.

There is a well-developed qualitative theory for interpolation by generalized K-polynomials to functions of the form (1.3) or even more generally, of the form

$$(1.4) \quad f_\alpha(x) := \int K(x, t)\alpha(t) d\mu(t),$$

where $\alpha(t)$ is a bounded real function and $d\mu(t)$ is a non-negative measure, the support of which contains infinitely many points, and may be unbounded. This theory is closely associated with generalized monosplines, generalized Gauss quadratures, optimal approximation and n-widths; see for example [**2**, **5**, **7**, **14**] and references therein.

Received January 6, 1988. The research of the first author was begun during a visit to N.R.I.M.S. while that of the second was completed during a visit to Tel Aviv University; Part-time at the Department of Mathematics, University of the Witwatersrand, Rep. of South Africa.

It is the purpose of this paper to use some of this qualitative theory to study convergence of interpolation by generalized polynomials of the form

$$(1.5) \quad \sum_{j=1}^{n} a_j K(x, \eta_j),$$

or

$$(1.6) \quad \sum_{j=1}^{n} \left\{ a_j K(x, \eta_j) + b_j \frac{\partial}{\partial t} K(x, t)_{t=\eta_j} \right\}$$

to functions of the form (1.4). While the interpolation points will be subject to mild restrictions, the $\{\eta_j\}$ will be chosen as Hobby-Rice nodes or Gauss nodes associated with the measure $d\mu(t)$ and the interpolation points. We use "nodes" to distinguish the $\{\eta_j\}$ from the interpolation points. This corresponds to choosing the exponents in exponential interpolation, or fixing the poles in rational interpolation, according to the interpolation points.

The novelty of our results lies in the generality of the kernel $K(x, t)$, the arbitrariness of $\alpha(t)$, and the use of the Hobby-Rice nodes. In the case of the Markov function (1.2), we extend convergence to the complex plane in an unusual way. In all cases, we obtain rates of convergence independent of $\alpha(t)$, $|\alpha(t)| \leqq 1$, using a Bernstein comparison argument. For simplicity of presentation, we treat interpolation only at simple (distinct) interpolation points, and merely comment on the conditions required for the (straightforward) generalization to multiple interpolation points.

This paper is organized as follows: In Section 2, we state our results associated with the Hobby-Rice nodes, and we present their proofs in Section 3. In Section 4, we present additional results specifically for the rational kernel. Finally, in Section 5, we present our results associated with the Gauss nodes.

2. Hobby-Rice nodes: simple node interpolation. In this section, we investigate convergence of interpolation by generalized polynomials of the form (1.5). First we define our notation:

Definition 2.1. Let I and J be (finite or infinite) real intervals, and let

$$K(x, t): I \times J \to \mathbf{R}$$

be continuous. We say that $K(x, t)$ is STP (strictly totally positive) if for each $n = 1, 2, 3, \ldots$, and $x_1, x_2, \ldots x_n \in I$, $t_1, t_2, \ldots t_n \in J$ with

$$x_1 < x_2 < \ldots < x_n \text{ and } t_1 < t_2 < \ldots < t_n,$$

we have

$$(2.1)\qquad K\begin{bmatrix} x_1 \dots x_n \\ t_1 \dots t_n \end{bmatrix} := \det(K(x_i, t_j))_{i,j=1}^n > 0.$$

Three important examples of STP kernels are

$$K(x, t) := e^{xt}, \ x, t \in \mathbf{R},$$

$$K(x, t) := 1/(1 - xt)^\gamma, \ x, t \in (-1, 1), \gamma > 0,$$

and the Gaussian kernel

$$K(x, t) := e^{-(x-t)^2}, \ x, t \in \mathbf{R}.$$

See [**9**, pp. 9-20] for these and further examples.

Throughout this section, we assume that $K(x, t)$ is STP and we use

$$(2.2)\qquad \hat{\xi}_n := (\xi_{n1}, \xi_{n2}, \dots, \xi_{nn})$$

to denote an n-tuple of interpolation points in I, with

$$(2.3)\qquad \xi_{n1} < \xi_{n2} < \dots < \xi_{nn},$$

$n = 1, 2, 3, \dots$. Further, throughout this section, $d\mu(t)$ denotes a non-atomic, non-negative Borel measure on J with infinitely many points in its support, such that for each $x \in I$,

$$(2.4)\qquad f(x) := \int_J K(x, t) d\mu(t)$$

is defined and finite. Given a (possibly complex-valued) function $\alpha(t)$, locally integrable with respect to $d\mu(t)$ and satisfying

$$(2.5)\qquad |\alpha(t)| \leqq 1, \ t \in J,$$

we set

$$(2.6)\qquad f_\alpha(x) := \int_J K(x, t)\alpha(t) d\mu(t), \quad x \in I.$$

Given the interpolation points $\hat{\xi}_n$, and an n-tuple of points in J,

$$(2.7)\qquad \hat{\eta}_n := (\eta_{n1}, \eta_{n2}, \dots, \eta_{nn}) : \eta_{n1} < \eta_{n2} < \dots < \eta_{nn},$$

there is a unique generalized polynomial [**9**, Chapter 1]

$$(2.8)\qquad U_n(x, \alpha) := \sum_{j=1}^n a_j K(x, \eta_j),$$

which interpolates to $f_\alpha(x)$ at $\hat{\xi}_n$, so that

$$(2.9)\qquad U_n(\xi_{nj}, \alpha) = f_\alpha(\xi_{nj}), \ j = 1, 2, \dots n.$$

Here we choose $\hat{\eta}_n$ to be the n-tuple of Hobby-Rice nodes associated with $d\mu(t)$ and $\hat{\xi}_n$: As usual, sign(x) is taken to be $+1$, 0, or -1, according to whether x is positive, zero or negative.

Definition 2.2. Given a non-atomic, non-negative Borel measure $d\mu(t)$ on J as above, and given interpolation points ξ_n as in (2.2) and (2.3), the Hobby-Rice nodes $\hat{\eta}_n$ are the unique nodes satisfying (2.7) and

$$(2.10) \quad \int_J K(\xi_{nl}, t) \operatorname{sign}\left\{\prod_{j=1}^{n} (t - \eta_{nj})\right\} d\mu(t) = 0, \ l = 1, 2, \dots n.$$

We shall use also the notation

$$(2.11) \quad \sigma_n(t) := \operatorname{sign}\left\{\prod_{j=1}^{n} (t - \eta_{nj})\right\}, \ t \in J.$$

See, for example, [**13**] for the existence and uniqueness of $\hat{\eta}_n$. When $\{K(\xi_{n1}, t), K(\xi_{n2}, t), \dots, K(\xi_{nn}, t), f(t)\}$ is a Chebyshev system, then by the well known characterization of best L_1-approximation, the generalized polynomial

$$P(t) = \sum_{j=1}^{n} a_j K(\xi_{nj}, t)$$

that interpolates to $f(t)$ in $\hat{\eta}_n$ is a best L_1-approximation to f, namely

$$\int_J |f(t) - P(t)| d\mu(t) = \min_{b_1, b_2 \dots b_n} \int_J \left| f(t) - \sum_{j=1}^{n} b_j K(\xi_{nj}, t) \right| d\mu(t).$$

Our use of the Hobby-Rice nodes not as interpolation points, but as "nodes" (or exponents or poles) is motivated by the following convergence result:

THEOREM 2.3 *Assume the notation and assumptions* (2.2) *to* (2.11) *and that for each* $x \in I$, *there exists* $j = j(n, x)$ *such that*

$$(2.12) \quad \lim_{n \to \infty} \xi_{nj} = x.$$

Then, uniformly in compact subsets of I,

$$(2.13) \quad \lim_{n \to \infty} U_n(x, \alpha) = f_\alpha(x).$$

The condition (2.12) states that each $x \in I$ should be a limit of interpolation points. The proof of Theorem 2.3 uses the continuity of $K(x, t)$, but this could be replaced by the condition that uniformly for x, y in compact subsets of I,

$$\lim_{x \to y} \int_J |K(x, t) - K(y, t)| d\mu(t) = 0.$$

When $K(x, t)$ is analytic in x, we can somewhat weaken (2.12). We use $|A|$ to denote the cardinality of a set A.

THEOREM 2.4. *Assume that* $K(x, t)$ *is analytic in* x *and continuous in* t *for* $x \in D$, $t \in J$, *where* D *is some connected open subset of* $\mathbf{C}$ *containing* I. *Assume further the notation* (2.2) *to* (2.11) *and that*

$$(2.14) \quad \int_J |K(x, t)|\,|d\mu(t)|$$

is uniformly bounded for x *in each compact subset of* D. *If there is a compact interval* I_1 *contained in the interior of* I, *such that*

$$(2.15) \quad \lim_{n\to\infty} |\{\xi_{n1}, \xi_{n2}, \dots, \xi_{nn}\} \cap I_1| = \infty,$$

then uniformly in compact subsets of I,

$$\lim_{n\to\infty} U_n(x, \alpha) = f_\alpha(x).$$

Only for the rational kernel can we extend the convergence in I to D; see Section 4. One may prescribe a rate of convergence independent of $\alpha(t)$ satisfying (2.5):

THEOREM 2.5. *With the notation* (2.2) *to* (2.11), *we have for each* $x \in I$ *and* $n = 1, 2, 3, \dots$,

$$(2.16) \quad |f_\alpha(x) - U_n(x, \alpha)| \leqq |f_{\sigma_n}(x)|,$$

where

$$(2.17) \quad f_{\sigma_n}(x) := \int_J K(x, t)\sigma_n(t)d\mu(t), \quad x \in I,$$

and where $\sigma_n(t)$ *is given by* (2.11). *Further,*

$$(2.18) \quad |f_{\sigma_n}(x)| = \min_{b_1, b_2 \dots b_n} \int_J \left|K(x, t) - \sum_{j=1}^{n} b_j K(\xi_{nj}, t)\right| d\mu(t), \quad x \in I,$$

and under the conditions of Theorem 2.3,

$$(2.19) \quad \lim_{n\to\infty} f_{\sigma_n}(x) = 0,$$

uniformly in compact subsets of I, *while under the conditions of Theorem* 2.4, (2.19) *holds uniformly in compact subsets of* D.

For kernels $K(x, t)$ such as e^{xt} of $1/(1 - xt)$, the conditions on $\hat{\xi}_n$ under which generalized polynomials

$$\sum_{j=1}^{n} b_j K(\xi_{nj}, t)$$

are dense in the space of continuous functions on an interval are classical. In these cases, one may use (2.18) and (2.16) to obtain rates of convergence, and to weaken the conditions on the interpolation points in Theorems 2.3 and 2.4; see Section 4.

https://doi.org/10.4153/CJM-1988-033-9 Published online by Cambridge University Press

The inequality (2.18) also shows that the choice of nodes $\hat{\eta}_n$ is optimal for the class $\{f_\alpha : |\alpha| \leqq 1\}$, given the interpolation points $\hat{\xi}_n$: When $\alpha = \sigma_n$, there is equality in (2.16); see the proof of Lemma 3.1.

For the exponential kernel, Sidi and Lubinsky [**15**, Theorems 4.1, 4.2, 5.5] investigated convergence and rates of convergence of exponential sums to $f_\alpha(x)$ for equally spaced interpolation points and Gauss nodes. While the $\alpha(t)$ in [**15**] is more general, and uniform convergence was obtained in unbounded sectors of the plane, the choice of interpolation points here is more general. With a careful choice of interpolation points, Braess and Saff [**6**] recently obtained what are evidently optimal rates of approximation by interpolating exponential sums to completely monotone functions, such as $f(x)$ of (2.4).

Finally, we note that we may allow consecutive interpolation points to coincide, provided that $K(x, t)$ is extended totally positive in x, up to an order matching the maximum multiplicity of the interpolation points.

3. Proof of the results of Section 2. Throughout this section, we assume the notation of Section 2, but also use abbreviations such as

$$K\begin{bmatrix} x & \hat{\xi}_n \\ t & \hat{\eta}_n \end{bmatrix} := K\begin{bmatrix} x & \xi_{n1} \ldots \xi_{nn} \\ t & \eta_{n1} \ldots \eta_{nn} \end{bmatrix}.$$

First, we establish a well-known error formula:

LEMMA 3.1. *For* $n = 1, 2, 3, \ldots$ *and* $x \in I$,

$$f_\alpha(x) - U_n(x, \alpha) = \int_J \Delta_n(x, t)\alpha(t)d\mu(t), \tag{3.1}$$

where

$$\Delta_n(x, t) := K\begin{bmatrix} x & \hat{\xi}_n \\ t & \hat{\eta}_n \end{bmatrix} \Big/ K\begin{bmatrix} \hat{\xi}_n \\ \hat{\eta}_n \end{bmatrix}. \tag{3.2}$$

Furthermore,

$$f_{\sigma_n}(x) = \int_J \Delta_n(x, t)\sigma_n(t)d\mu(t) \tag{3.3}$$

$$= \operatorname{sign}\left\{\prod_{j=1}^{n} (x - \xi_{nj})\right\} \int_J |\Delta_n(x, t)|\, d\mu(t). \tag{3.4}$$

Proof. Expanding

$$K\begin{bmatrix} x & \hat{\xi}_n \\ t & \hat{\eta}_n \end{bmatrix}$$

by its first row, we see that

$$\Delta_n(x, t) = K(x, t) - \sum_{j=1}^{n} c_j(t)K(x, \eta_j),$$

where $c_j(t)$, $j = 1, 2, \dots n$, are functions of t only. Hence the right-hand side of (3.1) may be expressed in the form $f_\alpha(x) - Q(x)$, where $Q(x)$ is a generalized polynomial of the same form as $U_n(x, \alpha)$. Further,

$$K\begin{bmatrix} x & \hat{\xi}_n \\ t & \hat{\eta}_n \end{bmatrix}$$

vanishes identically as a function of t, when $x \in \{\xi_{n1}, \xi_{n2}, \dots \xi_{nn}\}$. The uniqueness of the interpolating polynomial $U_n(x, \alpha)$ then yields (3.1).

To obtain (3.3), we set $\alpha := \sigma_n$ in (3.1). Since (2.10) implies that

$$f_{\sigma_n}(\xi_{nl}) = 0, \quad l = 1, 2, \dots n,$$

we have

$$U_n(x, \sigma_n) \equiv 0,$$

and so (3.3) holds. The fact that $K(x, t)$ is STP together with row and column interchanges show that

$$\text{(3.5)} \qquad \operatorname{sign}\left(K\begin{bmatrix} x & \hat{\xi}_n \\ t & \hat{\eta}_n \end{bmatrix}\right) = \operatorname{sign}\left\{\prod_{j=1}^{n} (x - \xi_{nj})\right\} \operatorname{sign}\left\{\prod_{j=1}^{n} (t - \eta_{nj})\right\}.$$

Then (3.4) follows.

It is easy to see that Theorems 2.3 and 2.4 follow from Theorem 2.5. In the proof of the latter, we shall need the following lemma:

LEMMA 3.2. *Let I_1 be a compact subinterval of I. Then*

$$f(x) := \int_J K(x, t)d\mu(t)$$

is uniformly convergent for x in I_1, in the sense that given $\epsilon > 0$, there is a compact subinterval J_1 of $\mathbf{R}$ such that

$$\text{(3.6)} \qquad \int_{J\setminus J_1} K(x, t)d\mu(t) \leqq \epsilon, \quad x \in I_1.$$

Further, $f(x)$ is continuous in I.

Proof. Let $I_1 := [a, b]$. Then for $x \in I_1$ and $s, t \in J$ such that $t > s$, we have as $K(x, t)$ is STP,

$$K\begin{bmatrix} x & b \\ s & t \end{bmatrix} \geqq 0,$$

whence

$$K(x, t) \leqq K(b, t)K(x, s)/K(b, s).$$

Fix s and set

$$C_1 := \max\{K(x, s)/K(b, s):x \in I_1\}.$$

Then

$$(3.7) \qquad \int_{J\cap(B,\infty)} K(x, t)d\mu(t) \leqq C_1 \int_{J\cap(B,\infty)} K(b, t)d\mu(t),$$

for all $B \in J$ such that $B > s$ and for all $x \in I_1$. Note that C_1 is independent of x and B.

Next, for $x \in I_1$ and $t, u \in J$ such that $t < u$, we have

$$K\begin{bmatrix} a & x \\ t & u \end{bmatrix} \geqq 0,$$

whence

$$K(x, t) \leqq K(a, t)K(x, u)/K(a, u).$$

Fix u and set

$$C_2 := \max\{K(x, u)/K(a, u):x \in I_1\}.$$

Then

$$(3.8) \qquad \int_{J\cap(-\infty,A)} K(x, t)d\mu(t) \leqq C_2 \int_{J\cap(-\infty,A)} K(a, t)d\mu(t)$$

for all $A \in J$ such that $A < u$ and for all $x \in I_1$. Here C_2 is independent of x and A.

Since the integrals $f(b)$ and $f(a)$ are finite, we can choose A and B such that the right-hand sides of (3.7) and (3.8) are each bounded by $\epsilon/2$. Setting $J_1 := [A, B]$ then yields (3.6).

Finally, the continuity of $f(x)$ follows as

$$\int_{J_1} K(x, t)d\mu(t)$$

is continuous in x for any compact interval J_1, and as the tail (3.6) of the integral may be made uniformly small for x in a compact interval.

Proof of Theorem 2.5. We first establish (2.16). Let

$$(3.9) \qquad \hat{\sigma}_n(x) := \operatorname{sign}\left\{\prod_{j=1}^{n} (x - \xi_{nj})\right\}, \quad x \in I,$$

and note that by (3.5),

$$\operatorname{sign}(\Delta_n(x, t)) = \hat{\sigma}_n(x)\sigma_n(t), \quad x \in I, t \in J.$$

Then by Lemma 3.1, and by (3.5),

$$f_{\sigma_n}(x) \pm (f_\alpha(x) - U_n(x, \alpha))$$
$$= \hat{\sigma}_n(x) \int_J |\Delta_n(x, t)|\{1 \pm \sigma_n(t)\alpha(t)\}d\mu(t).$$

https://doi.org/10.4153/CJM-1988-033-9 Published online by Cambridge University Press

As $|\sigma_n(t)\alpha(t)| \leqq 1$, and from (3.4), we deduce that for $x \in I$,

$$\operatorname{sign}\{f_{\sigma_n}(x) \pm (f_\alpha(x) - U_n(x, \alpha))\} \operatorname{sign}\{f_{\sigma_n}(x)\} \geqq 0.$$

Then (2.16) follows.

Next, if $b_1, \dots b_n \in \mathbf{R}$, then by (2.10),

$$(3.10) \quad \int_J \left| K(x, t) - \sum_{j=1}^{n} b_j K(\xi_{nj}, t) \right| d\mu(t)$$

$$\geqq \int_J \left\{ K(x, t) - \sum_{j=1}^{n} b_j K(\xi_{nj}, t) \right\} \hat{\sigma}_n(x)\sigma_n(t) d\mu(t)$$

$$= \hat{\sigma}_n(x) f_{\sigma_n}(x) = |f_{\sigma_n}(x)|.$$

If $x \in \{\xi_{n1}, \xi_{n2} \dots \xi_{nn}\}$, equality of both sides of (2.18) is immediate for then both sides are 0. If x is not in this set, choose $b_1, b_2, \dots b_n$ so that

$$P(t) := \sum_{j=1}^{n} b_j K(\xi_{nj}, t)$$

interpolates $K(x, t)$ at $t = \eta_{nj}$, $j = 1, 2, \dots n$. By strict total positivity of K, $K(x, t) - P(t)$ can vanish only at $t = \eta_{nj}$, $j = 1, 2, \dots n$. Hence for some $\delta(x) = \pm 1$,

$$\int_J |K(x, t) - P(t)| d\mu(t) = \int_J \{K(x, t) - P(t)\} \delta(x) \sigma_n(t) d\mu(t)$$
$$= \delta(x) f_{\sigma_n}(x) = |f_{\sigma_n}(x)|.$$

This and (3.10) yield (2.18).

Next, assume the conditions of Theorem 2.3, and let I_1 be a compact subinterval of I, and let $\epsilon > 0$. Note that if $j := j(n, x)$ is chosen so that ξ_{nj} is the closest interpolation point among $\hat{\xi}_n$ to $x \in I_1$, then

$$\lim_{n \to \infty} \xi_{nj} = x,$$

uniformly for $x \in I_1$. This is an easy consequence of (2.12) and the compactness of I_1. Thus if J_1 is a compact subinterval of J, then

$$(3.11) \quad \lim_{n \to \infty} \int_{J_1} |K(x, t) - K(\xi_{nj}, t)| d\mu(t) = 0,$$

uniformly for $x \in I_1$. With a suitable choice of J_1, this last statement and Lemma 3.2 show that we can choose $n_0 = n_0(\epsilon, I_1)$ such that

$$\int_J |K(x, t) - K(\xi_{nj}, t)| d\mu(t) \leqq \epsilon, \quad x \in I_1, \ n \geqq n_0(\epsilon).$$

In view of (2.18), we then have (2.19) uniformly in I_1.

Finally, suppose that the conditions of Theorem 2.4 are satisfied. It follows from well-known results [**12**, Theorem 17.21, p. 421] that $f_{\sigma_n}(x)$ is analytic in D, $n = 1, 2, 3, \dots$. Also from (2.17),

$$|f_{\sigma_n}(x)| \leqq \int_J |K(x, t)| d\mu(t), \quad x \in D, n = 1, 2, 3 \dots .$$

Thus $\{f_{\sigma_n}(x)\}_{n=1}^{\infty}$ is a normal family in D. Further, (2.15) ensures that the number of zeros of $f_{\sigma_n}(x)$ in the compact subinterval I_1 increases to ∞ as n approaches ∞. Hurwitz' Theorem then shows that any limit of some subsequence of $\{f_{\sigma_n}(x)\}_{n=1}^{\infty}$ has infinitely many zeros in I_1 and so vanishes identically in D. Then (2.19) follows pointwise in D, and the normality yields uniform convergence in compact subsets of D.

4. The Hobby-Rice nodes and the rational kernel. In this section, we let $I := J := (-1, 1)$, and

$$(4.1) \qquad K(x, t) := 1/(1 - xt),$$

and extend the convergence results of Section 2 to the complex plane. In this case

$$(4.2) \qquad f_\alpha(z) := \int_{-1}^{1} \frac{\alpha(t) d\mu(t)}{1 - zt}, \quad z \in \mathbf{C} \setminus \{(-\infty, -1] \cup [1, \infty)\},$$

while

$$U_n(z, \alpha) := \sum_{j=1}^{n} a_j/(1 - \eta_{nj} z),$$

is defined by the interpolation conditions (2.9), and has the form

$$P(z)/\prod_{j=1}^{n} (1 - \eta_{nj} z),$$

where $P(z)$ is a polynomial of degree at most $n - 1$.

There is a well-developed convergence theory for interpolation to Markov functions of the form (4.2), especially when $\alpha(t) \equiv 1$ (see, for example, [**10**, **11**, **16**, **18**] and references therein). While the results presented here are of a modest nature, we feel they are still of interest, since we place very few restrictions on $\alpha(t)$, $d\mu(t)$ and the interpolation points, while the poles are independent of $\alpha(t)$. Since we fix the poles given the interpolation points, the approximants are, in the terminology of C. Brezinski, "Padé-type approximants".

THEOREM 4.1. *Assume the notation and assumptions of* (2.2) *to* (2.11), *with* $K(\cdot, \cdot)$ *defined by* (4.1), *and let*

$$(4.3) \qquad A := \mathbf{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}.$$

Further assume that

(4.4) $$\lim_{n\to\infty} \sum_{k=1}^{n} \{1 - |\xi_{nk}|^{-1}(1 - \sqrt{1 - \xi_{nk}^2})\} = \infty.$$

Then, uniformly in compact subsets of A,

(4.5) $$\lim_{n\to\infty} U_n(z, \alpha) = f_\alpha(z).$$

When $\xi_{nk} = 0$, we interpret $|\xi_{nk}|^{-1}(1 - \sqrt{1 - \xi_{nk}^2})$ as 0. The condition (4.4) is known to be a necessary and sufficient condition for each continuous function $g(t)$ on $[-1, 1]$ to be uniformly approximable by rational functions of the form

$$\sum_{j=1}^{n} b_{nj}/(1 - \xi_{nj}t);$$

see [**1**, pp. 254-5]. In view of the inequality

$$|x|^{-1}\{1 - \sqrt{1 - x^2}\} \leqq |x|, \quad x \in (-1, 1),$$

we note that (4.4) is implied by the simpler condition

(4.6) $$\lim_{n\to\infty} \sum_{k=1}^{n} \{1 - |\xi_{nk}|\} = \infty.$$

This, in turn, is true if an unbounded number of $\xi_{n1}, \xi_{n2} \dots \xi_{nn}$ stay inside some compact subinterval of $(-1, 1)$, or if they do not approach ± 1 too rapidly.

Following is a rate of convergence:

THEOREM 4.2. *Under the conditions of Theorem* 4.1, *we have for* $z \in A$,

(4.7) $$|f_\alpha(z) - U_n(z, \alpha)| \leqq \begin{cases} (1 + |z|)(1 - |\mathrm{Re}\, z|)^{-1}|f_{\sigma_n}(z)|, & |\mathrm{Re}\, z| < 1, \\ (1 + |z|)(1 + |\mathrm{Re}\, z|/|\mathrm{Im}\, z|)|f_{\sigma_n}(z)|, & |\mathrm{Im}\, z| \neq 0. \end{cases}$$

Here

(4.8) $$f_{\sigma_n}(z) = \int_{-1}^{1} \frac{\sigma_n(t)d\mu(t)}{1 - tz}$$

satisfies

(4.9) $$\lim_{n\to\infty} f_{\sigma_n}(z) = 0,$$

uniformly in each compact subset of A.

https://doi.org/10.4153/CJM-1988-033-9 Published online by Cambridge University Press

Clearly, Theorem 4.1 is a corollary of Theorem 4.2. The proofs are based on the results of Section 2 and

LEMMA 4.3. *Let*

$$(4.10)\quad R_n(z) := \prod_{j=1}^{n} (z - \eta_{nj})/(1 - \xi_{nj}z), \quad n = 1, 2, 3 \dots .$$

Then for $z \in A$,

$$(4.11)\quad f_\alpha(z) - U_n(z, \alpha) = R_n(1/z)^{-1} \int_{-1}^{1} \frac{R_n(t)\alpha(t)}{1 - tz} d\mu(t),$$

and

$$(4.12)\quad |f_{\sigma_n}(z)| = |R_n(1/z)|^{-1} \left| \int_{-1}^{1} \frac{|R_n(t)|}{1 - tz} d\mu(t) \right|.$$

Proof. We see by partial fraction decomposition that

$$R_n(1/z)^{-1} \frac{R_n(t)}{1 - tz} = 1/(1 - tz) + \sum_{j=1}^{n} a_j(t)/(1 - \eta_{nj}z),$$

where $a_1(t), a_2(t), \dots a_n(t)$ are functions of t only. Further, this last expression vanishes identically as a function of t, when $z = \xi_{nj}$, $j = 1, 2, \dots n$. Thus the right-hand side of (4.11) has the same form as the left-hand side, and has the same zeros (and singularities). Uniqueness of the interpolant $U_n(z, \alpha)$ then yields (4.11). Since (2.10) implies that

$$f_{\sigma_n}(\xi_{nj}) = 0, \quad j = 1, 2, \dots n,$$

we have $U_n(z, \sigma_n) \equiv 0$. Further,

$$|R_n(t)| = R_n(t)\sigma_n(t), \quad t \in (-1, 1),$$

so (4.11) then yields (4.12).

We remark that various versions of (4.11) are well known in the literature.

Proof of Theorem 4.2. From (4.11) and (2.5), for $z \in A$,

$$(4.13)\quad |f_\alpha(z) - U_n(z, \alpha)| \leqq |R_n(1/z)|^{-1} \int_{-1}^{1} \frac{|R_n(t)|}{|1 - tz|} d\mu(t)$$

$$\leqq (1 + |z|)|R_n(1/z)|^{-1} \int_{-1}^{1} \frac{|R_n(t)|}{|1 - tz|^2} d\mu(t).$$

Next, note from (4.12) that

(4.14) $|f_{\sigma_n}(z)|$

$$= |R_n(1/z)|^{-1} \left| \int_{-1}^{1} \frac{|R_n(t)|}{|1 - tz|^2}(1 - t\bar{z})d\mu(t) \right|$$

$$= |R_n(1/z)|^{-1} \left| \int_{-1}^{1} \frac{|R_n(t)|}{|1 - tz|^2}\{1 - t \operatorname{Re} z + it \operatorname{Im} z\}d\mu(t) \right|$$

$$=: |R_n(1/z)|^{-1}|g_n(z) + ih_n(z)|,$$

where $g_n(z)$ and $h_n(z)$ are real valued. Next, if $|\operatorname{Re} z| < 1$,

$$1 - t \operatorname{Re} z \geqq 1 - |\operatorname{Re} z| > 0,$$

so from (4.13) and (4.14),

$$|f_\alpha(z) - U_n(z, \alpha)|$$

$$\leqq \frac{1 + |z|}{1 - |\operatorname{Re} z|}|R_n(1/z)|^{-1} \int_{-1}^{1} \frac{|R_n(t)|}{|1 - tz|^2}\{1 - t \operatorname{Re} z\}d\mu(t)$$

$$= \frac{1 + |z|}{1 - |\operatorname{Re} z|}|R_n(1/z)|^{-1}|g_n(z)|,$$

giving (4.7) in this case. If, on the other hand, $\operatorname{Im} z \neq 0$, we obtain from (4.13) and (4.14),

$$|f_\alpha(z) - U_n(z, \alpha)|$$

$$\leqq (1 + |z|)|R_n(1/z)|^{-1}$$

$$\times \int_{-1}^{1} \frac{|R_n(t)|}{|1 - tz|^2}\left\{1 - t \operatorname{Re} z - i\frac{\operatorname{Re} z}{\operatorname{Im} z}it \operatorname{Im} z\right\}d\mu(t)$$

$$\leqq (1 + |z|)|R_n(1/z)|^{-1}\left\{ |g_n(z)| + \left|\frac{\operatorname{Re} z}{\operatorname{Im} z}\right| |h_n(z)| \right\}.$$

Then (4.7) follows.

Because the conditions on the interpolation points are weaker in Theorem 4.1 than in Theorem 2.4, we cannot apply the latter, but proceed along similar lines: From (4.8),

$$|f_{\sigma_n}(z)| \leqq \int_{-1}^{1} \frac{d\mu(t)}{|1 - tz|}, \quad z \in A, n = 1, 2, 3, \ldots .$$

Thus $\{f_{\sigma_n}(z)\}_{n=1}^{\infty}$ is a normal family in A. Further, for $x \in (-1, 1)$, (2.18) in Theorem 2.5 shows that

$$|f_{\sigma_n}(x)|$$

$$= \min_{b_1, b_2, \ldots b_n} \int_{-1}^{1} \left|(1 - tx)^{-1} - \sum_{j=1}^{n} b_j(1 - \xi_{nj}t)^{-1}\right|d\mu(t).$$

Now condition (4.4) guarantees that one may uniformly approximate continuous functions on $[-1, 1]$ by sums of the form

$$\sum_{j=1}^{n} b_j(1 - \xi_{nj}t)^{-1};$$

see [**1**, pp. 254-5]. Although there $\xi_{nj} = \xi_j$, that is, the poles are independent of n, the proof goes through without alteration for the present case. Thus

$$\lim_{n\to\infty} f_{\sigma_n}(x) = 0, \quad x \in (-1, 1).$$

The convergence continuation theorems and uniform boundedness then yield the result.

Theorem 4.1 is an obvious consequence of Theorem 4.2. We close the section with an example in which the Hobby-Rice nodes may be computed as zeros of an orthogonal polynomial:

Example 4.4. Let

$$d\mu(t) := dt, \quad t \in [-1, 1],$$

$$\chi_n(t) := \prod_{j=1}^{n} (1 - \xi_{nj}t),$$

$$Q_n(t) := \prod_{j=1}^{n} (t - \eta_{nj}),$$

and

$$w_n(t) := \sqrt{1 - t^2}/\chi_n^2(t), \quad t \in [-1, 1].$$

In the present case, (2.10) takes the form

$$\int_{-1}^{1} \operatorname{sign}(Q_n(t))/(1 - t\xi_{nj})dt = 0, \quad j = 1, 2, \dots n.$$

Taking linear combinations of this last relation, we obtain

$$\int_{-1}^{1} P_{n-1}(t) \operatorname{sign}(Q_n(t))/\chi_n(t)dt = 0,$$

for each polynomial $P_{n-1}(t)$ of degree at most $n - 1$. A classical result of Bernstein (see [**1**, pp. 251-3] and use $\omega := \chi_n^2$ there) shows that $Q_n(t)$ is the monic orthogonal polynomial of degree n for the weight $w_n(t)$. Thus the Hobby-Rice nodes (the zeros of $Q_n(t)$) may be computed as the zeros of an orthogonal polynomial (namely $Q_n(t)$) satisfying for each polynomial $P_{n-1}(t)$ of degree at most $n - 1$,

$$\int_{-1}^{1} P_{n-1}(t)Q_n(t)w_n(t)dt = 0.$$

We now use this fact to bound the coefficients in the interpolant $U_n(x, \alpha)$. To this end, let $\lambda_{n1}, \lambda_{n2} \ldots \lambda_{nn}$ denote the Gauss-Christoffel numbers for the weight $w_n(x)$, and let $S_{n-1}[g](x)$ denote the nth partial sum of the orthonormal expansion of $g(x)$ in orthogonal polynomials with respect to $w_n(x)$, whenever defined. Note that $S_{n-1}[g]$ has degree at most $n - 1$. Given $\alpha(t)$, let us set

$$(4.15) \quad a_{nk} := \lambda_{nk}\chi_n(\eta_{nk})S_{n-1}[\alpha/(\chi_n w_n)](\eta_{nk}), \quad k = 1, 2, \ldots n.$$

Then, if $P_{n-1}(x)$ is a polynomial of degree at most $n - 1$, the Gauss quadrature formula for $w_n(x)$ yields

$$\sum_{k=1}^{n} a_{nk}P_{n-1}(\eta_{nk})/\chi_n(\eta_{nk})$$

$$= \int_{-1}^{1} P_{n-1}(t)S_{n-1}[\alpha/(\chi_n w_n)](t)w_n(t)dt$$

$$= \int_{-1}^{1} P_{n-1}(t)(\alpha/(\chi_n w_n))(t)w_n(t)dt$$

$$= \int_{-1}^{1} \{P_{n-1}(t)/\chi_n(t)\}\alpha(t)dt,$$

by definition of $w_n(t)$ and orthonormality. Since the above relation holds for each polynomial P_{n-1} of degree at most $n - 1$, we obtain

$$\sum_{k=1}^{n} a_{nk}/(1 - \xi_{nj}\eta_{nk}) = \int_{-1}^{1} \alpha(t)/(1 - \xi_{nj}t)dt, \quad j = 1, 2, \ldots n,$$

or

$$U_n(\xi_{nj}, \alpha) = f_\alpha(\xi_{nj}), \quad j = 1, 2, \ldots n,$$

where

$$U_n(x, \alpha) := \sum_{k=1}^{n} a_{nk}/(1 - x\eta_{nk}).$$

Thus in this special case, we have the explicit representation (4.15) for the coefficients in the interpolating polynomial. The theory of product integration rules, which suggested (4.15), **[15]** enables us to estimate the a_{nk}: Using the Gauss-quadrature formula, Cauchy-Schwarz' inequality, Bessel's inequality, and the bound $0 < 1 - \xi_{nn}t < 2$ for $t \in (-1, 1)$, we obtain

$$\sum_{k=1}^{n} |a_{nk}| \leqq \sqrt{2}\left\{\int_{-1}^{1} \chi_n^2(t)(1 - \xi_{nn}t)^{-1}w_n(t)dt\right\}^{1/2}$$

$$\times \left\{\int_{-1}^{1} (\alpha(t)/(\chi_n(t)w_n(t)\,)^2 w_n(t)dt\right\}^{1/2}$$

$$\leqq \sqrt{2}\left\{\int_{-1}^{1} \frac{\sqrt{1-t^2}}{1-|t|}dt\right\}^{1/2}\left\{\int_{-1}^{1} \alpha^2(t)/\sqrt{1-t^2}dt\right\}^{1/2}.$$

This last bound is independent of n and $\hat{\xi}_n$, and may be used to investigate convergence of $U_n(x, \alpha)$ when (2.5) is weakened to

$$\int_{-1}^{1} \alpha^2(t)/\sqrt{1-t^2}dt < \infty.$$

5. Gauss quadrature nodes: double node interpolation. In this section, we briefly discuss convergence of generalized polynomials of the form (1.6) that interpolate to functions of the form (1.4). First, we must define a suitable notion of extended total positivity:

Definition 5.1. Let I and J be real intervals, and let $K(x, t):I \times J \to \mathbf{R}$ be continuous. We say that $K(x, t)$ is ETP(2) in t (extended totally positive of order 2 in t) if

$$K_{01}(x, t) := \frac{\partial}{\partial t}K(x, t)$$

is continuous for $x \in I$, $t \in J$, and if for $n = 1, 2, 3, \ldots$, and $x_1, x_2, \ldots x_n \in I$; $t_1, t_2, \ldots t_n \in J$ satisfying

$$x_1 < x_2 < \ldots < x_n \text{ and } t_1 \leqq t_2 \leqq \ldots \leqq t_n,$$

with at most 2 consecutive t_j equal, we have

$$(5.1)\qquad K^*\begin{bmatrix} x_1 & \ldots & x_n \\ t_1 & \ldots & t_n \end{bmatrix} := \det(K_{0q_j}(x_i, t_j)\,)_{i,j=1}^{n} > 0,$$

where $q_j \in \{0, 1\}$ is defined as

$$(5.2)\qquad q_j := \max\{l{:}t_{j-l} = t_j\},\quad j = 1, 2, \ldots n.$$

Throughout this section, we assume that $K(x, t)$ is ETP(2) in t, and we use

$$(5.3)\qquad \hat{\xi}_{2n} := (\xi_{2n,1}, \xi_{2n,2}, \ldots, \xi_{2n,2n})$$

to denote a $2n$-tuple of interpolation points in I, with

$$(5.4)\qquad \xi_{2n,1} < \xi_{2n,2} < \ldots < \xi_{2n,2n}.$$

Further, throughout this section, $d\mu(t)$ denotes a (possibly atomic) non-negative Borel measure with infinitely many points in its support, such that for each $x \in I$, $f(x)$ defined by (2.4) is finite. Further, we define $f_\alpha(x)$ by (2.6) for $\alpha(t)$ satisfying (2.5).

Given the interpolation points $\hat{\xi}_{2n}$ and an n-tuple of points in I,

$$(5.5) \quad \hat{\tau}_n := (\tau_{n1}, \tau_{n2}, \dots \tau_{nn}): \tau_{n1} < \tau_{n2} < \dots < \tau_{nn},$$

there is a unique generalized polynomial [**9**, Chapter 1] of the form

$$(5.6) \quad V_{2n}(x, \alpha) := \sum_{j=1}^{n} \{a_j K(x, \tau_{nj}) + b_j K_{01}(x, \tau_{nj})\},$$

satisfying

$$(5.7) \quad V_{2n}(\xi_{2n,l}, \alpha) = f_\alpha(\xi_{2n,l}), \quad l = 1, 2, \dots 2n.$$

Here we choose $\hat{\tau}_n$ to be the unique n-tuple of Gauss nodes associated with $d\mu(t)$ and $\hat{\xi}_{2n}$:

Definition 5.2. Given a non-negative Borel measure $d\mu(t)$ on J with infinitely many points in its support and such that for each $x \in I$, $f(x)$ of (2.4) is finite, and given interpolation points $\hat{\xi}_{2n}$ as in (5.3) and (5.4), the *Gauss quadrature nodes* $\hat{\tau}_n$ are the unique nodes satisfying (5.5) and

$$(5.8) \quad \int_J K(\xi_{2n,l}, t) d\mu(t) = \sum_{j=1}^{n} \lambda_{nj} K(\xi_{2n,l}, \tau_{nj}), \quad l = 1, 2, \dots 2n.$$

Here the Gauss weights λ_{nj} satisfy

$$(5.9) \quad \lambda_{nj} > 0, \quad j = 1, 2, \dots n.$$

Note that (5.8) implies that

$$(5.10) \quad V_{2n}(x, 1) = \sum_{j=1}^{n} \lambda_{nj} K(x, \tau_{nj}).$$

Following is our analogue of Theorems 2.3 and 2.5:

THEOREM 5.3. *For each $x \in I$ and $n = 1, 2, 3, \dots,$*

$$(5.11) \quad |f_\alpha(x) - V_{2n}(x, \alpha)| \leqq |f(x) - V_{2n}(x, 1)|.$$

Furthermore, if for each $x \in I$, there exists $j = j(n, x)$ such that

$$\lim_{n \to \infty} \xi_{nj} = x,$$

then, uniformly in compact subsets of the interior of I,

$$(5.12) \quad \lim_{n \to \infty} V_{2n}(x, 1) = f(x).$$

https://doi.org/10.4153/CJM-1988-033-9 Published online by Cambridge University Press

The proof of this theorem is similar to that of Theorem 2.5. Here (2.18) is replaced by [**5**]:

$$(5.13)\quad |f(x) - V_{2n}(x, 1)| = \min \int_J \left| K(x, t) - \sum_{i=1}^{2n} b_i K(\xi_{2n,i}, t) \right| d\mu(t),$$

where the minimum is taken over all $\{b_i\}_{i=1}^{2n}$ such that for $t \in J$,

$$(5.14)\quad \operatorname{sign}\left\{\prod_{j=1}^{2n} (x - \xi_{2n,j})\right\}\left\{K(x, t) - \sum_{i=1}^{2n} b_i K(\xi_{2n,i}, t)\right\} \geqq 0.$$

The convergence follows as in Section 3, with (3.11) replaced by

$$(5.15)\quad \lim_{n\to\infty} \int_{J_1} \left| K(x, t) - K(\xi_{2n,j}, t)\frac{K(x, s)}{K(\xi_{2n,j}, s)} \right| d\mu(t) = 0,$$

where $s \in J\backslash J_1$ is fixed and $\xi_{2n,j}$ approaches x from only one side, to guarantee the correct sign of the difference in (5.15) according to (5.14).

We remark that when J is compact, and $K(x, t)$ is analytic in $x \in D \supset I$ and continuous in $t \in J$, then under conditions (2.14) and (2.15) of Theorem 2.4, one can show that

$$\lim_{n\to\infty} V_{2n}(z, 1) = f(z)$$

uniformly in compact subsets of D. This together with (5.11) yields the analogue of Theorem 2.4.

Finally, we note that for the rational kernel $K(x, t) := 1/(1 - tx)$, the results of Section 4 can be extended to the Gaussian nodes case by replacing $R_n(z)$ in (4.10) by

$$R_n(z) := \prod_{j=1}^{n} (z - \eta_{nj})^2 \Big/ \prod_{j=1}^{2n} (1 - \xi_{nj} z).$$

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