STABILITY OF LINEAR NEUTRAL DELAY-DIFFERENTIAL SYSTEMS

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Sufficient conditions are obtained for the stability of linear neutral delay-differential systems by using a delay-differential inequality.

1. INTRODUCTION

This paper extends the result [2] for linear delay-differential systems to the neutral case:

\[ \dot{x}(t) = Ax(t) + Bx(t - \tau) + Cx(t - \tau) \quad (t \geq 0) \]

where, \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \), \( A \), \( B \) and \( C \) denote real constant \( n \times n \) matrices with elements \( a_{ij}, b_{ij}, c_{ij} \) (\( i, j = 1, 2, \ldots, n \)) respectively, and \( \tau > 0 \) is a constant. We adopt the following norms for vectors \( x = (x_1, x_2, \ldots, x_n)^T \) and matrices \( A = (a_{ij})_{n \times n} \) respectively:

\[ \|x\|_1 = \max_i |x_i|, \quad \|x\|_2 = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}, \quad \|x\|_{\infty} = \sum_{i=1}^{n} |x_i|, \]

\[ \|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|, \quad \|A\|_2 = \{\lambda \max_{\lambda}[A^T A]\}^{1/2}, \quad \|A\|_{\infty} = \max_i \sum_{j=1}^{n} |a_{ij}|. \]

The measure \( \mu(A) \) of a matrix \( A \) is defined by

\[ \mu(A) = \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h}, \]

which for the corresponding norms reduces to

\[ \mu_1(A) = \max_j \{a_{ij} + \sum_{i=1, i \neq j}^{n} |a_{ij}|\}, \]

\[ \mu(A) = \frac{1}{2} \lambda_{\max}[A^T + A], \]

\[ \mu_{\infty}(A) = \max_i \{a_{ii} + \sum_{j=1, j \neq i}^{n} |a_{ij}|\}, \]

where \( A^T \) denotes the transposed of \( A \), \( \lambda_{\max}[B] \) denotes the largest eigenvalue of \( B \), and \( I \) denotes the unit matrix.

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2. A Delay-Differential Inequality

A delay-differential inequality was discussed in [1] and [4]. Here an extension is given.

**Theorem 2.1.** Let \( p_i(t) \geq 0 \) (\( i = 1, 2 \)) be continuously differentiable functions on \([-\tau, +\infty)\) which satisfy

\[
\begin{aligned}
\dot{P}_1(t) &\leq a_{11}p_1(t) + a_{12}p_2(t) + b_{11}\bar{p}_1(t) + b_{12}\bar{p}_2(t) \\
0 &\leq a_{21}p_1(t) + a_{22}p_2(t) + b_{21}\bar{p}_1(t) + b_{22}\bar{p}_2(t)
\end{aligned}
\quad (t \geq 0)
\]

where, \( a_{ij} \geq 0 (i \neq j), \ b_{ij} \geq 0, \ \bar{p}_i(t) = \sup_{-\tau \leq \theta \leq t} p_i(\theta) \quad (i, j = 1, 2). \) If \( a_{ii} + b_{ii} < 0 \) and the real parts of all eigenvalues of the matrix \((a_{ij} + b_{ij})_{2 \times 2}\) are negative. Then there exist constants \( M \geq 1, \alpha > 0 \) such that:

\[
p_i(t) \leq M \left( \sum_{j=1}^{2} \bar{p}_j(0) \right) e^{\alpha t} \quad t \in [-\tau, \infty).
\]

**Proof:** From the properties of a stable Metzler - Matrix [3], there exist constants \( \alpha_i > 0 \quad (i = 1, 2) \) such that:

\[
\begin{aligned}
(a_{11} + b_{11})\alpha_1 + (a_{12} + b_{12})\alpha_2 &< 0, \\
(a_{21} + b_{21})\alpha_1 + (a_{22} + b_{22})\alpha_2 &< 0.
\end{aligned}
\]

We choose two constants, \( \alpha > 0 \) and \( k > 0 \), such that

\[
\alpha \alpha_1 + a_{11}\alpha_1 + b_{11}\alpha_1 e^{\alpha \tau} + a_{12}\alpha_2 + b_{12}\alpha_2 e^{\alpha \tau} < 0,
\]

\[
a_{21}\alpha_1 + b_{21}\alpha_1 e^{\alpha \tau} + a_{22}\alpha_2 + b_{22}\alpha_2 e^{\alpha \tau} < 0,
\]

and

\[
k\alpha_i e^{-\alpha \tau} > 1 \quad (i = 1, 2).
\]

For a sufficiently small real number \( \varepsilon > 0 \), we define

\[
w_i(t) = k\alpha_i \left( \sum_{j=1}^{2} \bar{p}_j(0) + \varepsilon \right) e^{-\alpha t} \quad (i = 1, 2, \ t \geq -\tau).
\]

It is easy to check that \( p_i(t) < w_i(t) \quad (i = 1, 2, \ t \in [-\tau, 0]). \) We want to prove that:

\[
p_i(t) < w_i(t) \quad (i = 1, 2, \ t \in [0, +\infty)).
\]
If this inequality did not hold, then one of the following two situations would occur:

1. There exists a $t_1 > 0$, such that

$$p_1(t) = w_1(t_1) \quad \text{and} \quad p_i(t) \leq w_i(t) \quad (i = 1, 2, -\tau \leq t \leq t_1);$$

furthermore, $\dot{p}_1(t_1) \geq \dot{w}_1(t_1)$. Also, from (2.3), we have

$$\dot{p}_1(t_1) \leq a_{11}w_1(t_1) + a_{12}w_2(t_1) + b_{11} \sup_{t_1 - \tau \leq \theta \leq t_1} w_1(\theta) + b_{12} \sup_{t_1 - \tau \leq \theta \leq t_1} w_2(\theta),$$

$$= a_{11}w_1(t_1) + a_{12}w_2(t_1) + b_{11}w_1(t_1 - \tau) + b_{12}w_2(t_1 - \tau),$$

so

$$\dot{w}_1(t_1) = -k\alpha_1 \alpha \left( \sum_{j=1}^{2} \overline{p_j(0) + \varepsilon} \right) e^{-\alpha t_1},$$

$$> k(a_{11}w_1 + b_{11}\alpha_1 e^{\alpha \tau} + a_{12}\alpha_2 + b_{12}\alpha_2 e^{\alpha \tau}) \left( \sum_{j=1}^{2} \overline{p_j(0) + \varepsilon} \right) e^{-\alpha t_1},$$

$$= a_{11}w_1(t_1) + a_{12}w_2(t_1) + b_{11}w_1(t_1 - \tau) + b_{12}w_2(t_1 - \tau).$$

That is, $\dot{w}_1(t_1) > \dot{p}_1(t_1)$. This contradicts $\dot{p}_1(t_1) \geq \dot{w}_1(t_1)$.

2. There exists a $t_1 > 0$, such that

$$p_2(t_1) = w_2(t_1) \quad \text{and} \quad p_i(t) \leq w_i(t) \quad (i = 1, 2, -\tau \leq t \leq t_1).$$

Then from (2.3), we obtain

$$-a_{22}p_2(t_1) \leq a_{21}w_1(t_1) + b_{21}w_1(t_1 - \tau) + b_{22}w_2(t_1 - \tau)$$

$$= k \left( \sum_{j=1}^{2} \overline{p_j(0) + \varepsilon} \right) e^{-\alpha t_1} (a_{21}\alpha_1 + b_{21}\alpha_1 e^{\alpha \tau} + b_{22}\alpha_2 e^{\alpha \tau}),$$

$$< k \left( \sum_{j=1}^{2} \overline{p_j(0) + \varepsilon} \right) e^{-\alpha t} (-a_{22}\alpha_2),$$

$$= -a_{22}w_2(t_1).$$

Since $-a_{22} > 0$, we have $p_2(t_1) < w_2(t_1)$. This contradicts $p_2(t_1) = w_2(t_1)$.

Hence, we see that

$$p_i(t) < w_i(t) \quad (i = 1, 2, t \geq 0).$$

Let $M = k(\alpha_1 + \alpha_2)$ and $\varepsilon \to 0^+$. Then (2.2) is satisfied and the proof is complete.

Theorem 2.1 can be generalised immediately to the vector case:
THEOREM 2.2. Let \( p_i(t) \geq 0 \) \((i = 1, 2, \ldots, 2n)\) be continuously differentiable functions on \([-\tau, +\infty)\) such that \( p^*(t) = (p_1(t), \ldots, p_n(t), 0 \ldots 0)^T \) and \( p(t) = (p_1(t), \ldots, p_n(t), p_{n+1}(t) \ldots p_{2n}(t))^T \) satisfies

\[
p^*(t) \leq Ap(t) + Bp(t)
\]

where, \( p(t) = \left( \sup_{t-\tau \leq \theta \leq t} p_1(\theta), \ldots, \sup_{t-\tau \leq \theta \leq t} p_{2n}(\theta) \right)^T \), and \( A = (a_{ij})_{2n \times 2n}, \ B = (b_{ij})_{2n \times 2n}, \) with \( a_{ij} \geq 0 (i \neq j) b_{ij} \geq 0 \) \((i, j = 1, 2, \ldots, 2n)\). If \( a_{ii} + b_{ii} < 0 \) and \( \Re(\lambda(A + B)) < 0 \), then there exist constants \( M \geq 1, \alpha > 0 \) such that:

\[
(2.4) \quad p_i(t) \leq M \left( \sum_{j=1}^{2n} \sup_{-\tau \leq \theta \leq 0} p_j(\theta) \right) e^{-\alpha t}, \quad t \in [-\tau, +\infty) \quad i = 1, 2, \ldots, 2n
\]

3. A STABILITY THEOREM

By \( C^{(1)}[-\tau, 0] \), we mean the Banach space of all functions \( u(t) \) \((u \) an \( n\)-vector) which are continuously differentiable on \([-\tau, 0]\) with norm:

\[
\|u\|_r = \sup_{-\tau \leq \theta \leq 0} \|u(\theta)\| + \sup_{-\tau \leq \theta \leq 0} \|\dot{u}(\theta)\|
\]

Consider the neutral delay-differential system

\[
(3.1) \quad \dot{x}(t) = Ax(t) + Bx(t - \tau) + C\dot{x}(t - \tau).
\]

Let us define the stability of the solution \( x = 0 \) for (3.1) as follows. Suppose that \( \phi \) is a given continuously differentiable function on \([-\tau, 0]\) (that is \( \phi \in C^{(1)}[-\tau, 0]\)) and that \( x(t) = x(t, \phi) \) on \([-\tau, +\infty)\) denotes the unique solution of (3.1) with \( x(t) = \phi(t), \dot{x}(t) = \dot{\phi}(t) \) for \( t \in [-\tau, 0]\).

**DEFINITION 1:** The solution \( x = 0 \) of (3.1) is stable in \( C^{(1)}[-\tau, 0]\) if for each \( \varepsilon > 0 \) there exists a \( \delta = \delta(\varepsilon) > 0 \) such that \( \|\phi\|_r < \delta \) implies that the solution \( x(t), \phi) \) of (3.1) satisfies

\[
\|x(t, \phi)\| + \|\dot{x}(t, \phi)\| < \varepsilon \quad t \in [-\tau, +\infty).
\]

**DEFINITION 2:** The solution \( x = 0 \) of (3.1) is asymptotically stable in \( C^{(1)}[-\tau, 0]\) if it is stable in \( C^{(1)}[-\tau, 0]\) and there exists \( b_0 > 0 \) such that \( \|\phi\|_r \leq b_0 \) implies

\[
\lim_{t \to +\infty} (\|x(t), \phi\| + \|\dot{x}(t, \phi)\|) = 0.
\]
THEOREM 3.1. Suppose the coefficient matrices $A$, $B$, $C$ of (3.1) satisfy the following

$$
||c|| < 1 \text{ and } \mu(A) + \frac{||B|| + ||A|| \cdot ||C||}{1 - ||C||} < 0.
$$

Then the solution $x = 0$ of the system, (3.1) is asymptotically stable in $c^{(1)}[-\tau, 0]$ and there exist $M \geq 1$, $\alpha > 0$ such that

$$
||x(t, \phi)|| + ||\dot{x}(t, \phi)|| \leq 2M ||\phi|| e^{-\alpha t}
$$

for every solution $x(t, \phi)$ of (3.1) with $x(t) = \phi(t), \dot{x}(t) = \dot{\phi}(t)$ on $t \in [-\tau, 0]$.

PROOF: From the definition of the measure $\mu(A)$ we have for $t \in [0, +\infty)$,

$$
\frac{d}{dt} ||x(t)|| - \mu(A) ||x(t)|| = \lim_{h \to 0^+} \frac{1}{h} [||x(t + h)|| - ||(I + hA)|| \cdot ||x(t)||]
$$

$$
\leq \lim_{h \to 0^+} \frac{1}{h} [||x(t + h) - (I + hA)X(t)||]
$$

$$
\leq ||B|| ||x(t - \tau)|| + ||C|| \cdot ||\dot{x}(t - \tau)||,
$$

that is

$$
\frac{d}{dt} ||x(t)|| \leq \mu(A) ||x(t)|| + ||B|| \sup_{t - \tau \leq \theta \leq t} ||x(\theta)|| + ||C|| \sup_{t - \tau \leq \theta \leq t} ||\dot{x}(\theta)||.
$$

From (3.1), we have directly

$$
0 \leq ||\dot{x}(t)|| + ||A|| ||x(t)|| + ||B|| \sup_{t - \tau \leq \theta \leq t} ||x(\theta)|| + ||C|| \sup_{t - \tau \leq \theta \leq t} ||\dot{x}(\theta)||.
$$

Consider the functions $p_1(t), p_2(t)$ in Theorem 2.1 defined by

$$
p_1(t) = ||x(t)||, p_2(t) = ||\dot{x}(t)|| \quad t \in [-\tau, +\infty).
$$

It follows from (3.3) and (3.4) that

$$
\begin{cases}
\dot{p}_1(t) \leq \mu(A)p_1(t) + ||B|| \dot{p}_1(t) + ||C|| p_2(t), \\
0 \leq ||A|| p_1(t) - p_2(t) + ||B|| \dot{p}_1(t) + ||C|| \dot{p}_2(t).
\end{cases}
$$

We know from hypothesis (3.2) that the real parts of all eigenvalues of the matrix

$$
\left( \begin{array}{ccc} 
\mu(A) + ||B|| & ||C|| \\
||A|| + ||B|| & -1 + ||C|| 
\end{array} \right)_{2 \times 2}
$$

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are negative and $\mu(A) + \|B\| < 0, -1 + \|C\| < 0$. Therefore, there exist $M \geq 1$, $\alpha > 0$ such that

$$p_i(t) \leq M \left( \sum_{j=1}^{2} p_j(0) \right) e^{-\alpha t} \quad t \geq -\tau.$$ 

By Theorem 2.1, we have

$$\|x(t, \phi)\| + \|\dot{x}(t, \phi)\| \leq 2M \|\phi\|_e e^{-\alpha t}, \quad t \geq -\tau$$

and the proof is complete.

Consider the following system in component form

$$(3.5) \quad \dot{x}_i(t) = a_{ii}x_i(t) + \sum_{j \neq i}^{n} b_{ij}x_j(t - \tau) + \sum_{j=1}^{n} c_{ij}\dot{x}_j(t - \tau)$$

where, $a_{ii}, b_{ij}(i \neq j), c_{ij} \quad (i = 1, 2, \ldots, n)$ are constants and $a_{ii} < 0 \quad (i = 1, 2, \ldots, n)$. Imitating the proof of Theorem 3.1 by using the delay-differential inequality in vector form (Theorem 2.2), we can easily obtain the following theorem.

**Theorem 3.2.** Suppose that the coefficients of (3.5) satisfy the following

$$a_{ii} + \sum_{j \neq i}^{n} |b_{ij}| + \sum_{j=1}^{n} |c_{ij}| < 0 \text{ and } |a_{ij}| + \sum_{j \neq i}^{n} |b_{ij}| + \sum_{j=1}^{n} |c_{ij}| < 1$$

$(i = 1, 2, \ldots, n)$. Then the solution $x_i = 0$ of (3.5) is asymptotically stable in $C^{(1)}[-\tau, 0]$.

**Remark.** Using Theorem 2.2, we can discuss the stability of the trivial solution for more complex neutral delay-differential systems.

**References**


