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# A CHARACTERISATION OF ERGODIC MEASURES

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Consider a set $X$ together with a $\sigma$-algebra $\mathscr{B}$ of subsets of $X$. Let $G$ be a family of $\mathscr{B}$-measurable transformations on $X$, let $p(X)$ be the convex set of all probability measures on $\mathscr{B}$ and let $I$ be the convex set of all $G$-invariant probability measures in $p(X)$. For $\mu \in p(X)$ we define $\mathscr{B}_\mu = \{A \in \mathscr{B}: \mu(gA \,\Delta\, A) = 0 \text{ for all } g \in G\}$ and we define $\mathscr{B}_0 = \{A \in \mathscr{B}: gA = A \text{ for all } g \in G\}$. Then $\mathscr{B}_0 \subseteq \mathscr{B}_\mu$ and both are $\sigma$-subalgebras of $\mathscr{B}$. $G$ is said to act transitively on $X$ if for $x \in X$, $y \in X$, $gx = y$ for some $g \in G$.

Consider the following conditions on an element $\mu \in I$:

(a) $\mu$ is an extreme point of $I$,

(b) $\mu(\mathscr{B}_\mu) = \{0, 1\}$,

(c) $\mu(\mathscr{B}_0) = \{0, 1\}$.

Each of these conditions has been considered in the literature as a definition of ergodicity of $\mu$. Feldman has shown that (a) and (b) are equivalent (1966; page 81). Under certain conditions (b) and (c) are known to be equivalent (see Feldman (1966; page 84) for a discussion) and the result of this paper is one of this type. Our result was provided in the case where $G$ is a separable topological group by Varadarajan (1963).

THEOREM. *Let $G$ be a Hausdorff locally compact $\sigma$-compact topological group of $\mathscr{B}$-measurable transformations on $X$ such that the associated mapping $(g, x) \to gx$ on $G \times X$ to $X$ is jointly measurable when $G$ is equipped with the $\sigma$-algebra of Borel sets. Let $\mu \in I$. Then $\mu \in exI$ if and only if $\mu(\mathscr{B}_0) = \{0, 1\}$. If $G$ acts transitively on $X$, there is at most one $G$-invariant measure in $p(X)$.*

Before proving this theorem we make some definitions. A fixed left invariant Haar measure on $G$ will be denoted by $d\lambda$. For a function $\phi$ on $G$ and $g \in G$,

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$l_g\phi$ is defined on $G$ by: $l_g\phi(h) = \phi(gh)$ for $h \in G$. If $\phi \in L^1(G)$ and $f$ is a bounded real valued $\mathscr{B}$-measurable function on $X$, $\phi * f$ is defined by:

$$\phi * f(x) = \int_G \phi(g) f(g^{-1}x) d\lambda(g).$$

Then define $P(\phi,f) = \{x \in X: \phi * f(x) = 1\}$ and $Q(\phi,f) = \bigcap_{g \in G} P(l_g\phi,f)$. It follows from the definition of $Q(\phi,f)$ that $Q(\phi,f) = g(Q(\phi,f)))$ for all $g \in G$.

LEMMA. *Let $G$ be $\sigma$-compact, let $\phi$ be a continuous real valued function on $G$ having compact support and let $f$ be a bounded real valued, $\mathscr{B}$-measurable function on $X$. Then $Q(\phi,f) \in \mathscr{B}_0$.*

PROOF. $\phi$ is uniformly continuous on $G$, and a sequence $(g_n)$ can be chosen in $G$ so that $\{l_{g_i}\phi: i = 1,2,\cdots\}$ is uniformly dense in $\{l_g\phi: g \in G\}$. Let $K$ be the support of $\phi$, $g \in G$, $\varepsilon > 0$ and $x \in \bigcap_1^\infty P(l_{g_i}\phi,f)$. We may assume $\lambda(K) > 0$. Choose $i$ so that

$$\| l_g\phi - l_{g_i}\phi \| < \frac{\varepsilon}{2\lambda(K)} \| f \|^{-1}.$$

Then

$$\begin{aligned}
\left|(l_g\phi * f)(x) - 1\right| &= \left|(l_g\phi * f)(x) - (l_{g_i}\phi * f)(x)\right| \\
&\leqq \left(\int_G \left|\phi(gp) - \phi(g_ip)\right| d\lambda(p)\right) \cdot \| f \| \\
&= \left(\int_{g^{-1}K \cup g_i^{-1}K} \left|\phi(gp) - \phi(g_ip)\right| d\lambda(p)\right) \cdot \| f \| \\
&\leqq (\lambda(g^{-1}K) + \lambda(g_i^{-1}K)) \cdot \| l_g\phi - l_{g_i}\phi \| \cdot \| f \| \\
&\leqq \varepsilon, \text{ true for all } \varepsilon > 0, \text{ all } g \in G.
\end{aligned}$$

Hence $x \in \bigcap_{g \in G} P(l_g\phi,f)$ so that $Q(\phi,f) = \bigcap^\infty P(l_{g_i}\phi,f)$. Since each $P(l_{g_i}\phi,f) \in \mathscr{B}$, $Q(\phi,f) \in \mathscr{B}_0$.

PROOF OF THEOREM: There is a characterization of $exI$ due to Feldman (1966: page 81) which says: $\mu \in exI$ if and only if $\mu(\mathscr{B}_\mu) = \{0,1\}$.

Hence if $\mu \in exI$, $\mu(\mathscr{B}_0) = \{0,1\}$ since $\mathscr{B}_0 \subseteq \mathscr{B}_\mu$ and both $X$ and the void set are in $\mathscr{B}_0$. Conversely, if $\mu(\mathscr{B}_0) = \{0,1\}$ let $A \in \mathscr{B}_\mu$. Let $\chi_A$ be the characteristic function of $A$. Let $\phi$ be a continuous real valued function on $G$, having compact support and such that $\int_G \phi(g) d\lambda(g) = 1$. Let $A_0 = Q(\phi, \chi_A) \in \mathscr{B}_0$ by the Lemma. It now follows, by an adaptation of the argument of Varadarajan [5] p. 1¶, that $\mu(A) = \mu(A_0) \in \{0,1\}$. Hence $\mu(\mathscr{B}_\mu) = \{0,1\}$.

If $G$ acts transitively on $X$, then $\mathscr{B}_0$ is the trivial $\sigma$-algebra and $\mu(\mathscr{B}_0) = \{0,1\}$ for all $\mu \in p(X)$. In this case $I \subseteq exI \subseteq I$ so that $I$ has at most one element.

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