ON HANKEL CONVOLUTORS ON CERTAIN HANKEL TRANSFORMABLE FUNCTION SPACES

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Abstract. In this paper we introduce the spaces of Hankel convolutors. We characterize the dual spaces of certain Hankel transformable function spaces as spaces of Hankel convolutors. Here the Hankel convolution and the Hankel transformation play an important role.

1. Introduction and preliminaries. A. H. Zemanian ([**14**], [**15**] and [**16**]) investigated the Hankel transformation defined by

$$h_\mu(\phi)(y) = \int_0^\infty \sqrt{xy}\, J_\mu(xy)\phi(x)\, dx, \quad y \in (0,\infty),$$

on certain spaces of distributions. The results obtained by A. H. Zemanian for the h_μ-transformation are analogous to other well-known ones for the Fourier transformation ([**13**]). Throughout this paper μ always will denote a real number greater than $-1/2$ and the real interval $(0,\infty)$ will be represented by I.

In [**14**] there was introduced the space $\mathscr{H}_\mu$, that consists of all those complex valued and smooth functions $\phi(x)$, $(x \in I)$, such that

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in I} (1+x^2)^m \left| \left(\frac{1}{x}D\right)^k (x^{-\mu-1/2}\phi(x)) \right| < \infty,$$

for every $m,k \in \mathbb{N}$. $\mathscr{H}_\mu$ is endowed with the topology generated by the family $\{\gamma^\mu_{m,k}\}_{m,k\in\mathbb{N}}$ of seminorms. Then $\mathscr{H}_\mu$ becomes a Fréchet space. Moreover h_μ is an automorphism of $\mathscr{H}_\mu$ provided that $\mu \geq -1/2$.

For every $a > 0$, A. H. Zemanian [**15**] defined the space $\beta_{\mu,a}$ as follows: a complex valued and smooth function $\phi(x)$, $(x \in I)$, is in $B_{\mu,a}$ if and only if $\phi \in \mathscr{H}_\mu$ and $\phi(x) = 0$, for every $x \geq a$. We consider on $\beta_{\mu,a}$ the topology induced in it by $\mathscr{H}_\mu$ and then $\beta_{\mu,a}$ becomes a Fréchet space. The space $\beta_\mu = \bigcup_{a=1}^{\infty} \beta_{\mu,a}$ is endowed with the inductive limit topology. The Hankel transform of β_μ is characterized in [**15**].

Throughout this paper we shall denote by K the following set of functions

$$K = \{M \in C^2([0,\infty)),\ M(0) = M'(0) = 0,\ M'(\infty) = \infty \text{ and } M''(x) > 0,\ x \in I\}.$$

For every $M \in K$ we shall denote by $M^\times$ the Young dual function of M [**7**, p. 18]. Useful properties of functions in K can be found in [**6**].

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Let $M \in K$. We say that a complex valued and smooth function $\phi = \phi(x)$, $x \in I$, is in $\mathcal{H}_{\mu,M}$ when for every $m,k \in \mathbb{N}$ the quantity

$$\gamma_{\mu,M}^{m,k}(\phi) = \sup_{x \in I} e^{M(kx)} \left| \left(\frac{1}{x} D\right)^m [x^{-\mu-1/2}\phi(x)] \right|$$

is finite. $\mathcal{H}_{\mu,M}$ is endowed with the topology associated to the family $\{\gamma_{\mu,M}^{m,k}\}_{m,k \in \mathbb{N}}$ of seminorms. Then $\mathcal{H}_{\mu,M}$ becomes a Fréchet space. It is easy to see that $\mathcal{H}_{\mu,M}$ is contained in $\mathcal{H}_\mu$ and that inclusion is continuous. Moreover, if we define for every $m, k \in \mathbb{N}$ and $\phi \in \mathcal{H}_{\mu,M}$

$$\eta_{\mu,M}^{m,k}(\phi) = \sup_{x \in I} e^{M(kx)} x^{-\mu-1/2} |S_\mu^m \phi(x)|,$$

where $S_\mu = x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2}$, the family $\{\eta_{\mu,M}^{m,k}\}_{m,k \in \mathbb{N}}$ of seminorms generates on $\mathcal{H}_{\mu,M}$ the same topology as the one generated by $\{\gamma_{\mu,M}^{m,k}\}_{m,k \in \mathbb{N}}$. (See Proposition 27 §IV of [**12**].)

The space $\mathcal{Q}_{\mu,M}$ consists of all those complex valued functions Φ satisfying the following two conditions

(i) $z^{-\mu-1/2}\Phi(z)$ is an even, entire function, and

(ii) for every $m \in \mathbb{N}$ and $k \in \mathbb{N} - \{0\}$

$$p_{\mu,M}^{m,k}(\Phi) = \sup_{z \in \mathbb{C}} (1 + |z|^2)^m \, |z^{-\mu-1/2}\Phi(z)| \, e^{-M(|Im z|/k)} < \infty.$$

$\mathcal{Q}_{\mu,M}$ is equipped with the topology associated to the family $\{p_{\mu,M}^{m,k}\}_{m \in \mathbb{N}, k \in \mathbb{N}-\{0\}}$ of seminorms. Then $\mathcal{Q}_{\mu,M}$ becomes a Fréchet space.

In Section 3 we shall establish that the Hankel transformation is a homeomorphism between $\mathcal{H}_{\mu,M}$ onto $\mathcal{Q}_{\mu,M^x}$.

Our objective here is to develop for the spaces β'_μ and $\mathcal{H}'_{\mu,m}$ a theory similar to the one studied by P. Mikusinski and M. D. Taylor [**11**] for the spaces $\mathcal{D}'$ and $K\{M_p\}'$. In our investigation the Hankel convolution plays an important role. This Hankel convolution was introduced in [**8**] and [**9**]. It has been investigated recently in distributions spaces by J. J. Betancor and I. Marrero ([**2**], [**3**], [**4**], [**5**] and [**10**]).

In the sequel $\mathcal{E}$ will denote a complex Banach space. If f is an $\mathcal{E}$-valued measurable function on I and ϕ is a complex function on I, we define the $\#$-convolution $f \# \phi$ of f and ϕ by

$$(f \# \phi)(y) = \int_0^\infty f(x)(\tau_y \phi)(x)\,dx \quad (y \in I),$$

provided that the last Bochner integral exists. Here the Hankel translation τ_y ($y \in I$), is defined by

$$(\tau_y \phi)(x) = \int_0^\infty D_\mu(x, y, z)\phi(z)\,dx \quad (x, y \in I),$$

where

$$D_\mu(x, y, z) = \int_0^\infty t^{-\mu-1/2}(xt)^{1/2} J_\mu(xt)(yt)^{1/2} J_\mu(yt)(zt)^{1/2} J_\mu(zt)\,dt \quad (x, y, z \in I).$$

The space $\mathscr{C}_\mu$ consists of all those $\mathscr{E}$-valued and continuous functions $f(x)$ $(x \in I)$, such that $\lim_{x\to 0^+} x^{-\mu-1/2}f(x)$ exists. C_μ is endowed with the topology generated by the family $\{w_k^\mu\}_{k\in\mathbb{N}-\{0\}}$ of seminorms, where

$$w_k^\mu(f) = \sup_{x\in(0,k)} \|x^{-\mu-1/2}f(x)\| \quad (f \in \mathscr{C}_\mu \text{ and } k \in \mathbb{N}-\{0\}),$$

$\|\ \|$ being the norm in $\mathscr{E}$.

Now we denote by $\mathbf{H}$ either the spaces β_μ or $\mathscr{H}_{\mu,M}$.

We introduce the convolutive dual $\mathbf{H}^{\#}$ of $\mathbf{H}$. A continuous $\mathscr{E}$-valued function f in $\mathscr{C}_\mu$ belongs to $\mathbf{H}^{\#}$ if and only if for every $\phi \in \mathbf{H}$ we have

$$\int_0^\infty \|f(y)\|\,|(\tau_x\phi)(y)|\,dy < \infty \quad (x \in I),$$

and

$$\lim_{x\to 0^+} x^{-\mu-1/2} \textstyle\int_0^\infty \|f(y)\|\,|(\tau_x\phi(y))|\,dy = \alpha_\mu \int_0^\infty \|f(y)\|\,|\phi(y)|\,dy < \infty,$$

where

$$\alpha_\mu = \frac{1}{2^\mu\Gamma(\mu+1)}.$$

By a *convolutor* F on $\mathbf{H}$ we mean a linear mapping from $\mathbf{H}$ into $\mathbf{H}^{\#}$ such that

$$F(\phi\#\psi) = F(\phi)\#\psi, \quad \text{for each } \phi, \psi \in \mathbf{H}.$$

The space of all convolutors on $\mathbf{H}$ will be represented by $\mathscr{G}(\mathbf{H})$. We shall say that a sequence $\{F_n\}_{n\in\mathbb{N}} \subset \mathscr{G}(\mathbf{H})$ *converges to* $F \in \mathscr{G}(\mathbf{H})$ provided that for every $\phi \in \mathbf{H}$, $F_n(\phi) \to F(\phi)$, as $n \to \infty$, in $\mathscr{C}_\mu$.

$\mathbf{H}'$ will denote the space of continuous linear mappings from $\mathbf{H}$ into $\mathscr{E}$, and we shall consider on $\mathbf{H}'$ the topology of pointwise convergence.

In this paper we shall characterize the spaces β'_μ (Section 2) and $\mathscr{H}'_{\mu,M}$ (Section 4) as the spaces of convolutors of β_μ and $\mathscr{H}_{\mu,M}$, respectively.

Throughout this paper, C will denote a suitable positive constant (not necessarily the same in each occurrence).

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2. The space β'_μ. Firstly we describe the convolutive dual space of β_μ.

PROPOSITION 2.1. $\beta_\mu^{\#} = C_\mu$.

Proof. It is obvious that $\beta_\mu^{\#}$ is contained in $\mathscr{C}_\mu$. Let now $f \in \mathscr{C}_\mu$ and let $\phi \in \beta_{\mu,a}$, with $a > 0$. According to Corollary 3.3 of [**2**], $\tau_x\phi \in \beta_{\mu,a+x}$, for every $x \in I$. Hence one has

$$\int_0^\infty \|f(y)\|\,|(\tau_x\phi)(y)|\,dy = \int_0^{a+x} \|f(y)\|\,|(\tau_x\phi)(y)|\,dy < \infty \quad (x \in I).$$

Moreover

$$\lim_{x\to 0^+} x^{-\mu-1/2}(\tau_x\phi)(y) = \alpha_\mu\phi(y) \tag{1}$$

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uniformly in $y \in I$. In effect, by invoking (1.2) of [3]

$$x^{-\mu-1/2}(\tau_x\phi)(y) = h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(\phi)(t)](y) \quad (x, y \in I).$$

Since the functions $\sqrt{z}\,J_\mu(z)$ and $z^{-\mu}J_\mu(z)$ are bounded in I, and since $h_\mu(\phi) \in \mathscr{H}_\mu$ by Theorem 5.4-1 of [16], for every $b > 0$ and $m \in \mathbb{N}$ we have

$$\int_b^\infty |\sqrt{yt}\,J_\mu(yt)(xt)^{-\mu}J_\mu(xt)|\,|h_\mu(\phi)(t)|\,dt \leq C\gamma^\mu_{m,0}(h_\mu(\phi))\int_b^\infty \frac{t^{\mu+1/2}}{(1+t^2)^m}\,dt \quad (x, y \in I).$$

Hence, for each $\varepsilon > 0$ there exists $b_0 > 0$ such that

$$\left|\int_{b_0}^\infty \sqrt{yt}\,J_\mu(yt)[(xt)^{-\mu}J_\mu(xt) - \alpha_\mu]h_\mu(\phi)(t)\,dt\right| < \varepsilon \quad (x, y \in I). \tag{2}$$

Finally by taking into account that $\lim_{z\to 0^+} z^{-\mu}J_\mu(z) = \alpha_\mu$ we can deduce that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\int_0^{b_0} \sqrt{yt}\,J_\mu(yt)[(xt)^{-\mu}J_\mu(xt) - \alpha_\mu]h_\mu(\phi)(t)\,dt\right| < \varepsilon \tag{3}$$

when $y \in I$ and $0 < x < \delta$.

By combining (2) and (3) we get (1).

Then

$$\lim_{x\to 0^+} x^{-\mu-1/2}\int_0^\infty \|f(y)\|\,|(\tau_x\phi)(y)|\,dy = \alpha_\mu\int_0^\infty \|f(y)\|\,|\phi(y)|\,dy.$$

Thus the proof is complete. ■

In the next Proposition we characterize the convergence in β_μ in terms of convolutors.

Proposition 2.2. *Let $(\phi_n)_{n=0}^\infty$ be a sequence in β_μ. Then $\phi_n \to 0$ as $n \to \infty$ in β_μ, if and only if $F(\phi_n) \to 0$ as $n \to \infty$ in $\mathscr{C}_\mu$, for every $F \in \mathscr{G}(\beta_\mu)$.*

Proof. To prove that $\phi_n \to 0$ as $n \to \infty$ in β_μ implies that $F(\phi_n) \to 0$ as $n \to \infty$ in $\mathscr{C}_\mu$ for every $F \in \mathscr{G}(\beta_\mu)$, we invoke the closed graph theorem. Assume that F is a convolutor on β_μ and $(\psi_n)_{n=0}^\infty$ is a sequence in β_μ such that for certain $\psi \in B_\mu$ and $f \in C_\mu$

$$\psi_n \to \psi \text{ in } \beta_\mu \text{ and } F(\psi_n) \to f \text{ in } \mathscr{C}_\mu \text{ as } n \to \infty.$$

We have for every $\varphi \in \beta_\mu$

$$F(\psi_n)\#\varphi = F(\varphi)\#\psi_n \to F(\varphi)\#\psi = F(\psi)\#\varphi \text{ as } n \to \infty \text{ in } \mathscr{C}_\mu.$$

Indeed, since $\psi_n \to \psi$ as $n \to \infty$ in β_μ, there exists $a > 0$ such that $\psi_n \in \beta_{\mu,a}$ ($n \in \mathbb{N}$) and $\psi \in \beta_{\mu,a}$. Then, for every $n \in \mathbb{N}$, $m \in \mathbb{N} - \{0\}$, by invoking again Corollary 3.3 of [**2**] we can write

$$\|x^{-\mu-1/2}[F(\varphi)\#(\psi_n - \psi)](x)\| \leq \int_0^{a+m} \|F(\varphi)(y)\|\, x^{-\mu-1/2}\, |\tau_x(\psi_n - \psi)(y)|\, dy$$

$$\leq C \sup_{y \in I} |h_\mu[(xt)^{-\mu} J_\mu(xt) h_\mu(\psi_n - \psi)(t)](y)|,$$

for every $x \in (0, m)$.

Hence, since h_μ is an automorphism on $\mathscr{H}_\mu$ by Theorem 5.4-1 of [**16**] and $(xt)^{-\mu}J_\mu(xt)$ is a multiplier on $\mathscr{H}_\mu$, uniformly in $x \in (0, m)$, one concludes that

$$\sup_{x \in (0,m)} x^{-\mu-1/2} \|[F(\varphi)\#(\psi_n - \psi)](x)\| \to 0 \quad \text{as } n \to \infty.$$

Moreover, it is not hard to see that $F(\psi_n)\#\varphi \to f\#\varphi$ as $n \to \infty$ in $\mathscr{C}_\mu$. Hence $F(\psi)\#\varphi = f\#\varphi$, ($\varphi \in \beta_\mu$). Then, from (1) it is deduced that

$$\lim_{x \to 0^+} x^{-\mu-1/2} \int_0^\infty (F(\psi) - f)(y)(\tau_x\varphi)(y)\, dy = \alpha_\mu \int_0^\infty (F(\psi) - f)(y)\varphi(y)\, dy = 0,$$

for every $\varphi \in \beta_\mu$. Therefore $F(\psi) = f$.

Assume now that $(\phi_n)_{n=0}^\infty$ is a sequence in β_μ such that $F(\phi_n) \to 0$ as $n \to \infty$ in $\mathscr{C}_\mu$, for each $F \in \mathscr{G}(\beta_\mu)$. Since β_μ is the hyperstrict inductive limit of the family $\{\beta_{\mu,a}\}_{a>0}$, $\phi_n \to 0$ as $n \to \infty$ in β_μ if and only if the following two conditions hold.

(i) There exists $a > 0$ such that $\phi_n \in \beta_{\mu,a}$, for every $n \in \mathbb{N}$.

(ii) $\sup_{x \in (0,m)} |x^{-\mu-1/2} S_\mu^k \phi_n(x)| \to 0$, as $n \to \infty$, for every $k \in \mathbb{N}$, $m \in \mathbb{N} - \{0\}$.

Note that, by virtue of Proposition 2.2 of [**10**], for every $k \in \mathbb{N}$ the mapping F_k defined by

$$F_k : \beta_\mu \to \mathscr{C}_\mu,$$
$$\phi \to S_\mu^k \phi \,.\, u,$$

where $u \in \mathscr{E}$, is a convolutor in β_μ. Hence our assumption implies that (ii) holds. Suppose that $\phi_n \not\to 0$ as $n \to \infty$ in β_μ. Then there exist an increasing sequence $(x_n)_{n=0}^\infty$ in I and an increasing sequence $(q_n)_{n=0}^\infty$ in $\mathbb{N}$ such that $x_n \to \infty$ as $n \to \infty$, $|\phi_{q_n}(x_n)| > 0$, for each $n \in \mathbb{N}$, and $\phi_{q_k}(x_n) = 0$, for every $n, k \in \mathbb{N}$ with $k < n$.

We define the mapping F in β_μ by

$$F(\phi)(x) = \sum_{n=0}^\infty (\tau_x\phi)(x_n) v_n \ (\phi \in \beta_\mu \text{ and } x \in I),$$

$(v_n)_{n=0}^\infty$ being a sequence in $\mathscr{E}$. Thus F is in $\mathscr{G}(\beta_\mu)$. In effect, let $\phi \in \beta_{\mu,a}$ with $a > 0$. For every $x \in (0, m)$, where $m \in \mathbb{N} - \{0\}$, the sum defining $F(\phi)(x)$ is finite (and the number of non-zero terms does not depend on $x \in (0, m)$) because $\tau_x\phi \in \beta_{\mu,a+m}$, $x \in (0, m)$, by Corollary 3.3 of [**2**]. Then $\lim_{x \to 0^+} x^{-\mu-1/2} F(\phi)(x) = \alpha_\mu \sum_{n=0}^\infty \phi(x_n) v_n$. Also it is not very hard to establish that $F(\phi)$ is a continuous function on I. Thus we conclude that $F(\phi) \in \mathscr{C}_\mu$.

Moreover if $\phi, \psi \in \beta_\mu$, then since $(\tau_x\phi)(y) = (\tau_y\phi)(x) (x, y \in I)$ and since $\tau_x(\phi\#\psi) = (\tau_x\phi)\#\psi$ $(x \in I)$, one has

$$F(\phi\#\psi)(x) = \sum_{n=0}^{\infty} \tau_{x_n}(\phi\#\psi)(x)v_n = \sum_{n=0}^{\infty} [(\tau_{x_n}\phi)\#\psi](x)v_n = \left(\sum_{n=0}^{\infty} (\tau_{x_n}\phi)v_n\right)\#\psi(x)$$
$$= (F(\phi)\#\psi)(x) \quad (x \in I). \tag{4}$$

To justify the third equality in (4) we have taken into account that, according to Corollary 3.3 of [2], if $\phi, \psi \in \beta_{\mu,a}$ with $a > 0$, then for every $x \in I$ we have

$$[(\tau_{x_n}\phi)\#\psi](x) = \int_0^{a+x} (\tau_{x_n}\phi)(z)(\tau_x\psi)(z)\,dz.$$

Also by virtue of §2 of [9], we can write

$$(\tau_{x_n}\phi)(z) = \int_{|x_n - z|}^{x_n+z} D(x_n, z, t)\phi(t)\,dt = 0$$

when $x \in I$, $z \in (0, a+x)$ and n is large enough. Hence we conclude that for fixed $x \in I$ $[(\tau_{x_n}\phi)\#\psi](x) = 0$ when n is sufficiently large. Thus (4) is established.

On the other hand note that

$$\lim_{x\to 0^+} x^{-\mu-1/2}F(\phi_{q_k})(x) = \alpha_\mu \sum_{n=0}^{\infty} \phi_{q_k}(x_n)v_n \quad (k \in \mathbb{N}).$$

Hence, by choosing $(v_n)_{n=0}^{\infty}$ suitably, we obtain $F(\phi_{q_k}) \not\to 0$, as $k \to \infty$ in $\mathscr{C}_\mu$. Thus the proof is finished. ∎

We now establish that $\mathscr{G}(\beta_\mu)$ and β'_μ are isomorphic.

Proposition 2.3. *The spaces $\mathscr{G}(\beta_\mu)$ and β'_μ are isomorphic.*

Proof. The mapping J defined by

$$J : \mathscr{G}(\beta_\mu) \to \beta'_\mu,$$
$$F \to J(F) : \beta_\mu \to \mathscr{C},$$
$$\phi \to J(F)(\phi) = \lim_{x\to 0^+} \frac{1}{\alpha_\mu} x^{-\mu-1/2}F(\phi)(x),$$

is an algebraic and sequential isomorphism.

Let $F \in \mathscr{G}(\beta_\mu)$. If $(\phi_n)_{n=0}^{\infty} \subset \beta_\mu$ is such that $\phi_n \to 0$ as $n \to \infty$ in β_μ then, according to Proposition 2.2, $F(\phi_n) \to 0$ as $n \to \infty$ in $\mathscr{C}_\mu$. Hence $J(F)(\phi_n) \to 0$ as $n \to \infty$. Thus we have proved that $J(F) \in \beta'_\mu$.

It is obvious that J is a linear mapping. Moreover if $F \in \mathscr{G}(\beta_\mu)$ and $J(F) = 0$, then for every $\phi, \psi \in \beta_\mu$ one has

$$J(F)(\phi\#\psi) = \lim_{x\to 0^+} \frac{1}{\alpha_\mu} x^{-\mu-1/2}F(\phi\#\psi)(x) = \lim_{x\to 0^+} \frac{1}{\alpha_\mu} x^{-\mu-1/2}(F(\phi)\#\psi)(x)$$
$$= \frac{1}{\alpha_\mu} \lim_{x\to 0^+} \int_0^\infty F(\phi)(y)x^{-\mu-1/2}(\tau_x\psi)(y)\,dy = \int_0^\infty F(\phi)(y)\psi(y)\,dy = 0.$$

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Hence $F(\phi)=0$, for every $\phi \in \beta_\mu$. Therefore J is one to one.

Also J is onto. In effect, let $f \in \beta'_\mu$. We define the mapping F in β_μ through

$$F(\phi)(x)=(f\#\phi)(x) \quad (x \in I, \phi \in \beta_\mu).$$

By proceeding as in the proof of Proposition 4.1 of [2] we can see that $F(\phi) \in \mathscr{C}_\mu$, $\phi \in \beta_\mu$, and from Corollary 4.2 of [2] we infer that $F \in \mathscr{G}(\beta_\mu)$. Moreover

$$\lim_{x\to 0^+} x^{-\mu-1/2}(f\#\phi)(x)=\langle f,\phi\rangle$$

because $\lim_{x\to 0^+} x^{-\mu-1/2}\tau_x\phi=\alpha_\mu\phi$ in β_μ. Hence $J(F)=f$.

Let now $(F_n)_{n=0}^\infty \in \mathscr{G}(\beta_\mu)$ be such that $F_n \to 0$ as $n\to\infty$ in $\mathscr{G}(\beta_\mu)$. Then $F_n(\phi)\to 0$ as $n\to\infty$ in $\mathscr{C}_\mu$, for every $\phi \in \beta_\mu$. Hence we immediately deduce that $J(F_n)(\phi)\to 0$ as $n\to\infty$, for every $\phi\in\beta_\mu$. Thus it is established that J is sequentially continuous.

Finally to see that J^{-1} is sequentially continuous it is sufficient to note that if $f_n\to 0$ as $n\to\infty$ in β'_μ then $f_n\#\phi\to 0$ as $n\to\infty$ in $\mathscr{C}_\mu$, for every $\phi\in\beta_\mu$. (See Proposition 2.11 and Proposition 4.1 of [2].) ■

3. Hankel transformation and Hankel convolution on $\mathscr{H}_{\mu,M}$. We begin this section by establishing that the Hankel transformation maps $\mathscr{H}_{\mu,M}$ onto $\mathcal{Q}_{\mu,M^\times}$ homeomorphically.

PROPOSITION 3.1. *The Hankel transformation h_μ is an homeomorphism from $\mathscr{H}_{\mu,M}$ onto $\mathcal{Q}_{\mu,M^\times}$. Moreover h_μ^{-1}, the inverse of h_μ, is given by*

$$h_\mu^{-1}(\Phi)(y)=\int_0^\infty \sqrt{xy}\,J_\mu(xy)\Phi(x)\,dx \quad (\Phi\in\mathcal{Q}_{\mu,M^\times}).$$

Proof. Let firstly ϕ be in $\mathscr{H}_{\mu,M}$. By invoking (5.3b) of [6] we write

$$\int_0^\infty |(zt)^{-\mu}J_\mu(zt)|\,|t^{\mu+1/2}\phi(t)|\,dt \le C\int_0^\infty e^{t|Im\,z|}t^{\mu+1/2}\,|\phi(t)|\,dt \quad (z\in\mathbb{C}).$$

Hence if $|\mathrm{Im}\,z|\le k$, then from Lemma 2.4 of [6] it follows that

$$\int_0^\infty |(zt)^{-\mu}J\mu(zt)t^{\mu+1/2}\phi(t)|\,dt \le C\int_0^\infty e^{kt}t^{\mu+1/2}\,|\phi(t)|\,dt \le Ce^{M^\times(k+1)}\gamma^{0,1}_{\mu,M}(\phi)\int_0^\infty e^{-t}t^{2\mu+1}\,dt.$$

Thus we prove that $z^{-\mu-1/2}h_\mu(\phi)(z)$ admits a continuous extension to $\mathbb{C}$. Moreover, in a similar way, we can see that such a extension is holomorphic. It is clear that $z^{-\mu-1/2}h_\mu(\phi)(z)$ is even. According to Lemma 5.4-1 of [16], for every $i\in\mathbb{N}$ one has

$$z^{2i-\mu-1/2}h_\mu(\phi)(z)=(-1)^i\int_0^\infty (zt)^{-\mu}J_\mu(zt)t^{\mu+1/2}S^i_\mu\phi(t)\,dt \quad (z\in\mathbb{C}).$$

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Hence, by invoking again (5.3.b), Lemma 2.4 and (2.2) of [6], we obtain

$$|z^{2i-\mu-1/2}h_\mu(\phi)(z)| \leq C\int_0^\infty e^{t|\mathrm{Im}\,z|}t^{\mu+1/2}|S_\mu^i\phi(t)|\,dt \leq Ce^{M^\times(|\mathrm{Im}\,z|/k)}\int_0^\infty e^{M(kt)}t^{\mu+1/2}|S_\mu^i\phi(t)|\,dt$$

$$\leq Ce^{M^\times(|\mathrm{Im}\,z|/k)}\eta_{\mu,M}^{i,k+1}(\phi)\int_0^\infty e^{-M(t)}t^{2\mu+1}\,dt \quad (i \in \mathbb{N}, k \in \mathbb{N}-\{0\} \text{ and } z \in \mathbb{C}).$$

Thus we have proved that

$$p_{\mu,M^\times}^{m,k}(h_\mu\phi) \leq C\sum_{i=0}^m \eta_{\mu,M}^{i,k+1}(\phi) \quad (m \in \mathbb{N}, k \in \mathbb{N}-\{0\}).$$

Then h_μ is a continuous mapping from $\mathcal{H}_{\mu,M}$ into $\mathcal{Q}_{\mu,M^\times}$.

Let now Φ be in $\mathcal{Q}_{\mu,M^\times}$. According to Lemma 6.1 of [6]

$$t^{-\mu-1/2}h_\mu(\Phi)(t) = \frac{1}{2}\int_{-\infty}^\infty (t(\xi+i\eta))^{-\mu}H_\mu^{(1)}(t(\xi+i\eta))\Phi(\xi+i\eta)(\xi+i\eta)^{\mu+1/2}\,d\xi \quad (t \in I), \tag{5}$$

for every $\eta \in I$. Here $H_\mu^{(1)}$ denotes the Hankel function of the first class and order μ.

Let $m \in \mathbb{N}$. From (5.1.b) of [6] we infer that, for every t, $\eta \in I$,

$$\left(\frac{1}{t}D\right)^m[t^{-\mu-1/2}h_\mu(\Phi)(t)] = \frac{(-1)^m}{2}\int_{-\infty}^\infty (t(\xi+i\eta))^{-\mu-m}H_{\mu+m}^{(1)}(t(\xi+i\eta))\Phi(\xi+i\eta)$$

$$\times(\xi+i\eta)^{2m+\mu+1/2}\,d\xi.$$

Assume now that $\mu \geq 1/2$. According to (5.3.c) of [6], we have

$$\left|\left(\frac{1}{t}D\right)^m(t^{-\mu-1/2}h_\mu(\Phi)(t))\right| \leq C\bigg(\int_{|t(\xi+i\eta)|\leq 1}|\Phi(\xi+i\eta)|\,|\xi+i\eta|^{-\mu+1/2}t^{-2\mu-2m}\,d\xi$$

$$+\int_{|t(\xi+i\eta)|>1}e^{-t\eta}t^{-\mu-m-1/2}|\Phi(\xi+i\eta)|\,|\xi+i\eta|^m\,d\xi\bigg) \quad (t, \eta \in I).$$

Hence if $t \geq 1$, then

$$\left|\left(\frac{1}{t}D\right)^m(t^{-\mu-1/2}h_\mu(\Phi)(t)\right| \leq C\bigg(\int_{|t(\xi+i\eta|\leq 1}e^{-t\eta}|\Phi(\xi+i\eta)|\,|\xi+i\eta|^{-\mu+1/2}\,d\xi$$

$$+\int_{|t(\xi+i\eta)|>1}e^{-t\eta}|\Phi(\xi+i\eta)|\,|\xi+i\eta|^m\,d\xi\bigg) \quad (\eta \in I).$$

If $k \in \mathbb{N}-\{0\}$, then by invoking now Lemma 2.4 of [6] and by taking $\eta \in I$ such that $M'(kt) = \dfrac{\eta}{k}$, we obtain

$$\left|\left(\frac{1}{t}D\right)^m(t^{-\mu-1/2}h_\mu(\Phi)(t)\right| \leq Cp_{\mu,M^\times}^{l,k}(\Phi)e^{-M(kt)} \quad (t \geq 1), \tag{6}$$

where $l \in \mathbb{N}$ is such that $2l > m+\mu+3/2$.

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On the other hand if $t \in (0,1)$, then by using (7) §5.1 of **[16]** we can write

$$\left(\frac{1}{t}D\right)^m (t^{-\mu-1/2}h_\mu(\Phi)(t)) = (-1)^m \int_0^\infty (xt)^{-\mu-m}J_{\mu+m}(xt)x^{\mu+1/2+2m}\Phi(x)\,dx.$$

Hence, for every $k \in \mathbb{N}-\{0\}$, one has

$$\left|\left(\frac{1}{t}D\right)^m (t^{-\mu-1/2}h_\mu(\Phi)(t)\right| \le Cp^{l+m,k}_{\mu,m}(\Phi)e^{-M(kt)}\ (t \in (0,1)), \tag{7}$$

where $l \in \mathbb{N}$ with $l > \mu+1$.

From (6) and (7) we conclude that h_μ is a continuous mapping from $\mathcal{Q}_{\mu,M^\times}$ into $\mathcal{H}_{\mu,M}$.

When $-1/2 < \mu < 1/2$ we can proceed in a similar way by using (5.3.d) of **[6]**. Thus the proof is finished. ■

In the following Lemma we prove that $z^{-\mu}J_\mu(z)$ defines a multiplier of $\mathcal{Q}_{\mu,M}$.

LEMMA 3.1. *Let $a > 0$. The mapping $\Phi(z) \to (az)^{-\mu}J_\mu(az)\Phi(z)$ is continuous from $\mathcal{Q}_{\mu,M}$ into itself.*

Proof. According to (5.3.b) and Lemma 2.4 of **[6]** one has, for each $k \in \mathbb{N}-\{0\}$,

$$|(az)^{-\mu}J_\mu(az)| \le Ce^{a|\mathrm{Im}\,z|} \le Ce^{M(|\mathrm{Im}\,z|/2k)} \quad (z \in \mathbb{C}).$$

Hence, for every $m \in \mathbb{N}$ and $k \in \mathbb{N}-\{0\}$, we obtain from (2.2) of **[6]**

$$\begin{aligned}(1+|z|^2)^m e^{-M(|\mathrm{Im}\,z|/k)}\,|(az)^{-\mu}J_\mu(az)\Phi(z)| &\le C(1+|z|^2)^m e^{M(|\mathrm{Im}\,z|/2k)-M(|\mathrm{Im}\,z|/k)}\,|\Phi(z)| \\ &\le C(1+|z|^2)^m e^{-M(|\mathrm{Im}\,z|/2k)}\,|\Phi(z)| \quad (z \in \mathbb{C}).\end{aligned}$$

Thus the desired result follows. ■

Since $\mathcal{H}_{\mu,M}$ is contained in $\mathcal{H}_\mu$, by invoking Proposition 2.1 of **[10]** we deduce that $\tau_x\phi \in \mathcal{H}_\mu$, for every $\phi \in \mathcal{H}_{\mu,M}$. Moreover, according to (2.1) of **[10]**, for every $\phi \in \mathcal{H}_{\mu,M}$ we have

$$h_\mu(\tau_x\phi)(z) = x^{\mu+1/2}(xz)^{-\mu}J_\mu(xz)h_\mu(\phi)(z) \quad (x \in I \text{ and } z \in \mathbb{C}).$$

From Lemma 3.1 and Proposition 3.1 we immediately deduce the following result.

PROPOSITION 3.2. *For every $x \in I$, τ_x defines a continuous linear mapping from $\mathcal{H}_{\mu,M}$ into itself.* ■

Also it is clear that $z^{-\mu-1/2}\Phi$ is a multiplier of $\mathcal{Q}_{\mu,M}$, provided that $\Phi \in \mathcal{Q}_{\mu,M}$. Hence, from the interchange formula (1.3) of **[10]** and Proposition 3.1, we deduce the next result.

PROPOSITION 3.3. *The Hankel convolution is a continuous bilinear mapping from $\mathcal{H}_{\mu,M} \times \mathcal{H}_{\mu,M}$ into $\mathcal{H}_{\mu,M}$.*

The following result will be useful in the sequel. ■

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PROPOSITION 3.4. *Let $\phi \in \mathcal{H}_{\mu,M}$. The mapping T_ϕ defined by*

$$T_\phi : [0, \infty) \to \mathcal{H}_{\mu,M}$$
$$x \to x^{-\mu-1/2}\tau_x\phi, \quad \text{if } x \in I,$$
$$\alpha_\mu\phi, \quad \text{if } x = 0,$$

is continuous.

Proof. Let $x_0 \in I$. According to (2.1) of [**10**] and Proposition 3.1, to see that T_ϕ is continuous in x_0 it is sufficient to prove that

$$[(xz)^{-\mu}J_\mu(xz) - (x_0z)^{-\mu}J_\mu(x_0z)]\Phi(z) \to 0 \text{ as } x \to x_0 \text{ in } \mathcal{Q}_{\mu,M^\times}. \tag{8}$$

By invoking (5.3.b) of [**6**] we conclude that

$$|(xz)^{-\mu}J_\mu(xz) - (x_0z)^{-\mu}J_\mu(x_0z)| \le C(e^{x|\operatorname{Im} z|} + e^{x_0|\operatorname{Im} z|})$$
$$\le Ce^{(x_0+1)|\operatorname{Im} z|} \quad (z \in \mathbb{C} \text{ and } x \in (0, x_0+1)).$$

Let $m \in \mathbb{N}$ and $k \in \mathbb{N} - \{0\}$. By invoking Lemma 2.4 and (2.2) of [**6**] we obtain

$$(1+|z|^2)^m\, |z^{-\mu-1/2}\Phi(z)[(xz)^{-\mu}J_\mu(xz) - (x_0z)^{-\mu}J_\mu(x_0z)]|\, e^{-M^\times(|\operatorname{Im} z|/k)}$$
$$\le C(1+|z|^2)^m\, |z^{-\mu-1/2}\Phi(z)|\, e^{(x_0+1)|\operatorname{Im} z| - M^\times(|\operatorname{Im} z|/k)}$$
$$\le C(1+|z|^2)^m\, |z^{-\mu-1/2}\Phi(z)|\, e^{M^\times(|\operatorname{Im} z|/2k) - M^\times(|\operatorname{Im} z|/k)}$$
$$\le C(1+|z|^2)^{-1} p^{m+1,2k}_{\mu,M^\times}(\Phi) \quad (z \in \mathbb{C} \text{ and } x \in (0, x_0+1)).$$

Hence, if $\varepsilon > 0$ there exists $r > 0$ such that for every $x \in (0, x_0+1)$ and $|z| \ge r$

$$(1+|z|^2)^m\, |z^{-\mu-1/2}\Phi(z)[(xz)^{-\mu}J_\mu(xz) - (x_0z)^{-\mu}J_\mu(x_0z)]|\, e^{-M^\times(|\operatorname{Im} z|/k)} < \varepsilon. \tag{9}$$

Moreover, since the function $z^{-\mu}J_\mu(z)$ is uniformly continuous in each compact subset of the complex plane, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in (0, \infty)$ with $|x - x_0| < \delta$ and $|z| \le r$ we have

$$(1+|z|^2)^m\, |z^{-\mu-1/2}\Phi(z)[(xz)^{-\mu}J_\mu(xz) - (x_0z)^{-\mu}J_\mu(x_0z)]|\, e^{-M^\times(|\operatorname{Im} x|/k)} < \varepsilon. \tag{10}$$

By combining (9) and (10) we establish (8).

To see that T_ϕ is continuous at the origin we can proceed in a similar way. ∎

4. The space $\mathcal{H}'_{\mu,M}$. In this section we investigate some new properties of the spaces $\mathcal{H}_{\mu,M}$ and $\mathcal{H}'_{\mu,M}$. Also we characterize $\mathcal{H}'_{\mu,M}$ as the space of Hankel convolutors on $\mathcal{H}_{\mu,M}$.

If $F \in \mathcal{H}'_{\mu,M}$ and $\phi \in \mathcal{H}_{\mu,M}$, we define the function $F\#\phi$ from I into $\mathscr{E}$ by

$$(F\#\phi)(x) = \langle F(y), (\tau_x\phi)(y)\rangle \quad (x \in I).$$

Note that a consequence of Proposition 3.4 is that $F\#\phi \in C_\mu$, for every $F \in \mathcal{H}'_{\mu,M}$ and $\phi \in \mathcal{H}_{\mu,M}$.

Next we establish an interesting result on pointwise convergence in $\mathcal{H}'_{\mu,M}$.

PROPOSITION 4.1. *Let $\{F_n\}_{n=0}^\infty$ be a sequence in $\mathcal{H}'_{\mu,M}$. The following two conditions are equivalent:*

(i) *$\langle F_n, \phi\rangle \to 0$ as $n \to \infty$ in $\mathscr{E}$, for every $\phi \in \mathcal{H}_{\mu,M}$;*

(ii) *$F_n\#\phi \to 0$ as $n \to \infty$ in $\mathscr{C}_\mu$, for every $\phi \in \mathcal{H}_{\mu,M}$.*

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Proof. We shall prove first that (i) implies (ii).

Let $\phi \in \mathscr{H}_{\mu,M}$ and $m \in \mathbb{N}$. According to Proposition 3.4, the set $\{x^{-\mu-1/2}\tau_x\phi\}_{x\in[0,m]}$ is compact in $\mathscr{H}_{\mu,M}$. By invoking the Banach-Steinhauss theorem we conclude that

$$\sup_{x\in[0,m]} \|\langle F_n, x^{-\mu-1/2}\tau_x\phi\rangle\| = \sup_{x\in[0,m]} x^{-\mu-1/2}\|(F_n\#\phi)(x)\| \to 0$$

as $n \to \infty$. Hence $F_n\#\phi \to 0$ as $n \to \infty$ in $\mathscr{C}_\mu$.

To see that (ii) implies (i) it is sufficient to note that, according to Proposition 3.4,

$$\lim_{x\to 0^+} x^{-\mu-1/2}\langle F_n, \tau_x\phi\rangle = \langle F_n, \alpha_\mu\phi\rangle,$$

where $n \in \mathbb{N}$ and $\phi \in \mathscr{H}_{\mu,M}$. ■

We now characterize the convolutive dual $\mathscr{H}^\#_{\mu,M}$ of $\mathscr{H}_{\mu,M}$. An $\mathscr{E}$-valued function f defined on I is said to be in $L_1(I,\mathscr{E})$ if and only if $\|f\|$ belongs to $L_1(I)$.

Proposition 4.2. $\mathscr{H}^\#_{\mu,M} = \left\{f \in \mathscr{C}_\mu : \dfrac{x^{\mu+1/2}f}{e^{M(px)}} \in L_1(I,\mathscr{E}), \textit{ for some } p \in \mathbb{N}\right\}.$

Proof. Let $f \in \mathscr{C}_\mu$ be such that $\dfrac{x^{\mu+1/2}f}{e^{M(px)}} \in L_1(I,\mathscr{E})$, where $p \in \mathbb{N}$ and let $\phi \in \mathscr{H}_{\mu,M}$. We can write, for every $x \in I$,

$$\int_0^\infty \|f(y)\|\,|\tau_x\phi(y)|\,dy \le \sup_{y\in I} e^{M(py)}|y^{-\mu-1/2}(\tau_x\phi)(y)| \int_0^\infty \frac{y^{\mu+1/2}\|f(y)\|}{e^{M(py)}}\,dy.$$

Hence, according to Proposition 3.2, one infers that

$$\int_0^\infty \|f(y)\|\,|\tau_x\phi(y)|\,dy < \infty \quad (x \in I).$$

Moreover, by virtue of Proposition 3.4, we have

$$\lim_{x\to 0^+} x^{-\mu-1/2}\int_0^\infty \|f(y)\|\,|\tau_x\phi(y)|\,dy = \alpha_\mu \int_0^\infty \|f(y)\|\,|\phi(y)|\,dy < \infty.$$

Hence $f \in \mathscr{H}^\#_{\mu,M}$. Conversely, assume now $f \in \mathscr{H}^\#_{\mu,M}$ and suppose that there is no $p \in \mathbb{N}$ such that $\dfrac{x^{\mu+1/2}f}{e^{M(px)}} \in L_1(I,\mathscr{E})$. By defining $g(x) = \|f(x)\|$, $(x \in I)$, it is obvious that we cannot find $p \in \mathbb{N}$ for which $\dfrac{x^{\mu+1/2}g(x)}{e^{M(px)}} \in L_1(I)$. We choose $\phi \in \beta_\mu$ such that $\phi \not\equiv 0$, $\phi(x) \ge 0$, $(x \in I)$, and $\phi(x) = 0$, $(x > 1)$.

For every $n \in \mathbb{N} - \{0\}$, $p \in \mathbb{N}$ one has

$$\int_0^\infty g(x)\left(\frac{\chi_{(0,n)}t^{\mu+1/2}}{e^{M(pt)}}\#\phi\right)(x)\,dx = \int_0^\infty g(x)\int_0^1 \phi(y)\tau_x\left(\frac{\chi_{(0,n)}t^{\mu+1/2}}{e^{M(pt)}}\right)(y)\,dy\,dx.$$

Moreover

$$\tau_x\left(\frac{\chi_{(0,n)}t^{\mu+1/2}}{e^{M(pt)}}\right)(y) = \int_0^{x+y} \frac{\chi_{(0,n)}(z)z^{\mu+1/2}}{e^{M(pz)}} D_\mu(x,y,z)\,dz \quad (x,y \in I).$$

Hence since $D_\mu(x,y,z) \geq 0$, $(x,y,z \in I)$ by (3) of §2 of [**9**], and since

$$\int_0^\infty D_\mu(x,y,z)z^{\mu+1/2}\,dz = \alpha_\mu(xy)^{\mu+1/2} \quad (x,y \in I)$$

by (2) of Section 2 of [**9**], we get

$$\begin{aligned}\int_0^\infty g(x)\left(\frac{\chi_{(0,n)}t^{\mu+1/2}}{e^{M(pt)}} \# \phi\right)(x)\,dx &\geq \int_0^{n-1} g(x)\left(\frac{t^{\mu+1/2}}{e^{M(pt)}} \# \phi\right)(x)\,dx \\ &= \int_0^{n-1} g(x) \int_0^1 \phi(y) \int_0^{x+y} \frac{z^{\mu+1/2}}{e^{M(pz)}} D_\mu(x,y,z)\,dz\,dy\,dx \\ &\geq \int_0^{n-1} g(x) \int_0^1 \frac{\phi(y)}{e^{M[p(x+y)]}} \int_0^\infty z^{\mu+1/2} D_\mu(x,y,z)\,dz\,dy\,dx \\ &= \alpha_\mu \int_0^{n-1} x^{\mu+1/2}g(x) \int_0^\infty \frac{\phi(y)y^{\mu+1/2}}{e^{M[p(x+y)]}}\,dy\,dx \\ &\geq C \int_0^{n-1} \frac{x^{\mu+1/2}g(x)}{e^{M(2px)}}\,dx \int_0^\infty y^{\mu+1/2}\phi(y)\,dy \quad (n \in \mathbb{N}-\{0\}).\end{aligned}$$

In the last inequality we have used the facts that M is an increasing function and that $\phi(x) = 0$, for every $x > 1$.

Therefore

$$\lim_{n\to\infty} \int_0^\infty g(x)\left(\frac{\chi_{(0,n)}t^{\mu+1/2}}{e^{M(pt)}} \# \phi\right)(x)\,dx = \infty, \quad \text{for every } p \in \mathbb{N}.$$

Consequently there exists a sequence of positive numbers $\{a_p\}_{p\in\mathbb{N}}$ such that

$$\int_0^\infty g(x)\left(\frac{\chi_{(0,a_p)}t^{\mu+1/2}}{e^{M(pt)}} \# \phi\right)(x)\,dx > p.$$

Assume that $a_p > a_{p+1}$, $p \in \mathbb{N}$. Moreover since $\phi \geq 0$, we have also $\tau_x\phi \geq 0$ $(x \in I)$ and, since M is an increasing function, it is clear that for every $0 \leq a < b$

$$\int_a^b \frac{t^{\mu+1/2}}{e^{M(pt)}}(\tau_x\phi)(t)\,dt \geq \int_a^b \frac{t^{\mu+1/2}}{e^{M[(p+1)t]}}(\tau_x\phi)(t)\,dt \quad (x \in I).$$

Then, for every $p \in \mathbb{N}$,

$$\int_0^\infty g(x)(\psi_p \# \phi)(x)\,dx > p,$$

where $\psi_p(x) = x^{\mu+1/2}$, $(0 < x < a_0)$, $\psi_p(x) = \dfrac{x^{\mu+1/2}}{e^{M(lx)}}$, $(a_{l-1} \leq x < a_l,\ l = 1,\ldots,p)$, and $\psi_p(x) = 0$, $(x \geq a_p)$.

Hence, by defining $\psi(x)=x^{\mu+1/2}$, $(0<x<a_0)$, $\psi(x)=\dfrac{x^{\mu+1/2}}{e^{M(lx)}}$, $(a_{l-1}\le x<a_l$, $l=1,2,\dots)$, we have that

$$\int_0^\infty g(x)(\psi\#\phi)(x)=\infty. \tag{11}$$

On the other hand $z^{-\mu-1/2}h_\mu(\psi)$ is a multiplier of $\mathcal{Q}_{\mu,M^\times}$. Indeed, for every $k\in\mathbb{N}-\{0\}$, we can write

$$|z^{-\mu-1/2}h_\mu(\psi)(z)|\le C\int_0^\infty e^{x\,|\mathrm{Im}\,z|}x^{\mu+1/2}\,|\psi(x)|\,dx$$

$$\le Ce^{M^\times(|\mathrm{Im}\,z|/k)}\int_0^\infty e^{M(kx)}x^{\mu+1/2}\,|\psi(x)|\,dx \quad (z\in\mathbb{C}).$$

Moreover the last integral is finite. Hence, by proceeding as in the proof of Lemma 3.1, we conclude that $z^{-\mu-1/2}h_\mu(\psi)\Phi$ is in $\mathcal{Q}_{\mu,M^\times}$, for every $\Phi\in\mathcal{Q}_{\mu,M^\times}$. Then by invoking Theorem 2.d [9] one deduces that $\psi\#\phi\in\mathcal{H}_{\mu,M}$.

Since $f\in\mathcal{H}^\#_{\mu,M}$ we have

$$\lim_{x\to 0^+}\int_0^\infty g(y)\,|x^{-\mu-1/2}\tau_x(\psi\#\phi)(y)|\,dy=\alpha_\mu\int_0^\infty g(y)\,|(\psi\#\phi)(y)|\,dy<\infty,$$

which contradicts (11). This completes the proof. ■

An immediate consequence of Proposition 4.2 is the following result.

Corollary 4.1. *If $f\in\mathcal{H}^\#_{\mu,M}$, then $f\#\phi\in\mathcal{C}_\mu$, for every $\phi\in\mathcal{H}_{\mu,M}$.*

Proof. Let $f\in\mathcal{H}^\#_{\mu,M}$. According to Proposition 4.2 there exists $p\in\mathbb{N}$ such that $\dfrac{x^{\mu+1/2}f}{e^{M(px)}}\in L_1(I,\mathscr{E})$. Then, by proceeding as in the proof of Proposition 4.2, for every $\phi\in\mathcal{H}_{\mu,M}$ and $x,y\in I$ we obtain

$$\|(f\#\phi)(x)-(f\#\phi)(y)|\le\int_0^\infty\frac{z^{\mu+1/2}\,\|f(z)\|}{e^{M(pz)}}\,dz\,\sup_{z\in I}|e^{M(pz)}z^{-\mu-1/2}[(\tau_x\phi)(z)-(\tau_y\phi)(z)]|.$$

Proposition 3.4 leads to the following: for every $\phi\in\mathcal{H}_{\mu,M}$ and $x\in I$, $(f\#\phi)(y)\to(f\#\phi)(x)$ as $y\to x$ in $\mathscr{E}$.

Moreover, by invoking again Proposition 3.4, we obtain

$$\lim_{x\to 0^+}x^{-\mu-1/2}(f\#\phi)(x)=\alpha_\mu\int_0^\infty f(y)\phi(y)\,dy,\quad\text{for each }\phi\in\mathcal{H}_{\mu,M}.$$

Thus the proof is complete.

In the following Proposition we characterize the convergence in $\mathcal{H}_{\mu,M}$ in terms of convolutors.

Proposition 4.3. *Let $(\phi_n)_{n=0}^\infty$ be a sequence in $\mathcal{H}_{\mu,M}$. Then $\phi_n\to 0$ as $n\to\infty$ in $\mathcal{H}_{\mu,M}$ if and only if $F(\phi_n)\to 0$ as $n\to\infty$ in $\mathcal{C}_\mu$, for every $F\in\mathcal{G}(\mathcal{H}_{\mu,M})$.*

Proof. We prove firstly that if $\phi_n \to 0$ as $n \to \infty$ in $\mathcal{H}_{\mu,M}$ then $F(\phi_n) \to 0$ as $n \to \infty$ in $\mathcal{C}_\mu$, for every $F \in \mathcal{G}(\mathcal{H}_{\mu,M})$. To see this we shall use the closed graph theorem. Let $F \in \mathcal{G}(\mathcal{H}_{\mu,M})$. Assume that $(\psi_n)_{n=0}^\infty$ is a sequence in $\mathcal{H}_{\mu,M}$ such that $\psi_n \to \psi$ as $n \to \infty$ in $\mathcal{H}_{\mu,M}$ and $F(\psi_n) \to f$ as $n \to \infty$ in $\mathcal{C}_\mu$, where $\psi \in \mathcal{H}_{\mu,M}$ and $f \in \mathcal{H}^\#_{\mu,M}$. We have to show that $F(\psi) = f$.

For every $g \in \mathcal{H}^\#_{\mu,M}$, $g \# \psi_n \to g \# \psi$ as $n \to \infty$ in $\mathcal{C}_\mu$. In effect, let $g \in \mathcal{H}^\#_{\mu,M}$. There exists $p \in \mathbb{N}$ such that $\dfrac{x^{\mu+1/2} g}{e^{M(px)}} \in L_1(I, \mathscr{E})$.

Then for every x in I we have

$$\|x^{-\mu-1/2}[g\#(\psi_n - \psi)](x)\| \le \int_0^\infty \frac{y^{\mu+1/2}\,\|g(y)\|}{e^{M(py)}}\,dy \sup_{y\in I} |e^{M(py)}(xy)^{-\mu-1/2}\tau_x(\psi_n - \psi)(y)|. \tag{12}$$

According to (1.2) of [3] we have

$$x^{-\mu-1/2}\tau_x(\psi_n - \psi)(y) = h_\mu((xt)^{-\mu}J_\mu(xt)h_\mu(\psi_n - \psi)(t))(y) \quad (x,y \in I).$$

Hence, since h_μ is a homeorphism from $\mathcal{H}_{\mu,M}$ to $\mathcal{Q}_{\mu,M^\times}$ (Proposition 3.1) and since $(xt)^{-\mu}J_\mu(xt)$ is a multiplier in $\mathcal{Q}_{\mu,M^\times}$ uniformly in $x \in (0,m)$, where $m \in \mathbb{N} - \{0\}$ (see Lemma 3.1), (12) allows us to deduce that, for every $m \in \mathbb{N} - \{0\}$,

$$\sup_{x\in(0,m)} \|x^{-\mu-1/2}(g\#(\psi_n - \psi))(x)\| \to 0, \quad \text{as } n \to \infty.$$

Let $\varphi \in \beta_\mu$. We can write

$$F(\psi_n)\#\varphi = F(\varphi)\#\psi_n \to F(\varphi)\#\psi = F(\psi)\#\varphi \quad \text{as } n \to \infty \text{ in } \mathcal{C}_\mu.$$

Also for every $n \in \mathbb{N}$ and $x \in (0,m)$, where $m \in \mathbb{N} - \{0\}$, one has

$$\|x^{-\mu-1/2}[(F(\psi_n) - f)\#\varphi](x)\| \le \int_0^{a+m} \|[F(\psi_n) - f](y)\|\, x^{-\mu-1/2}\,|(\tau_x\varphi)(y)|\,dy$$

$$\le (a+m)^{\mu+1/2}\int_0^{a+m} y^{-\mu-1/2}\,\|[F(\psi_n) - f](y)\|\,|h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(\varphi)(t)](y)|\,dy.$$

Here $a > 0$ is such that $\varphi(x) = 0$, for each $x > a$. Hence, for every $m \in \mathbb{N} - \{0\}$, there exists $C > 0$ such that

$$\sup_{x\in(0,m)} \|x^{-\mu-1/2}((F(\psi_n) - f)\#\varphi)(x)\| \le C \sup_{y\in(0,a+m)} \|y^{-\mu-1/2}(F(\psi_n) - f)(y)\| \quad (n \in \mathbb{N}).$$

Thus we establish that $F(\psi_n)\#\varphi \to f\#\varphi$ as $n \to \infty$ in $\mathcal{C}_\mu$. Therefore $F(\psi)\#\varphi = f\#\varphi$ and it is not hard to see that

$$\lim_{x\to 0^+} x^{-\mu-1/2}\int_0^\infty F(\psi)(y)(\tau_x\varphi)(y)\,dy = \alpha_\mu \int_0^\infty F(\psi)(y)\varphi(y)\,dy$$

$$= \alpha_\mu \int_0^\infty f(y)\varphi(y)\,dy = \lim_{x\to 0^+} x^{-\mu-1/2}\int_0^\infty f(y)(\tau_x\varphi)(y)\,dy.$$

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Hence $F(\psi)=f$. Assume now that $F(\phi_n)\to 0$ as $n\to\infty$ in $\mathscr{C}_\mu$, for every $F\in\mathscr{G}(\mathscr{H}_{\mu,M})$. We shall prove that $\phi_n\to 0$ as $n\to\infty$ in $\mathscr{H}_{\mu,M}$. It is easy to see that $\phi_n\to 0$ as $n\to\infty$ in $\mathscr{H}_{\mu,M}$ if and only if the following two conditions hold.

(i) $(\phi_n)_{n=0}^\infty$ is bounded in $\mathscr{H}_{\mu,M}$; i.e., for every $m,k\in\mathbb{N}$, there exists $C_{m,k}>0$ such that $\eta_{\mu,M}^{m,k}(\phi_n)\le C_{m,k}$, for every $n\in\mathbb{N}$.

(ii) For every $l\in\mathbb{N}-\{0\}$, $k\in\mathbb{N}$, $\sup_{x\in(0,l)}|x^{-\mu-1/2}S_\mu^k\phi_n(x)|\to 0$, as $n\to\infty$.

Note that, by virtue of Proposition 2.2 of **[10]**, for every $k\in\mathbb{N}$ the mapping F_k from $\mathscr{H}_{\mu,M}$ into $\mathscr{H}_{\mu,M}^{\#}$, defined by

$$F_k(\phi)=S_\mu^k\phi\,.\,u\ (\phi\in\mathscr{H}_{\mu,M}),$$

where $u\in\mathscr{E}$, is in $\mathscr{G}(\mathscr{H}_{\mu,M})$. Hence our assumption implies that the condition (ii) holds.

Suppose that $\phi_n\not\to 0$ as $n\to\infty$ in $\mathscr{H}_{\mu,M}$ or, equivalently, that $(\phi_n)_{n=0}^\infty$ is not bounded in $\mathscr{H}_{\mu,M}$. Then there exist $m,k\in\mathbb{N}$ such that the set

$$\left\{\sup_{x\in I}e^{M(kx)}x^{-\mu-1/2}|S_\mu^m\phi_n(x)|\right\}_{n\in\mathbb{N}}$$

is not bounded. Since $\sup_{x\in(0,1)}e^{M(kx)}x^{-\mu-1/2}|S_\mu^m\phi_n(x)|\to 0$ as $n\to\infty$ we can find $w\ge 2$ for which

$$\sup_{x\in(0,1)}e^{M(kx)}x^{-\mu-1/2}|S_\mu^m\phi_n(x)|\le w,$$

for every $n\in\mathbb{N}$. Also there exist $n_1\in\mathbb{N}$ and $x_1\in I$ (necessarily $x_1\notin(0,1)$) such that $e^{M(kx_1)}x_1^{-\mu-1/2}|S_\mu^m\phi_{n_1}(x_1)|\ge w$, and there exists $C_1>0$ such that

$$x_1^{-\mu-1/2}|S_\mu^m\phi_n(x_1)|\le C_1,\quad n\in\mathbb{N}.$$

Suppose now that we have found for $s\in\mathbb{N}-\{0\}$, $\{x_\alpha\}_{\alpha=1}^s$, $\{C_\alpha\}_{\alpha=1}^s$ and $\{\phi_{n_\alpha}\}_{\alpha=1}^s$ satisfying the following three conditions for every $\alpha=1,\ldots,s$:

(iii) $\sup_{n\in\mathbb{N}}|x_\alpha^{-\mu-1/2}S_\mu^m\phi_n(x_\alpha)|\le C_\alpha$,

(iv) $\left|\sum_{i=1}^s e^{M(kx_i)}x_i^{-\mu-1/2}S_\mu^m\phi_{n_\alpha}(x_i)\right|\ge 1+\sum_{j=s-\alpha+1}^\infty\frac{1}{2^j}$,

(v) $e^{-M(x_\alpha)}\le\dfrac{1}{\alpha^2}$.

We are going to define x_{s+1}, C_{s+1} and $\phi_{n_{s+1}}$ such that $\{x_\alpha\}_{\alpha=1}^{s+1}$, $\{C_\alpha\}_{\alpha=1}^{s+1}$ and $\{\phi_{n_\alpha}\}_{\alpha=1}^{s+1}$ satisfy the above three conditions, where s is replaced by $s+1$. We choose $\rho_s>0$ for which $e^{M(x)}\ge(s+1)^2$, for every $x\notin(0,\rho_s)$. There exists $C>0$ such that

$$e^{M\{(k+1)x\}}x^{-\mu-1/2}|S_\mu^m\phi_{n_i}(x)|\le C\quad(x\in I\text{ and }i=1,\ldots,s).$$

Hence $e^{M(kx)}x^{-\mu-1/2}|S_\mu^m\phi_{n_i}(x)|\le\dfrac{C}{e^{M(x)}}$ $(x\in I,\ i=1,\ldots,s)$ and we can find ρ_s sufficiently large that

$$e^{M(kx)}x^{-\mu-1/2}|S_\mu^m\phi_{n_i}(x)|\le\frac{1}{2^{s-i+1}}\quad(x\notin(0,\rho_s),\,i=1,\ldots,s).$$

We choose $w \geq 2 + e^{M(kx_1)}C_1 + \ldots + e^{M(kx_s)}C_s$ satisfying

$$e^{M(kx)}x^{-\mu-1/2}|S_\mu^m\phi_n(x)| \leq w \quad (x \in (0,\rho_s),\ n \in \mathbb{N}).$$

Since the set $\{\sup_{x\in I} e^{M(kx)}x^{-\mu-1/2}|S_\mu^m\phi_n(x)|\}_{n\in\mathbb{N}}$ is not bounded, there exist $n_{s+1} \in \mathbb{N}$ and $x_{s+1} \in I$ for which

$$e^{M(kx_{s+1})}x_{s+1}^{-\mu-1/2}|S_\mu^m\phi_{n_{s+1}}(x_{s+1})| \geq w.$$

It is clear that $x_{s+1} \notin (0,\rho_s)$. Hence $e^{M(x_{s+1})} \geq (s+1)^2$ and (v) holds. Also we choose $C_{s+1} > 0$ such that $\sup_{n\in\mathbb{N}} |x_{s+1}^{-\mu-1/2}S_\mu^m\phi_n(x_{s+1})| \leq C_{s+1}$. Thus (iii) holds. Also if $1 \leq \alpha \leq s$, then

$$\begin{aligned}
&|e^{M(kx_1)}x_1^{-\mu-1/2}S_\mu^m\phi_{n_\alpha}(x_1) + \ldots + e^{M(kx_{s+1})}x_{s+1}^{-\mu-1/2}S_\mu^m\phi_{n_\alpha}(x_{s+1})| \\
&\geq \left|\sum_{i=1}^{s} e^{M(kx_i)}x_i^{-\mu-1/2}S_\mu^m\phi_{n_\alpha}(x_i)\right| - |e^{M(kx_{s+1})}x_{s+1}^{-\mu-1/2}S_\mu^m\phi_{n_\alpha}(x_{s+1})| \\
&\geq 1 + \sum_{j=s-\alpha+1}^{\infty} \frac{1}{2^j} - \frac{1}{2^{s-\alpha+1}} = 1 + \sum_{j=(s+1)-\alpha+1} \frac{1}{2^j}.
\end{aligned}$$

Moreover

$$\begin{aligned}
&|e^{M(kx_1)}x_1^{-\mu-1/2}S_\mu^m\phi_{n_{s+1}}(x_1) + \ldots + e^{M(kx_{s+1})}x_{s+1}^{-\mu-1/2}S_\mu^m\phi_{n_{s+1}}(x_{s+1})| \\
&\geq e^{M(kx_{s+1})}x_{s+1}^{-\mu-1/2}|S_\mu^m\phi_{n_{s+1}}(x_{s+1})| - \sum_{i=1}^{s} e^{M(kx_i)}x_i^{-\mu-1/2}|S_\mu^m\phi_{n_{s+1}}(x_i)| \\
&\geq w - \sum_{i=1}^{s} e^{M(kx_i)}C_i \geq 2.
\end{aligned}$$

Hence (iv) holds. Thus we have constructed three sequences $\{x_\alpha\}_{\alpha=1}^\infty$, $\{C_\alpha\}_{\alpha=1}^\infty$ and $\{\phi_{n_\alpha}\}_{\alpha=1}^\infty$ satisfying (iii) and (v), for every $\alpha = 1,2,\ldots$, and satisfying (iv), for every $s = 1,2,\ldots$ and $\alpha = 1,2,\ldots,s$.

We now define the sequence $\{F_n\}_{n=1}^\infty$ in $\mathcal{G}(\mathcal{H}_{\mu,M})$ where, for every $n \in \mathbb{N}$ with $n \geq 1$ and $\phi \in \mathcal{H}_{\mu,M}$, we have

$$[F_n(\phi)](x) = \sum_{\alpha=1}^{n} e^{M(kx_\alpha)}x_\alpha^{-\mu-1/2}\tau_{x_\alpha}(S_\mu^m\phi)(x)\,.\,u \quad (x \in I),$$

u being a unit vector in $\mathscr{E}$.

Our next purpose is to show that for every $\phi \in \mathcal{H}_{\mu,M}$ the sequence $\{F_n(\phi)\}_{n=1}^\infty$ converges in $\mathscr{C}_\mu$. Let $\phi \in \mathcal{H}_{\mu,M}$. For every $n,l \in \mathbb{N} - \{0\}$, according to (1.2) of [**3**] we have

$$\begin{aligned}
&x^{-\mu-1/2}\|[F_{n+l}(\phi)](x) - [F_n(\phi)](x)\| \leq \sum_{\alpha=n+1}^{n+l} e^{M(kx_\alpha)}x_\alpha^{-\mu-1/2}|\tau_{x_\alpha}(S_\mu^m\phi)(x)|\,x^{-\mu-1/2} \\
&\leq \sum_{\alpha=n+1}^{n+l} e^{-M(x_\alpha)}e^{M[(k+1)x_\alpha]}x_\alpha^{-\mu-1/2}|h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(S_\mu^m\phi)(t)](x_\alpha)| \\
&\leq \sum_{\alpha=n+1}^{n+l} \frac{1}{\alpha^2} \sup_{z\in(0,\infty)} e^{M[(k+1)z]}z^{-\mu-1/2}|h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(S_\mu^m\phi)(t)](z)| \quad (x \in I).
\end{aligned}$$

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Hence, since for every $r \in \mathbb{N} - \{0\}$, $(xt)^{-\mu}J_\mu(xt)$ is a uniform multiplier in $x \in (0, r)$ (see Lemma 3.1), $\{F_n(\phi)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathscr{C}_\mu$ and, since $\mathscr{C}_\mu$ is a Banach space, $\{F_n(\phi)\}_{n=1}^\infty$ converges in $\mathscr{C}_\mu$. We define

$$F(\phi) = \lim_{n\to\infty} F_n(\phi) \quad (\phi \in \mathscr{H}_{\mu,M}),$$

where the limit is understood in $\mathscr{C}_\mu$.

Moreover $F(\phi) \in \mathscr{H}^\#_{\mu,M}$, for every $\phi \in \mathscr{H}_{\mu,M}$. In effect, let $\phi \in \mathscr{H}_{\mu,M}$. By (1.2) of [3] together with (5.3.b), (2.2) of [6] and Proposition 3.1, there exists $l \in \mathbb{N}$ such that

$$e^{M[(k+1)x_\alpha]}(xx_\alpha)^{-\mu-1/2}\,|\tau_{x_\alpha}(S^m_\mu\phi)(x)|$$

$$\leq \sup_{z\in I} e^{M[(k+1)z]}z^{-\mu-1/2}\,|h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(S^m_\mu\phi)(t)](z)| \leq Ce^{M(lx)} \quad (x \in I,\ \alpha = 1, 2, \ldots).$$

Then

$$\int_0^\infty \frac{x^{\mu+1/2}\,\|[F(\phi)](x)\|}{e^{M((l+1)x)}}\,dx \leq C\sum_{\alpha=1}^\infty \frac{1}{\alpha^2}.$$

Proposition 4.2 implies that $F(\phi) \in \mathscr{H}^\#_{\mu,M}$. To see that $F \in \mathscr{G}(\mathscr{H}_{\mu,M})$ we have to prove that $F(\phi)\#\psi = F(\phi\#\psi)$, for every ϕ, $\psi \in \mathscr{H}_{\mu,M}$. Let ϕ, $\psi \in \mathscr{H}_{\mu,M}$. For every $l \in \mathbb{N} - \{0\}$ we can write

$$\|x^{-\mu-1/2}[(F_n(\phi) - F(\phi))\#\psi](x)\|$$

$$\leq \int_0^\infty \|(F_n(\phi) - F(\phi))(y)\|\, x^{-\mu-1/2}\,|(\tau_x\psi)(y)|\,dy$$

$$\leq \int_0^\infty \frac{y^{\mu+1/2}\,\|(F_n(\phi) - F(\phi))(y)\|}{e^{M((l+1)y)}}\,dy\,\sup_{z\in I} e^{M((l+1)z)}z^{-\mu-1/2}\,|h_\mu[(xt)^{-\mu}J_\mu(xt)h_\mu(\psi)(t)](z)|$$

$$\leq C\sum_{\alpha=n+1}^\infty \frac{1}{\alpha^2}, \quad \text{for each } x \in (0, l).$$

Hence $F_n(\phi)\#\psi \to F(\phi)\#\psi$ as $n \to \infty$ in $\mathscr{C}_\mu$. Moreover $F_n(\phi)\#\psi = F_n(\phi\#\psi)$ ($n \in \mathbb{N}$, $n \geq 1$). Therefore $F(\phi)\#\psi = F(\phi\#\psi)$. Finally we note that, for every $l \in \mathbb{N}$, we have

$$x^{-\mu-1/2}\,\|[F(\phi_{n_l})](x)\| = x^{-\mu-1/2}\left|\sum_{\alpha=1}^\infty e^{M(kx_\alpha)}x_\alpha^{-\mu-1/2}\tau_{x_\alpha}(S^m_\mu\phi_{n_l})(x)\right| \quad (x \in I).$$

Then for every $l \in \mathbb{N}$ we have

$$\lim_{x\to 0^+} x^{-\mu-1/2}\,\|[F(\phi_{n_l})](x)\| = \left|\sum_{\alpha=1}^\infty e^{M(kx_\alpha)}x_\alpha^{-\mu-1/2}\lim_{x\to 0^+} x^{-\mu-1/2}\tau_x(S^m_\mu\phi_{n_l})(x_\alpha)\right|$$

$$= \alpha_\mu\left|\sum_{\alpha=1}^\infty e^{M(kx_\alpha)}x_\alpha^{-\mu-1/2}S^m_\mu\phi_{n_l}(x_\alpha)\right| \geq \alpha_\mu.$$

Hence $F(\phi_n) \not\to 0$ as $n \to \infty$ in C_μ, which contradicts our assumption. ∎

We now prove that the spaces $\mathscr{G}(\mathscr{H}_{\mu,M})$ and $\mathscr{H}'_{\mu,M}$ are isomorphic.

Proposition 4.4. *The spaces $\mathscr{G}(\mathscr{H}_{\mu,M})$ and $\mathscr{H}'_{\mu,M}$ are isomorphic.*

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Proof. We define the mapping J through

$$J:\mathcal{G}(\mathcal{H}_{\mu,M})\to\mathcal{H}'_{\mu,M}$$
$$F\to J(F):\mathcal{H}_{\mu,M}\to\mathcal{E}$$
$$\phi\to J(F)(\phi)=\lim_{x\to 0^+}\frac{1}{\alpha_\mu}x^{-\mu-1/2}F(\phi)(x).$$

We are going to prove that J is an algebraic and sequential isomorphism. To see that J is a linear one to one mapping we can proceed as in the proof of Proposition 2.3. Also J is onto. In effect, for every $f\in\mathcal{H}'_{\mu,M}$, we define

$$[F(\phi)](x)=(f\#\phi)(x)\quad(\phi\in\mathcal{H}_{\mu,M}\text{ and }x\in I).$$

From Proposition 3.4, we can infer that $f\#\phi\in\mathcal{C}_\mu$ and that

$$\lim_{x\to 0^+}\frac{1}{\alpha_\mu}x^{-\mu-1/2}[F(\phi)](x)=\langle F,\phi\rangle.$$

Therefore, according to Proposition 4.2, $F(\phi)\in\mathcal{H}^{\#}_{\mu,M}$. Moreover, by proceeding as in Proposition 4.7 of [**10**] we can see that $F(\phi)\#\psi=F(\phi\#\psi)$, for every $\phi,\psi\in\mathcal{H}_{\mu,M}$. Hence $F\in\mathcal{G}(\mathcal{H}_{\mu,M})$ and $J(F)=f$. On the other hand, proceeding as in Proposition 2.3 it follows that J is sequentially continuous. Moreover, let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{H}'_{\mu,M}$ such that $f_n\to 0$ as $n\to\infty$ in $\mathcal{H}'_{\mu,M}$. Then by Proposition 4.1 we deduce that $f_n\#\phi\to 0$ as $n\to\infty$ in $\mathcal{C}_\mu$. Hence J^{-1} is sequentially continuous. This completes the proof. ■

Finally we note that, proceeding in a similar way to that presented here, analogous results can be obtained for the space $\mathcal{H}_\mu$.

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