Since HS is || and = UO,
UOHS forms a $||^m$ whose diagonals mutually bisect.
But UH is bisected in M;
hence SO passes through and is bisected in M.

V. If the median AH be drawn it will cut OS in G so that OG = 2GS.

Through U draw UP || OG;
then UP = $\frac{1}{2}$OG.

AG is bisected in P, hence a line PQ drawn through P || AO bisects OG and is = $\frac{1}{2}$AO = UO = SH.

Hence PQHS is a $||^m$ whose diagonals mutually bisect.

Hence QG = GS,

or GS = $\frac{1}{3}$OS.

HG is also equal to $\frac{1}{3}$AH.

Hence G is the point of intersection of the medians of the △ ABC.

VI. O, M, G, and S, respectively the ortho-centre, the centre of the medioscribed circle, the centroid and the centre of the circumscribed circle, are collinear.

Sixth Meeting, April 13*th*, 1888.

W. J. Macdonald, Esq., M.A., F.R.S.E., President, in the Chair.

Extension of a theorem of Abel's in summation to integration.

By George A. Gibson, M.A.

The extension referred to was first given, I believe, by M. Ossian Bonnet in *Liouville's Journal*, vol. xiv., pp. 249 *et seq.* M. Jordan, in his *Cours d'Analyse*, tom. 2, § 82, seems to refer to this article in attributing to M. Bonnet the discovery of the "Second Theorem of the Mean," though he does not explicitly say so. In a course of lectures on "Simple and Multiple Integrals," delivered at Berlin University in the summer of 1885 by Professor Kronecker, which I attended, the "Second Theorem of the Mean" was shown to be a case of Abel's theorem, though I do not remember that Professor Kronecker mentioned M. Bonnet's name in connection with it. I have therefore thought that it might not be without interest to the members of this society to direct attention to the theorem, and to M. Bonnet's connection with it. The proof given is very similar to that in Jordan's *Cours d'Analyse.*

The theorem of Abel in question is Theorem III. of the introduction to his memoir on the Binomial Series (*Collected Works*, vol. I., p. 222). For present purposes we may state the theorem as follows:—If $A_0, A_1, \dots A_m$ bea series of quantities* such that for all values of r from 1 to m, $A_r - A_{r-1}$ retains the same sign, and if P_r (where $P_r = B_0 + B_1 + \dots + B_r$, and B_0, B_1, &c., are any quantities whatever) always lies between M and N, then the sum

$$Q = A_0B_0 + A_1B_1 + \dots + A_mB_m$$

will always lie between MA_0 and NA_0.

We have $P_0 = B_0,\ P_1 - P_0 = B_1 \dots P_m - P_{m-1} = B_m$

$$\therefore Q = A_0P_0 + A_1(P_1 - P_0) + \dots + A_r(P_r - P_{r-1}) + \dots + A_m(P_m - P_{m-1})$$
$$= (A_0 - A_1)P_0 + (A_1 - A_2)P_1 + \dots + (A_{r-1} - A_r)P_r + \dots \dots \dots$$
$$+ (A_{m-1} - A_m)P_{m-1} + A_mP_m.$$

Suppose $M > P_r > N$ and we see that Q lies in value between MA_0 and NA_0 since $A_r - A_{r-1}$ is always of the same sign. We may therefore write $Q = [N + \theta(M - N)]A_0$, where θ is a positive proper fraction.

If we now suppose the A*s* and B*s* to be functions of a variable x, we may get some of the ordinary theorems of integration.

Let $A_r = f(x_r)$ where $f(x)$ always varies in the same sense from x to x_m, and let $B_r = \phi(x_r)h_r$ where h_r is infinitesimal.

$\therefore\ P_r = \int_{x_0}^{x_r}\phi(x)dx$ and M and N are the greatest and least values of $\int_{x_0}^{x_r}\phi(x)dx\ (r = 1, 2 \dots m)$

$$\therefore\ Q = \int_{x_0}^{x_m} f(x)\phi(x)dx = [N + \theta(M - N)]f(x_0) = f(x_0)\int_{x_0}^{\xi}\phi(x)dx$$

where ξ lies between x_0 and x_m. [It is evident that $M + \theta(M - N)$ can be put in the form $\int_{x_0}^{\xi}\phi(x)dx.$]

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* $A_0, A_1, \dots \dots A_m$ are supposed to be positive quantities.

This result is very like that given by M. Bonnet.

Again, we have

$$Q = \sum_{r=0}^{r=m} A_0B_0 = \sum_{r=1}^{r=m}(A_{r-1} - A_r)P_{r-1} + A_mP_m$$

and, as before, $\sum_{r=1}^{r=m}(A_{r-1} - A_r)P_{r-1}$ lies between $M(A_0 - A_m)$ and $N(A_0 - A_m)$

$$\therefore\ Q = [N + \theta(M - N)](A_0 - A_m) + A_mP_m$$

$$\therefore \int_{x_0}^{x_m} f(x)\phi(x)dx = [N + \theta(M-N)](f(x_0) - f(x_m)) + f(x_m)\int_{x_0}^{x_m}\phi(x)dx$$

$$= (f(x_0) - f(x_m))\int_{x_0}^{\xi}\phi(x)dx + f(x_m)\int_{x_0}^{x_m}\phi(x)dx$$

$$= f(x_0)\int_{x_0}^{\xi}\phi(x)dx + f(x_m)\int_{\xi}^{x_m}\phi(x)dx$$

which is the ordinary form of the "Second Theorem of the Mean."

Lastly, we may note that the theorem of "Integration by parts" is virtually given in Abel's theorem, for we have

$$Q = A_0P_0 + \sum_{r=1}^{r=m} A_r(P_r - P_{r-1}) = \sum_{r=1}^{r=m}(A_{r-1} - A_r)P_r + A_mP_m$$

Let $A_r = f(x_r)$ and $P_r = \phi(x_r)$

$$\therefore \qquad A_{r-1} - A_r = -\frac{df(x_r)}{dx}h_r, \quad P_r - P_{r-1} = \frac{d\phi(x_r)}{dx_r}h_r$$

$$\therefore \quad f(x_0)\phi(x_0) + \int_{x_0}^{x_m} f(x)\phi'(x)dx = f(x_m)\phi(x_m) - \int_{x_0}^{x_m} f'(x)\phi(x)dx$$

i.e., $$\int_{x_0}^{x_m} f(x)\phi'(x)dx = f(x_m)\phi(x_m) - f(x_0)\phi(x_0) - \int_{x_0}^{x_m} f'(x)\phi(x)dx.$$

On the inscription of a triangle of given shape in a given triangle.

By R. E. Allardice, M.A.

§ 1. *To inscribe in a triangle ABC a triangle similar to the triangle DEF, and having its sides parallel to those of DEF.*

In order to inscribe in the triangle ABC (fig. 21), a triangle

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