MODULES OVER PRÜFER DOMAINS WHICH SATISFY THE RADICAL FORMULA

DILEK BUYRUK
Department of Mathematics, Abant Izzet Baysal University, 14280 Bolu/Turkey
e-mail: dilekbuyruk@yahoo.com

and DILEK PUSAT-YILMAZ
Department of Mathematics, Izmir Institute of Technology, 35430 Urla, Izmir/Turkey
e-mail: dilekyilmaz@iyte.edu.tr

(Received 6 June, 2006; revised 26 September, 2006; accepted 13 October, 2006)

Abstract. In this paper we prove that if $R$ is a Prüfer domain, then the $R$-module $R \oplus R$ satisfies the radical formula.

2000 Mathematics Subject Classification. 13A15, 13C99, 13F05, 13F30.

1. Introduction. Let $M$ be a module over a commutative ring $R$ and $N$ be a submodule of $M$. The prime radical of $N$ in $M$, $\text{rad}_M(N)$, is defined to be the intersection of all prime submodules of $M$ containing $N$. If there is no prime submodule containing $N$, then $\text{rad}_M(N) = M$. In particular $\text{rad}_M(M) = M$.

Let $M$ be an $R$-module and $N$ a submodule of $M$. The envelope of $N$ in $M$ which is denoted by $E_M(N)$ is defined to be the set

$$\{rm : r \in R, m \in M \text{ such that } r^n m \in N \text{ for some } n \in \mathbb{Z}^+\}.$$ 

We say that the submodule $N$ satisfies the radical formula in $M$ ($N$ s.t.r.f. in $M$) if $\text{rad}_M(N) = (E_M(N))$. A module $M$ s.t.r.f. if for every submodule $N$ of $M$, the prime radical of $N$ is the submodule generated by its envelope, that is, $\text{rad}_M(N) = (E_M(N))$. A ring $R$ s.t.r.f. provided that for every $R$-module $M$, $M$ s.t.r.f.

The question of what kind of rings and modules s.t.r.f. has drawn the attention of many authors. In [2], Jenkins and Smith proved that Dedekind domains s.t.r.f.. In [3], Leung and Man proved that the only Noetherian rings which s.t.r.f. are of dimension at most one and they gave a complete characterization of Noetherian rings which s.t.r.f.. Now we are looking for non-Noetherian rings which s.t.r.f.. For that reason we investigate whether modules over Prüfer domains s.t.r.f.. We prove in Theorem 2.4 that if $R$ is a Prüfer domain, then the $R$-module $R \oplus R$ s.t.r.f. Throughout $R \oplus R$ will be denoted by $R^2$. The following is given in [6].

**Proposition 1.1.** Let $R$ be a ring. If the $R_M$-module $R_M \oplus R_M$ s.t.r.f. for any maximal ideal $M$ of $R$, then the $R$-module $R^2$ s.t.r.f..

2. Results.

**Lemma 2.1.** Let $R$ be a commutative ring, and $N$ be a submodule of $R^2$. If $N = I \oplus J$ for some ideals $I, J$ of $R$, then $N$ s.t.r.f. in $R^2$. 
Therefore, for any prime ideal $P$ of $R$ containing $I$, $P \oplus R$ is a prime submodule of $R^2$ containing $N$. Thus, $\text{rad}_R(N) \subseteq \sqrt{I} \oplus R$. Similarly, $\text{rad}_R(N) \subseteq R \oplus \sqrt{J}$. Hence $\text{rad}_R(N) \subseteq \sqrt{I} \oplus \sqrt{J}$. Now take $(x, y) \in \sqrt{I} \oplus \sqrt{J}$. That is $x^k \in I$ and $y^t \in J$ for some $k, t \in \mathbb{Z}^+$ and $x^k(1, 0), y^t(0, 1) \in N$. Then $(x, y) = x(1, 0) + y(0, 1) \in \langle E_R(N) \rangle$. Thus, we always have the other inclusion, $\text{rad}_R(N) = \sqrt{I} \oplus \sqrt{J} = \langle E_R(N) \rangle$ and $N$ s.t.r.f. in $R^2$.

**Lemma 2.2.** Let $D$ be a valuation domain and let $N$ be a $D$-submodule of $D^2$. If $(m, n)$ is an element of $N$ such that $(m^k, 0) \in N$ or $(0, n^k) \in N$ for some $k, k' \in \mathbb{Z}^+$, then $\sqrt{Dm} \oplus \sqrt{Dn} \subseteq \langle D^2(N) \rangle$.

**Proof.** Suppose $(m^k, 0) \in N$ for some $k \in \mathbb{Z}^+$. Then for any $x \in \sqrt{Dm}$ there exists $d \in D, l \in \mathbb{Z}^+$ such that $x^l = dm$. Hence $x^{kl}(1, 0) = d^km^k(1, 0) \in N$ implies that $\sqrt{Dm}(1, 0) \subseteq \langle D^2(N) \rangle$.

If $Dm \subseteq Dn$, then $m/n \in D$. Take a nonzero $s \in \sqrt{Dn}$ that is $s' = rm$ for some nonzero $r \in D$ and $t \in \mathbb{Z}^+$. Since $s'((m/n), 1) = (rm, rn) \in N$, and $(s(m/n))^{k}(1, 0) = r^k(m/n)^{k-1}(m^k, 0) \in N$ we have $(0, s) = s((m/n), 1) - s(m/n)(1, 0) \in \langle D^2(N) \rangle$.

Therefore $\sqrt{Dn}(0, 1) \subseteq \langle D^2(N) \rangle$.

If $Dn \subseteq Dm$, then $n/m \in D$ and $(0, n^k) = n^{k-1}(m, n) - (n/m)^{k-1}(m^k, 0) \in N$. Thus $\sqrt{Dn}(0, 1) \subseteq \langle D^2(N) \rangle$.

In any case, $\sqrt{Dm} \oplus \sqrt{Dn} \subseteq \langle D^2(N) \rangle$.

If $(0, n^k) \in N$, then the proof can be carried out in a similar way.

**Theorem 2.3.** Let $D$ be a valuation domain with unique maximal ideal $M$. Then $D^2$ s.t.r.f.

**Proof.** Let $N$ be a nonzero submodule of $D^2$ where $N$ is generated by $S = \{(a_i, b_i)_{i \in I}\}$. Consider the canonical projections $\pi_k : D^2 \rightarrow D$ given by $\pi_k(d_1, d_2) = d_k$, where $d_1, d_2 \in D, \lambda = 1, 2$. We have $\pi_1(N) = \{(a_i)_{i \in I}\}$ and $\pi_2(N) = \{(b_i)_{i \in I}\}$.

Case 1. If $N = \pi_1(N) \oplus \pi_2(N)$, then $N$ satisfies the radical formula in $D^2$ by Lemma 2.1.

Case 2. If $\pi_1(N) + \pi_2(N)$ is a finitely generated ideal of $D$, then $\pi_1(N) + \pi_2(N) = \sum_{\text{finite}} Da_i + \sum_{\text{finite}} Db_i$. Since $D$ is a valuation ring and the ideals of $D$ are totally ordered, we may assume $\pi_1(N) + \pi_2(N) = Db_k$ for some $k \in I$. Then $Da_k \subseteq Db_k$, hence $a_k/b_k \in D$. Note that $\{(a_k/b_k, 1), (1, 0)\}$ forms a basis for $D^2$.

Define

$$
\phi : D \oplus D \rightarrow D \oplus D
$$

$$(a_k/b_k, 1) \rightarrow (0, 1)
$$

$$(1, 0) \rightarrow (1, 0).
$$

Clearly $\phi$ is an isomorphism and $\phi(N) = B \oplus Db_k$ where $B$ is an ideal of $D$ and $B \cong N \cap (D \oplus 0)$. By case 1, $\phi(N)$ s.t.r.f. in $D^2$. Then by [5, Theorem 1.5], $N$ s.t.r.f. in $D^2$.

Case 3. Suppose $\pi_1(N) + \pi_2(N)$ is not a finitely generated ideal, but $\pi_1(N)$ or $\pi_2(N)$ is finitely generated. We may assume that $\pi_1(N)$ is finitely generated. Clearly $\pi_1(N)$ is nonzero. (If it is zero then $N$ satisfies Case 1 and result is clear.) Then $\pi_1(N) = Da_t$ for some $t \in I$. Since $\pi_1(N) + \pi_2(N)$ is not finitely generated there are infinitely many
principal ideals generated by elements of \( \pi_2(N) \) containing \( D a_i \). Then there exists an element \((a_i, b_i) \in S \) such that \( D a_i + D b_i \nsubseteq D b_s, s \in I \). Hence \( D a_i b_i \subseteq D a_i b_s \), that is \( a_i b_i/a_i b_s \) is a unit in \( D \). Then

\[
b_i(a_i, b_i) - b_i(a_i, b_s) = (1 - a_i b_i/a_i b_s)(a_i b_s, 0) \in N, \text{ so } (a_i b_s, 0) \in N.
\]

Note that \( a_i^2(1, 0) \in D(a_i b_s, 0) \subseteq N \), and hence \( (a_i^2, 0) \in N \) for any \( i \in I \). By Lemma 2.2, 
\[
\sqrt{D a_i} \oplus \sqrt{D b_i} \subseteq \langle E_D^2(N) \rangle \text{ for all } i \in I.
\]

Hence,
\[
\text{rad}(N) \subseteq \sqrt{\pi_1(N) \oplus \pi_2(N)} = \bigcup_{i \in I} \sqrt{D a_i} \oplus \bigcup_{i \in I} \sqrt{D b_i} \subseteq \langle E_D^2(N) \rangle.
\]

**Case 4.** Let \( N \) be a submodule of \( D \oplus D \) where \( \pi_1(N) \) and \( \pi_2(N) \) are not finitely generated. Recall that \( S = \{(a_i, b_i)_{i \in I}\} \) is the set of generators of \( N \). We order the index set \( I \) as follows: \( i \leq j \) if and only if \( D a_i \subseteq D a_j \). Define \( P_i = \sqrt{D a_i} \) and \( Q_i = \sqrt{D b_i} \) then 
\[
\sqrt{\pi_1(N)} = \bigcup_i P_i \text{ and } \sqrt{\pi_2(N)} = \bigcup_i Q_i.
\]

**Subcase 1.** For any \( i \in I \), if one of the following is satisfied, then \( P_i \oplus Q_i \subseteq \langle E_D^2(N) \rangle \) and \( N \) s.t.r.f. in \( D^2 \).

- (a) There exists \( j > i \) such that \( D b_j \nsubseteq D b_i \).
- (b) There exists \( k < i \) such that \( D b_k \nsubseteq D b_i \).
- (c) There exists \( j_0 > i \) such that \( D a_j b_{j_0} \neq D a_{j_0} b_{j_0} \) while \( D b_i \subseteq D b_j \) for all \( j > i \),
- (d) There exists \( j_1 > i \) such that \( D a_j b_{j_1} \subseteq D(t_{j_1} - 1) \) while for all indices \( j > i \), \( D b_i \subseteq D b_j \) and \( a_i b_j = u_i a_i b_j \) for some unit \( u_i \).

**Proof of Subcase 1.** It is enough to prove for any \( i \), \( P_i \oplus Q_i \subseteq \langle E_D^2(N) \rangle \). Assume both \( a_i \) and \( b_i \) are nonzero, otherwise the result is clear.

Let condition (a) be satisfied. Then by assumption we have \( (a_i, b_j) \in S \) such that

\[
D a_i b_j \nsubseteq D a_i b_i \subseteq D a_j b_i \text{ and } a_i b_j/a_i b_i \in \mathcal{M}.
\]

- If \( D a_i \subseteq D b_i \), then \( b_i(a_i, b_i) - b_i(a_i, b_i) = a_i b_i(a_i b_i/a_i b_i - 1)(1, 0) \in N \). Since \( a_i b_i/a_i b_i - 1 \) is a unit, \((a_i b_i, 0) \in N \) and so \((a_i^2, 0) \in D a_i b_i(1, 0) \subseteq N \). Hence by Lemma 2.2, \( P_i \oplus Q_i \subseteq \langle E_D^2(N) \rangle \).

- If \( D b_i \subseteq D a_i \), then \( a_i a_i(b_i, b_j) - a_i(b_i, b_i) = a_i b_i(1 - a_i b_i/a_i b_i)(0, 1) \in N \) and \( 1 - a_i b_i/a_i b_i \) is a unit implies \( a_i b_i(0, 1) \in N \), and \( a_i b_i(0, 1) \subseteq \langle E_D^2(N) \rangle \) so \( (0, b_i) \in D a_i b_i(0, 1) \subseteq N \) and \( Q_i(0, 1) \subseteq \langle E_D^2(N) \rangle \). By Lemma 2.2, \( P_i \oplus Q_i \subseteq \langle E_D^2(N) \rangle \). Hence,

\[
P_i \oplus Q_i \subseteq P_i \oplus Q_i \subseteq \langle E_D^2(N) \rangle.
\]

Let condition (b) be satisfied. That is there is a \( k < i \) such that \( D b_k \nsubseteq D b_i \). Assume \( a_k \neq 0 \) (otherwise result is clear), then

\[
D a_k b_i \nsubseteq D a_k b_k \subseteq D a_i b_k \text{ and } a_k b_i/a_k b_k \in \mathcal{M}.
\]

- If \( D a_i \subseteq D b_k \), then \( b_k(a_k, b_i) - b_k(a_i, b_k) = a_i b_k(1 - a_k b_i/a_k b_k)(1, 0) \in N \). Since \( 1 - a_k b_i/a_k b_k \) is a unit, \((a_k b_k, 0) \in N \). Then \((a_i^2, 0) \in D a_i b_k(1, 0) \subseteq N \) and \( P_i(1, 0) \subseteq \langle E_D^2(N) \rangle \). By Lemma 2.2, \( P_k \oplus Q_k \subseteq \langle E_D^2(N) \rangle \) and hence \( P_i \oplus Q_k \subseteq \langle E_D^2(N) \rangle \).

- If \( D b_i \subseteq D a_k \), then \( a_k a_k(b_i, b_k) - a_k(b_i, b_k) = a_k b_k(a_k b_i/a_k b_k - 1)(0, 1) \in N \) and \( 1 - a_k b_i/a_k b_k \) is a unit implies \( a_k b_k(1, 0) \in N \) and \( (0, b_i^2) \in D a_k b_k(0, 1) \subseteq N \). Hence we have \( P_i \oplus Q_k \subseteq \langle E_D^2(N) \rangle \) by Lemma 2.2.
Let (c) be satisfied. That is there exists $j_0 \in I$ such that $j_0 > i$ and $D_{a_ib_j} \neq D_{a_ib_i}$. Then one of these ideals of $D$ is contained in the other. Let $D_{a_ib_j} \subseteq D_{a_ib_i}$ that is $a_{ib_j}/a_{ib_i} \in M$ and $(a_{ib_i}, 0) \in N$; again we have two cases, such that $D_{a_{ib_j}} \subseteq D_{b_i}$ or $D_{b_i} \subseteq D_{a_{ib_j}}$. $D_{a_{ib_j}} \subseteq D_{b_i}$ implies $(a_{ib_i}^2, 0) \in D_{a_{ib_j}}(1, 0) \subseteq N$ and $D_{b_i} \subseteq D_{a_{ib_j}}$ implies $(0, b_i^2) \in N$. Using Lemma 2.2, we have $\mathcal{P}_i \oplus \mathcal{Q}_i \subseteq \langle E_{D^2}(N) \rangle$.

Let (d) be satisfied. That is, $a_i/a_j = u_j(b_i/b_j)$ for some unit $u_j$ for all $j > i$. By the above argument, $a_i/b_j(1 - u_j)(1, 0)$ and $a_i/b_j(1 - u_j)(0, 1) \in N$ for all $j > i$. By assumption, there is $j_1 \in I$ with $j_1 > i$ such that $D_{a_jb_j} \subseteq D(1 - u_j)$ for some $j_1 > i$. Then $(a_{j_1}^2, 0)$ or $(0, b_i^2) \in N$. By Lemma 2.2, we have $\mathcal{P}_j \oplus \mathcal{Q}_j \subseteq \langle E_{D^2}(N) \rangle$ Since this equality holds for all $i \in I$, we have

$$
\text{rad}(N) = \sqrt{\pi_1(N)} \oplus \sqrt{\pi_2(N)} = \bigcup \mathcal{P}_i \oplus \bigcup \mathcal{Q}_i \subseteq \langle E_{D^2}(N) \rangle.
$$

Subcase 2. Suppose the possibilities in subcase 1 do not occur for some $i \in I$, then there exists $j > i$ such that $D_{b_i} \subseteq D_{b_j}$, $a_i/b_j = u_ja_jb_j$ for some unit $u_j$ and $D(1 - u_j) \subseteq D_{a_jb_j}$. In this case $N$ s.t.r.f. in $D^2$.

Proof of Subcase 2. Similar to the proof of subcase 1, we have $a_j/b_j(1 - u_j)(1, 0)$ and $a_j/b_j(1 - u_j)(0, 1) \in N$ that is $(1 - u_j)^2, 0 \in N$ for all $j > i$. Now we may assume that $D_{a_i} \subseteq D_{b_i}$ then define

$$
\phi : D \oplus D \rightarrow D \oplus D \quad (\frac{a_j}{b_j}, 1) \rightarrow (0, 1) \quad (1, 0) \rightarrow (1, 0)
$$

Consider $N = N_1 + N_2$ where $N_1 = \langle (a_k, b_k)_{k \leq i} \rangle$ and $N_2 = \langle (a_k, b_k)_{k > i} \rangle$. If $k \leq i$, then $\phi(a_k, b_k) = (a_k - (b_k/b_j)a_j, 0) + \phi((b_k/b_j)a_i, b_k) = (a_k - (b_k/b_j)a_j, 0) + b_k(0, 1)$, and $\phi(a_i, b_i) = b_i\phi(a_i/b_j, 1) = b_i(0, 1) = (0, b_i) \in \phi(N_1)$ that implies $0 \oplus D_{b_i} \subseteq \phi(N_1)$ and $(0, b_k) = (b_k/b_j)(0, b_i) \in \phi(N_1)$ where $k \leq i$ and $(a_k - (b_k/b_j)a_i, 0) = \phi(a_k, b_k) - b_k(0, 1) \in \phi(N_1)$. Thus $\phi(N_1) = B \oplus D_{b_i}$.

where $B$ is an ideal of $D$ such that $B \cong (D \oplus 0) \cap \langle (a_i, b_i)_{i \leq i} \rangle$ by case 2. If $k > i$, then $\phi(a_k, b_k) = \phi(a_k - u_k a_k, 0) + \phi(u_k a_k, b_k) = (a_k - u_k a_k, 0) + b_k(0, 1) = (a_k - u_k a_k, b_k)$. Hence

$$
\phi(N_2) = \langle (a_k(1 - u_k), b_k)_{i < k} \rangle.
$$

So $\phi(N) = (B \oplus D_{b_i}) + \langle (a_k(1 - u_k), b_k)_{k > i} \rangle$. Since $(1 - u_j)^2, 0 \in N$ we have $(a_j(1 - u_j)^2, 0) = \phi(a_j(1 - u_j)^2, 0) \in \phi(N)$. By Lemma 2.2,

$$
\sqrt{D_{a_j}(1 - u_j)} \oplus \sqrt{D_{b_j}} \subseteq \langle E_{D^2}(\phi(N)) \rangle,
$$

for all $j > i$. Combining this result by Lemma 2.1, we have

$$
\text{rad}(\phi(N)) = \sqrt{\pi_1(\phi(N))} \oplus \sqrt{\pi_2(\phi(N))} \subseteq \langle E_{D^2}(\phi(N)) \rangle.
$$

Thus $\phi(N)$ s.t.r.f. in $D^2$ and by [5, Theorem 1.5], $N$ s.t.r.f. in $D^2$.

Theorem 2.4. Let $R$ be a Prüfer domain, then the $R$-module $R^2$ satisfies the radical formula.
Proof. For any maximal ideal $\mathcal{M}$ of $R$, $R_{\mathcal{M}}$ is a valuation ring. Then by Theorem 2.3, the $R_{\mathcal{M}}$-module $R_{\mathcal{M}} \oplus R_{\mathcal{M}}$ satisfies the radical formula. By Proposition 1.1, $R^2$ satisfies the radical formula.

REFERENCES