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THE DE BRUIJN–NEWMAN CONSTANT IS NON-NEGATIVE

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Abstract

For each $t \in \mathbb{R}$, we define the entire function

$$H_t(z) := \int_0^\infty e^{tu^2} \Phi(u) \cos(zu)\, du,$$

where Φ is the super-exponentially decaying function

$$\Phi(u) := \sum_{n=1}^\infty (2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}) \exp(-\pi n^2 e^{4u}).$$

Newman showed that there exists a finite constant Λ (the *de Bruijn–Newman constant*) such that the zeros of H_t are all real precisely when $t \geqslant \Lambda$. The Riemann hypothesis is equivalent to the assertion $\Lambda \leqslant 0$, and Newman conjectured the complementary bound $\Lambda \geqslant 0$. In this paper, we establish Newman's conjecture. The argument proceeds by assuming for contradiction that $\Lambda < 0$ and then analyzing the dynamics of zeros of H_t (building on the work of Csordas, Smith and Varga) to obtain increasingly strong control on the zeros of H_t in the range $\Lambda < t \leqslant 0$, until one establishes that the zeros of H_0 are in local equilibrium, in the sense that they locally behave (on average) as if they were equally spaced in an arithmetic progression, with gaps staying close to the global average gap size. But this latter claim is inconsistent with the known results about the local distribution of zeros of the Riemann zeta function, such as the pair correlation estimates of Montgomery.

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1. Introduction

Let $H_0 \colon \mathbb{C} \to \mathbb{C}$ denote the function

$$H_0(z) := \frac{1}{8}\xi\left(\frac{1}{2} + \frac{iz}{2}\right), \tag{1}$$

where ξ denotes the Riemann xi function

$$\xi(s) := \frac{s(s-1)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{2}$$

and ζ is the Riemann zeta function. Then H_0 is an entire even function with functional equation $H_0(\overline{z}) = \overline{H_0(z)}$, and the Riemann hypothesis is equivalent to the assertion that all the zeros of H_0 are real.

It is a classical fact (see [29, page 255]) that H_0 has the Fourier representation

$$H_0(z) = \int_0^\infty \Phi(u)\cos(zu)\,du,$$

where Φ is the super-exponentially decaying function

$$\Phi(u) := \sum_{n=1}^{\infty}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u})\exp(-\pi n^2 e^{4u}). \tag{3}$$

The sum defining $\Phi(u)$ converges absolutely for negative u also. From Poisson summation, one can verify that Φ satisfies the functional equation $\Phi(u) = \Phi(-u)$ (that is, Φ is even).

De Bruijn [4] introduced the more general family of functions $H_t \colon \mathbb{C} \to \mathbb{C}$ for $t \in \mathbb{R}$ by the formula

$$H_t(z) := \int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\,du. \tag{4}$$

As noted in [11, page 114], one can view H_t as the evolution of H_0 under the backward heat equation $\partial_t H_t(z) = -\partial_{zz} H_t(z)$. As with H_0, each of the H_t is an entire even function with functional equation $H_t(\overline{z}) = \overline{H_t(z)}$. From results of Pólya [20], it is known that if H_t has purely real zeros for some t, then $H_{t'}$ has purely real zeros for all $t' > t$. De Bruijn showed that the zeros of H_t are purely real for $t \geqslant 1/2$. Strengthening these results, Newman [17] showed that there is an absolute constant $-\infty < \Lambda \leqslant 1/2$, now known as the *De Bruijn–Newman constant*, with the property that H_t has purely real zeros if and only if $t \geqslant \Lambda$. The Riemann hypothesis is then clearly equivalent to the upper bound $\Lambda \leqslant 0$. Newman conjectured the complementary lower bound $\Lambda \geqslant 0$ and noted that this conjecture asserts that if the Riemann hypothesis is true, it is only 'barely so'. As progress towards this conjecture, several lower bounds on Λ were established; see Table 1.

Table 1. Previous lower bounds on Λ. Dates listed are publication dates. The final four results use the method of Csordas, Smith and Varga [11].

Lower bound on Λ	References
$-\infty$	Newman 1976 [17]
-50	Csordas–Norfolk–Varga 1988 [8]
-5	te Riele 1991 [22]
-0.385	Norfolk–Ruttan–Varga 1992 [18]
-0.0991	Csordas–Ruttan–Varga 1991 [10]
-4.379×10^{-6}	Csordas–Smith–Varga 1994 [11]
-5.895×10^{-9}	Csordas–Odlyzko–Smith–Varga 1993 [9]
-2.63×10^{-9}	Odlyzko 2000 [19]
-1.15×10^{-11}	Saouter–Gourdon–Demichel 2011 [23]

We also mention that the upper bound $\Lambda \leqslant 1/2$ of de Bruijn [4] was sharpened slightly by Ki, Kim and Lee [13] to $\Lambda < 1/2$. (In press: this bound has recently been improved to $\Lambda \leqslant 0.22$ in [21].) See also [5, 25] on work on variants of Newman's conjecture and [3, Ch. 5] for a survey.

The main result of this paper is to affirmatively settle Newman's conjecture:

THEOREM 1. *One has $\Lambda \geqslant 0$.*

We now discuss the methods of the proof. Starting from the work of Csordas–Smith–Varga [11], the best lower bounds on Λ were obtained by exploiting the following repulsion phenomenon: if Λ was significantly less than zero, then adjacent zeros of H_0 (or of the Riemann ξ function) cannot be too close to each other (as compared with the other nearby zeros). See [11, Theorem 1] for a precise statement. In particular, a negative value of Λ gives limitations on the quality of 'Lehmer pairs' [14], which roughly speaking refer to pairs of adjacent zeros of the Riemann zeta function that are significantly closer to each other than the average spacing of zeros at that level. The lower bounds on Λ in [9, 11, 19, 23] then follow from numerically locating Lehmer pairs of increasingly high quality. (See also [26] for a refinement of the Lehmer pair concept used in the above papers.)

In principle, one could settle Newman's conjecture by producing an infinite sequence of Lehmer pairs of arbitrarily high quality. As suggested in [19], we were able to achieve this under the Gaussian Unitary Ensemble (GUE) hypothesis on the asymptotic distribution of zeros of the Riemann zeta function; we do not detail this computation here as it is superseded by our main result. (A sketch of the argument may be found at `terrytao.wordpress.com/2018/01/20`.) However, without the GUE hypothesis, the known upper bounds on narrow gaps between zeros (for example, [7]) do not appear to be sufficient to make this strategy work, even if one assumes the Riemann hypothesis (which one can do for Theorem 1 without loss of generality). Instead, we return to the analysis in [11] and strengthen the repulsion phenomenon to a *relaxation to local equilibrium* phenomenon: if Λ is negative, then the zeros of H_0 are not only repelled from each other but will nearly always be arranged locally as an approximate arithmetic progression, with the gaps between zero mostly staying very close to the global average gap that is given by the Riemann–von Mangoldt formula. (To illustrate the equilibrium nature of arithmetic progressions under backward heat flow, consider the entire functions $F_t(z) := e^{tu^2}\cos(zu)$ for some fixed real $u > 0$. These functions all have zeros on the arithmetic progression $\{\frac{2\pi(k+\frac{1}{2})}{u} : k \in \mathbb{Z}\}$ and solve the backward heat equation $\partial_t F_t = -\partial_{zz} F$.)

To obtain the local relaxation to equilibrium under the hypothesis that $\Lambda < 0$ requires a sequence of steps in which we obtain increasingly strong control on the distribution of zeros of H_t for $\Lambda < t \leqslant 0$ (actually for technical reasons, we will need to move t away from Λ as the argument progresses, restricting instead to ranges such as $\Lambda/2 \leqslant t \leqslant 0$ or $\Lambda/4 \leqslant t \leqslant 0$). The first step is to obtain Riemann–von Mangoldt type formulae for the number of zeros of H_t in an interval such as $[0, T]$ or $[T, T+\alpha]$, where $T \geqslant 2$ and $0 < \alpha \leqslant o(T)$. When $t = 0$, we can obtain asymptotics of $\frac{T}{4\pi}\log\frac{T}{4\pi} - \frac{T}{4\pi} + O(\log T)$ and $\frac{\alpha}{4\pi}\log T + o(\log T)$ by the classical Riemann–von Mangoldt formula and a result of Littlewood, respectively; this gives good control on the zeros down to length scales $\alpha \asymp 1$. For $\Lambda < t < 0$, we were only able to obtain the weaker bounds of $\frac{T}{4\pi}\log\frac{T}{4\pi} - \frac{T}{4\pi} + O(\log^2 T)$ and $\frac{\alpha}{4\pi}\log T + o(\log^2 T)$, respectively, down to length scales $\alpha \asymp \log T$, but it turns out that these bounds still (barely) suffice for our arguments; see Section 3. A key input in the proof of the Riemann–von Mangoldt type formula will be some upper and lower bounds for $H_t(x - iy)$ when y is comparable to $\log x$; see Lemma 4 for a precise statement. The main tool used to prove these bounds is the saddle point method, in which various contour integrals are shifted until they resemble the integral for the gamma function, to which the Stirling approximation may be applied.

It was shown in [11] that in the region $\Lambda < t \leqslant 0$, the zeros $x_j(t)$ of H_t are simple, and furthermore evolve according to the system of ordinary differential

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equations (ODEs)

$$\partial_t x_k(t) = 2 \sum_{j:\, j \neq k} \frac{1}{x_k(t) - x_j(t)}; \tag{5}$$

see Theorem 11 for a more precise statement. One can view this equation as describing the dynamics of a system of 'particles' x_j, in which every pair of particles x_j, x_k experiences a repulsion that is inversely proportional to their separation. (We caution however that the dynamics here are not Newtonian in nature since (5) prescribes the velocity $\partial_t x_k$ of each particle rather than the acceleration $\partial_t^2 x_k$. Nevertheless we found the physical analogy to be helpful in locating the arguments used in this paper.) By refining the analysis in [11], we can obtain a more quantitative lower bound on the gap $x_{j+1}(t) - x_j(t)$ between adjacent 'particles' (zeros), in particular establishing a bound of the form

$$\log \frac{1}{x_{j+1}(t) - x_j(t)} \ll \log^2 j \log\log j$$

for all large j in the range $\Lambda/2 \leqslant t \leqslant 0$; see Proposition 13 for a more precise statement. While far from optimal, this bound almost allows one to define the *Hamiltonian*

$$\mathcal{H}(t) := \sum_{j,k:\, j \neq k} \log \frac{1}{|x_j(t) - x_k(t)|},$$

although in practice we will have to apply some spatial cutoffs in j, k to make this series absolutely convergent. For the sake of this informal overview, we ignore this cutoff issue for now. The significance of this quantity is that system (5) can (formally, at least) be viewed as the gradient flow for the Hamiltonian $\mathcal{H}(t)$. In particular, there is a formal monotonicity formula

$$\partial_t \mathcal{H}(t) = -4E(t), \tag{6}$$

where the *energy* $E(t)$ is defined as

$$E(t) := \sum_{j,k:\, j \neq k} \frac{1}{|x_j(t) - x_k(t)|^2}.$$

Again, in practice, one needs to apply spatial cutoffs to j, k to make this quantity finite, and one then has to treat various error terms arising from this cutoff, which among other things 'renormalizes' the summands $\frac{1}{|x_j(t)-x_k(t)|^2}$ so that the renormalized energy vanishes when the zeros are arranged in the equilibrium state of an arithmetic progression; we ignore these issues for the current discussion. A further formal calculation indicates that $E(t)$ is monotone nonincreasing in time (so that $\mathcal{H}(t)$ is formally convex in time, as one would expect for the gradient flow of a convex Hamiltonian). Exploiting (a variant

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of) equation (6), we are able to control integrated energies that resemble the quantities $\int_{\Lambda/2}^{0} E(t)\,dt$; see first the weak preliminary integrated energy bound in Proposition 15 and then the final integrated energy bound in Theorem 17. By exploiting local monotonicity properties of the energy (and using a pigeonholing argument of Bourgain [2]), we can then obtain good control (a truncated version) of the energy $E(t)$ at time $t = 0$, which intuitively reflects the assertion that the 'particles' $x_j(t)$ are close to local equilibrium at time $t = 0$. This implies that the zeros of the Riemann zeta function behave locally like an arithmetic progression on the average. However, this can be ruled out by the existing results on the local distribution of zeros, such as pair correlation estimates of Montgomery [15]. As it turns out, it will be convenient to make use of a closely related estimate of Conrey, Ghosh, Goldston, Gonek and Heath-Brown [6].

It may be possible to use the methods of this paper to also address the generalized Newman conjecture introduced in [25], but we do not pursue this direction here.

REMARK 2. It is interesting to compare this with the results in [13, Theorem 1.14], which show that regardless of the value of Λ, the zeros of H_t will be spaced like an arithmetic progression on average for any *positive* t.

REMARK 3. In press: We note that in forthcoming work, Alex Dobner has found a proof that $\Lambda \geqslant 0$, which avoids the heat equation approach we have used here. Dobner's approach instead relies on a Riemann–Siegel type approximation for H_t in order to demonstrate the existence of zeros off the critical line. There is also some very intriguing numerical work of Rudolph Dwars (see the comments to `terrytao.wordpress.com/2018/12/28`) that suggests that many of the zeros of $H_t, t < 0$ away from the critical line organize around deterministic curves.

1.1. Notation. Throughout the rest of the paper, we will assume for the sake of contradiction that Newman's conjecture fails:

$$\Lambda < 0.$$

In particular, this implies the Riemann hypothesis (which, as mentioned previously, is equivalent to the assertion $\Lambda \leqslant 0$).

We will have a number of logarithmic factors appearing in our upper bounds. To avoid the minor issue of the logarithm occasionally being negative, we will use the modified logarithm

$$\log_+(x) := \log(2 + |x|)$$

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for several of these bounds. We also use the standard branch of the complex logarithm, with imaginary part in the interval $(-\pi, \pi]$, and the standard branch $z^{1/2} := \exp(\frac{1}{2}\log z)$ of the square root, defined using the standard branch of the complex logarithm.

Let $\Lambda < t \leqslant 0$; then the zeros of H_t are all real and symmetric around the origin. It is a result of Csordas, Smith and Varga [11, Corollary 1] that the zeros are also distinct and avoid the origin. Thus we can express the zeros of H_t as $(x_j(t))_{j\in\mathbb{Z}^*}$, where $\mathbb{Z}^* := \mathbb{Z}\backslash\{0\}$ are the nonzero integers,

$$0 < x_1(t) < x_2(t) < \dots,$$

and $x_{-j}(t) = -x_j(t)$ for all $j \geqslant 1$.

For any real numbers $j_- \leqslant j_+$, we use $[j_-, j_+]_{\mathbb{Z}^*}$ to denote the discrete interval

$$[j_-, j_+]_{\mathbb{Z}^*} := \{j \in \mathbb{Z}^* : j_- \leqslant j \leqslant j_+\}.$$

We use the usual asymptotic notation $X \ll Y$, $Y \gg X$, or $X = O(Y)$ to denote a bound of the form $|X| \leqslant CY$ for some absolute constant C, and write $X \asymp Y$ for $X \ll Y \ll X$. Note that as Λ is also an absolute constant, C can certainly depend on Λ; thus, for instance, $|\Lambda| \asymp 1$. If we need the implied constant C to depend on other parameters, we will indicate this by subscripts, thus, for instance, $X = O_\kappa(Y)$ denotes the estimate $|X| \leqslant C_\kappa Y$ for some C depending on κ. If the quantities X, Y depend on an asymptotic parameter such as T, we write $X = o_{T\to\infty}(Y)$ to denote a bound of the form $|X| \leqslant c(T)Y$, where $c(T)$ is a quantity that goes to zero as $T \to \infty$.

For X and Y depending on an asymptotic parameter T, we will also use the notation $X \lessapprox Y$ or $X = \tilde{O}(Y)$ for $X \ll Y \log^{O(1)} T$ in the last two sections of this paper.

Furthermore, in sums that will appear which depend on a parameter T, we say that indices j, k are *nearby*, and write $j \sim_T k$ if one has $0 < |j - k| < (T^2 + |j| + |k|)^{0.1}$.

We will use a marked sum to indicate principle value summation:

$$\sideset{}{'}\sum_j \cdots = \lim_{J\to\infty} \sum_{|j|\leqslant J} \cdots .$$

In cases where there is any chance of confusion for the range of summation, we record the index being summed and use a colon to indicate its range; for example, we write $\sum_{j:\, j\neq k}$ to indicate that the summation is over j, and j is to not equal k (where k is fixed outside the sum). Semicolons are used to separate additional conditions.

We use the phrase *for almost every t* throughout this paper to denote that a relation holds for all t except a set of null Lebesgue measure.

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2. Asymptotics of H_t

In this section, we establish some upper and lower bounds on $H_t(z)$ and its logarithmic derivative $\frac{H_t'}{H_t}(z)$. We will be able to obtain reasonable upper bounds in the regime where $z = x - iy$ with $y = O(\log_+ x)$, and obtain more precise asymptotics when $y \asymp \log_+ x$ (as long as the ratio $y/\log_+ x$ is large enough); this will be the key input for the Riemann–von Mangoldt type asymptotics in the next section. More precisely, we show the following.

LEMMA 4. *Let $z = x - i\kappa \log_+ x$ for some $x \geqslant 0$ and $0 \leqslant \kappa \leqslant C$, and let $\Lambda < t \leqslant 0$. Then one has*

$$H_t(z) \ll \exp\left(-\frac{\pi x}{8} + O_C(\log_+^2 x)\right). \tag{7}$$

(The reader is advised not to take the numerous factors of π, $\sqrt{2}$ and so on appearing in this section too seriously, as the exact numerical values of these constants are not of major significance in the rest of the arguments.) Furthermore, there is an absolute constant $C' > 0$ (not depending on C) such that if $\kappa \geqslant C'$, then one has the refinement

$$H_t(z) = \exp\left(-\frac{\pi x}{8} + O_C(\log_+^2 x)\right), \tag{8}$$

as well as the additional estimate

$$\frac{H_t'}{H_t}(z) = \frac{i}{4}\log\left(\frac{iz}{4\pi}\right) + O_C\left(\frac{\log_+ x}{x}\right), \tag{9}$$

using the standard branch of the complex logarithm.

REMARK 5. With a little more effort, one could replace the hypothesis $\Lambda < t$ here by $-C < t$; in particular (in contrast to the remaining arguments in this paper), these results are nonvacuous when $\Lambda \geqslant 0$. However, we will need to assume $\Lambda < t$ in the application of these estimates in the next section, particularly with regard to the proof of (49). Our proof methods also allow for a more precise version of asymptotic (8) (as one might expect given the level of precision in (9)), but such improvements do not seem to be helpful for the rest of the arguments in this paper. In the $t = 0$ case, one can essentially obtain Dirichlet series expansions for $\frac{1}{H_0(z)}$ or $\frac{H_0'}{H_0}(z)$, which allow one to also obtain bounds such as (8) or (9) when the imaginary part of z is much smaller than $\log_+ x$. However, in the $t < 0$ case, there does not appear to be any usable series expansions for $\frac{1}{H_t}(z)$ or $\frac{H_t'}{H_t}(z)$ that could be used to prove (8) or (9). Instead, we will prove these estimates by computing $H_t(z)$ to a high degree of accuracy, which we can only do when y is greater than or equal to a large multiple of $\log_+ x$ in order to ensure that the series expansions we have for $H_t(z)$ converge rapidly.

We begin by treating the easy case $t = 0$, in which we can exploit identity (1). We have the very crude bound

$$\zeta(\sigma + i\tau) \ll (1 + |\tau|)^{O(1)} \tag{10}$$

whenever $\sigma \geqslant 1/2$ and $\tau \in \mathbb{R}$ (this follows for instance from [29, Theorem 4.11]). In the region $\sigma \geqslant 1/4$, we also have the Stirling approximation (see for example, [1, 6.1.41])

$$\begin{aligned} \Gamma(\sigma + i\tau) = \exp\Bigg(\left(\sigma + i\tau - \frac{1}{2}\right) \log(\sigma + i\tau) - (\sigma + i\tau) \\ + \log\sqrt{2\pi} + O\left(\frac{1}{|\sigma + i\tau|}\right)\Bigg), \end{aligned} \tag{11}$$

where we use the standard branch of the logarithm; in particular,

$$\Gamma(\sigma + i\tau) \ll \exp\left(\left(\sigma - \frac{1}{2}\right) \log|\sigma + i\tau| - \tau \arctan\frac{\tau}{\sigma} - \sigma \right). \tag{12}$$

As $\arctan\frac{\tau}{\sigma} = \frac{\pi}{2}\operatorname{sgn}(\tau) + O(\frac{\sigma}{\sigma + |\tau|})$, we have in particular that

$$\Gamma(\sigma + i\tau) \ll \exp\left(-\frac{\pi}{2}|\tau| + O(\sigma \log_+(|\sigma| + |\tau|)) \right).$$

Inserting these bounds into (1) and (2), we obtain the crude upper bound

$$H_0(x - iy) \ll \exp\left(-\frac{\pi|x|}{8} + O((1 + y)\log_+(|x| + y)) \right) \tag{13}$$

for $x \in \mathbb{R}$ and $y \geqslant 0$. This gives the $s = 0$ case of (7). As is well known, when $\sigma \geqslant 2$ (say), we can improve (10) to

$$|\zeta(\sigma + i\tau)| \asymp 1$$

and so we obtain the improvement

$$H_0(x - iy) = \exp\left(-\frac{\pi|x|}{8} + O((1 + y)\log_+(|x| + y)) \right)$$

when $y \geqslant C' \log_+ x$ (in fact, in this case it would suffice to have $y \geqslant 4$, say). This gives the $s = 0$ case of (8). Finally, from taking logarithmic derivatives of (1) and (2), one has

$$\frac{H_0'}{H_0}(z) = \frac{i}{2}\left(\frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \frac{\zeta'}{\zeta}(s) \right),$$

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where $s := \frac{1}{2} + \frac{iz}{2}$. From taking log-derivatives of (11) using the Cauchy integral formula, one has the well-known asymptotic

$$\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) = \log\frac{s}{2} + O\left(\frac{1}{|s|}\right)$$

for the digamma function $\frac{\Gamma'}{\Gamma}$, and from the Dirichlet series expansion $\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} \ll \sum_{n=2}^{\infty}\frac{\log n}{n^{\mathrm{Re}\, s}}$, one can easily establish the bound

$$\frac{\zeta'}{\zeta}(s) \ll \frac{1}{|s|}$$

in the regime $C'\log_+ x \leqslant y \leqslant C\log x$. Putting all this together, one obtains (9) in this case.

Henceforth, we address the $t < 0$ case. We begin with the proof of the upper bound (7). Here it will be convenient to exploit the fundamental solution for the (backward) heat equation to relate H_t with H_0. Indeed, for any $t < 0$, we have the classical heat equation (or Gaussian) identity

$$e^{tu^2}\exp(izu) = \frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} e^{-r^2/4}\exp(i(z + r|t|^{1/2})u)\,dr \tag{14}$$

for any complex numbers z, u; replacing z, r by $-z, -r$ and averaging, we conclude that

$$e^{tu^2}\cos(zu) = \frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} e^{-r^2/4}\cos\left((z + r|t|^{1/2})u\right)\,dr.$$

Multiplying by $\Phi(u)$, integrating u from 0 to infinity and using Fubini's theorem, we conclude that

$$H_t(z) = \frac{1}{\sqrt{4\pi}}\int_{\mathbb{R}} e^{-r^2/4}H_0(z + r|t|^{1/2})\,dr. \tag{15}$$

Applying (13), the triangle inequality and the hypothesis $\Lambda < t \leqslant 0$, we conclude that

$$\begin{aligned} H_t(x - iy) \ll \exp&\left(-\frac{\pi|x|}{8} + O((1+y)\log_+(|x|+y))\right) \\ &\times \int_{\mathbb{R}} \exp\left(-\frac{r^2}{4} + O((1+y+|r|)\log_+ r)\right)\,dr. \end{aligned}$$

Using $(1 + |r|)\log_+ r \leqslant \varepsilon r^2 + O_\varepsilon(1)$ and $y\log_+ r \ll \varepsilon r^2 + O_\varepsilon(y^2)$ for any absolute constant $\varepsilon > 0$, we have

$$-\frac{r^2}{4} + O(|r|) + O((1 + y + |r|)(1 + \log_+ r)) \leqslant -\frac{r^2}{8} + O((1+y)^2),$$

thus arriving at the bound

$$H_t(x - iy) \ll \exp\left(-\frac{\pi|x|}{8} + O((1 + y)\log_+ |x| + (1 + y)^2)\right).$$

Since $y = O_C(\log_+ x)$, this gives (7).

To prove the remaining two bounds (8) and (9), it is convenient to cancel off the $t = 0$ case that has already been established, and reduce to showing that

$$\frac{H_t}{H_0}(z) = \exp\left(O_C(\log_+^2 x)\right) \tag{16}$$

and

$$\frac{H_t'}{H_t}(z) - \frac{H_0'}{H_0}(z) \ll_C \frac{\log_+ x}{x} \tag{17}$$

when $\kappa \geqslant C'$. To prove these estimates, the heat equation approach is less effective due to the significant oscillation present in H_0. Instead, we will use the method of steepest descent (also known as the saddle point method) to shift contours to where the phase is stationary rather than oscillating. We allow all implied constants to depend on C. We may assume that x is larger than any specified constant C'' (depending on C) as the case $x = O_C(1)$ follows trivially from compactness, since the zeros of H_t for $t \geqslant \Lambda$ are all real, so that $H_t(z)$ is bounded away from zero in this region of interest.

Now suppose that $z = x - iy$, where $y = \kappa \log_+ x$ for some $C' \leqslant \kappa \leqslant C$; in particular, C is large since C' is. As Φ is even, we may write (4) as

$$H_t(z) = \frac{1}{2}\int_{\mathbb{R}} e^{tu^2}\Phi(u)e^{izu}\,du.$$

From (3) and Fubini's theorem (which can be justified when $t < 0$), we conclude that

$$H_t(z) = \frac{1}{2}\sum_{n=1}^{\infty} 2\pi^2 n^4 I_t(\pi n^2, 9 + y + ix) - 3\pi n^2 I_t(\pi n^2, 5 + y + ix), \tag{18}$$

where $I_t(b, \zeta)$ denotes the oscillatory integral

$$I_t(b, \zeta) := \int_{\mathbb{R}} \exp(tw^2 - be^{4w} + \zeta w)\,dw, \tag{19}$$

which is an absolutely convergent integral for $t < 0$ whenever $\operatorname{Re} b > 0$.

We therefore need to obtain good asymptotics on $I_t(b, \zeta)$ for $b \geqslant 1$ and ζ in the region

$$\Omega := \{y + ix : x \geqslant C''; C' \log_+ x \leqslant y \leqslant 2C \log_+ x\}. \tag{20}$$

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Observe that the phase $tw^2 - be^{4w} + \zeta w$ has a stationary point at the origin when $4b = \zeta$. In general, $4b$ will not equal ζ; however, for any complex number w_0 in the strip

$$\left\{ w_0 \in \mathbb{C} : 0 \leqslant \mathrm{Im}(w_0) < \frac{\pi}{8} \right\}, \tag{21}$$

we see from shifting the contour in (19) to the horizontal line $\{w + w_0 : w \in \mathbb{R}\}$ that we have the identity

$$I_t(b, \zeta) = \exp(tw_0^2 + \zeta w_0) I_t(be^{4w_0}, \zeta + 2tw_0) \tag{22}$$

whenever $b > 0$ (so that be^{4w_0} has a positive real part). We will thus be able to reduce to the stationary phase case $4b = \zeta$ if we can solve the equation

$$4be^{4w_0} = \zeta + 2tw_0 \tag{23}$$

in strip (21). This we do in the following lemma. (One could also write w_0 explicitly in terms of the Lambert W-function as $w_0 = -\frac{\zeta}{2t} + \frac{1}{4} W(-\frac{8b}{t} \exp(-\frac{2\zeta}{t}))$, but we will not use this expression in this paper, and in fact will not explicitly invoke any properties of the W-function in our arguments.)

LEMMA 6. *If $b \geqslant 1$ and $\zeta \in \Omega$, then there exists a unique $w_0 = w_0(b, \zeta)$ in strip (21) such that (23) holds. Furthermore, we have the following estimates:*

(i) $\mathrm{Re}(4be^{4w_0}) \geqslant 1$.

(ii) *(Precise asymptotic for small and medium b) If $\zeta = y + ix$ and $b \leqslant x \exp(100 \frac{x^{1/2}}{|t|})$, then*

$$w_0 = \frac{1}{4} \log \frac{x}{4b} + O^{\mathbb{R}}\left(\frac{1}{x}\right) + i\left(\frac{\pi}{8} - \frac{y}{4x} - \frac{t \log \frac{x}{4b}}{8x} + O_C^{\mathbb{R}}\left(\frac{\log_+^2 x}{x^{3/2}}\right)\right),$$

where the superscript in the $O()$ notation indicates that these quantities are real-valued.

(iii) *(Crude bound for huge b) If $\zeta = y + ix$ and $b > x \exp(\frac{x^{1/2}}{|t|})$, then* $\mathrm{Re} w_0$ *is negative; in fact, we have*

$$-\mathrm{Re} w_0 \geqslant \tfrac{1}{8} \log_+ b.$$

Proof. The function $w_0 \mapsto 4be^{4w_0} - 2tw_0$ traverses the graph $\{a + i(\frac{\pi|t|}{4} + 4be^{2a/|t|}) : a \in \mathbb{R}\}$ on the upper edge $\{\frac{a}{2|t|} + i\frac{\pi}{8} : a \in \mathbb{R}\}$ of strip (21), while the lower edge of the strip is of course mapped to the real axis. Since $|t| \leqslant \Lambda$

and C, C' are large, the region Ω lies between these two curves, and so from the argument principle (and observing that the map $w_0 \mapsto 4be^{4w_0} - 2tw_0$ sends the line segments $\{-R + i\beta : 0 < \beta < \pi/8\}$ and $\{R + i\beta : 0 < \beta < \pi/8\}$ well to the left and right of ζ, respectively, for R large enough), for every $\zeta \in \Omega$, there exists exactly one w_0 in strip (21) such that $4be^{4w_0} - 2tw_0 = \zeta$, which is of course equivalent to (23). The uniqueness implies that the holomorphic function $w_0 \mapsto 4be^{4w_0} - 2tw_0$ has a nonzero derivative at this value of w_0.

Now write $\zeta = y + ix$ as per (20), and write $w_0 = \alpha + i\beta$ for some $\alpha \in \mathbb{R}$ and $0 < \beta < \pi/8$. Taking real and imaginary parts in (23), we have the system of equations

$$4be^{4\alpha} \cos 4\beta = y + 2t\alpha \tag{24}$$

and

$$4be^{4\alpha} \sin 4\beta = x + 2t\beta. \tag{25}$$

To prove (i), suppose for contradiction that $\mathrm{Re}(4be^{4w_0}) < 1$; thus

$$4be^{4\alpha} \cos 4\beta \leqslant 1. \tag{26}$$

Since $t, \beta = O(1)$, we see from (25) that $4be^{4\alpha} \sin 4\beta \ll x$, and hence from $\sin^2 4\beta + \cos^2 4\beta = 1$, we have

$$4be^{4\alpha} \ll x$$

and hence (since $b \geqslant 1$) $\alpha \leqslant \frac{1}{4} \log_+ x + O(1)$. In particular, $-2t\alpha \leqslant \frac{|t|}{2} \log_+ x + O(1)$. Inserting this into (24) and using (26), one then has

$$y \leqslant \frac{|t|}{2} \log_+ x + O(1),$$

which contradicts (20) since $|t| \leqslant \Lambda$ and C' is large.

Now we show (ii). From (25) and $\sin 4\beta \leqslant 1$, $t, \beta = O(1)$, one has

$$4be^{4\alpha} \geqslant x - O(1)$$

and hence on taking logarithms (and using the fact that $b \geqslant 1$ and x is large)

$$\alpha \geqslant \frac{1}{4} \log \frac{x}{4b} - O\left(\frac{1}{x}\right). \tag{27}$$

On the other hand, from squaring (24) and (25) and summing, we have

$$(4be^{4\alpha})^2 = (y + 2t\alpha)^2 + (x + 2t\beta)^2. \tag{28}$$

Crudely bounding $x+2t\beta = O(x)$, $y = O(x)$, $b \geqslant 1$ and $t = O(1)$, we conclude that

$$e^{8\alpha} \ll x^2 + \alpha^2$$

, which implies that $\alpha \leqslant O(\log_+ x)$. From the hypothesis $b \leqslant x \exp(100\frac{x^{1/2}}{|t|})$ and (27), we also have $\alpha \geqslant -O(x^{1/2}/t)$; thus $t\alpha \ll x^{1/2}$. Returning to (28) and using $2t\beta = O(1)$ and $y \ll x^{1/2}$, we conclude that

$$(4be^{4\alpha})^2 = x^2 + O(x).$$

So on taking square roots

$$4be^{4\alpha} = x + O(1) \tag{29}$$

and hence on taking logarithms, we have the matching upper bound

$$\alpha \leqslant \frac{1}{4}\log\frac{x}{4b} + O\left(\frac{1}{x}\right)$$

to (27). In particular,

$$y + 2t\alpha = y + \frac{t\log\frac{x}{4b}}{2} + O\left(\frac{1}{x}\right).$$

Inserting this and (29) into (24), we have

$$\cos 4\beta = \frac{y}{x} + \frac{t\log\frac{x}{4b}}{2x} + O\left(\frac{1}{x^{3/2}}\right)$$

and hence (by Taylor expansion of the arc cosine function)

$$4\beta = \frac{\pi}{2} - \frac{y}{x} - \frac{t\log\frac{x}{4b}}{2x} + O_C\left(\frac{\log_+^2 x}{x^{3/2}}\right),$$

giving (ii).

Finally, we prove (iii). From identity (28) and crudely bounding $y, t\beta = O(x)$, we have

$$(4be^{4\alpha})^2 \ll x^2 + t^2|\alpha|^2$$

and hence either

$$e^{-4\alpha} \gg \frac{b}{x}$$

or

$$e^{-4\alpha} \gg \frac{b}{|t||\alpha|}.$$

Under the hypothesis $b > x\exp(\frac{x^{1/2}}{|t|})$, so that $1/|t|$ and x are $O(b^{1/10})$ (say), so both options force $-\alpha \geqslant \frac{1}{8}\log b$ as claimed. □

We combine the above lemma with the following asymptotic.

LEMMA 7. *Let b be a complex number with* $\mathrm{Re}\, b \geqslant 1$. *Then*

$$I_t(b, 4b) = \sqrt{\frac{\pi}{8}} \exp(-b) \left(\frac{1}{\sqrt{b}} + O\left(\frac{1}{|b|^{3/2}} \right) \right) \tag{30}$$

using the standard branch of the square root.

Proof. One could establish this from Laplace's method, but we will instead use the Stirling approximation (11). (We thank Alex Dobner for pointing out some issues in the original proof of this lemma and suggesting a repaired proof, which is reproduced here.) Writing

$$e^{tw^2} = \int_{\mathbb{R}} e^{4i\xi w} \, d\mu(\xi)$$

, where μ is the Gaussian probability measure

$$d\mu(\xi) := \frac{2}{\sqrt{\pi |t|}} e^{-4\xi^2/|t|}$$

of mean zero and variance $|t|/8$, and applying Fubini's theorem, we obtain

$$I_t(b, 4b) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \exp(-be^{4w} + 4(b + i\xi)w) \, dw \right) d\mu(\xi).$$

Making the change of variables $r = be^{4w}$ (and contour shifting or analytic continuation) and the definition $\Gamma(s) = \int_0^\infty e^{-r} r^{s-1} \, dr$ of the Γ function, we see that

$$\int_{\mathbb{R}} \exp(-be^{4w} + 4(b + i\xi)w) \, dw = \frac{1}{4} \exp(-(b + i\xi) \log b) \Gamma(b + i\xi)$$

and hence

$$I_t(b, 4b) = \frac{1}{4} \int_{\mathbb{R}} \exp(-(b + i\xi) \log b) \Gamma(b + i\xi) \, d\mu(\xi).$$

We divide the integral into regions $|\xi| \leqslant 10|t|^{1/2}|b|^{1/2}$ and $|\xi| > 10|t|^{1/2}|b|^{1/2}$. By applying the Stirling approximation, the integral over the first region becomes

$$\frac{1}{4} \int_{|\xi| \leqslant 10|t|^{1/2}|b|^{1/2}} \left(1 + O\left(\frac{1}{|b + i\xi|} \right) \right) \frac{\sqrt{2\pi}}{\sqrt{b + i\xi}}$$

$$\times \exp\left((b+i\xi)\log\left(1+\frac{i\xi}{b}\right)-b-i\xi\right)\,d\mu(\xi).$$

Note that we may assume that $|b|$ is sufficiently large so that $|\xi| < |b|/2$ in this region (the small $|b|$ case of the lemma follows trivially from compactness), and so we have $\frac{1}{\sqrt{b+i\xi}} = \left(1+O\left(\frac{|\xi|}{|b|}\right)\right)\frac{1}{\sqrt{b}}$ and $(b+i\xi)\log(1+\frac{i\xi}{b}) = i\xi + O\left(\frac{|\xi|^2}{|b|}\right)$. Substituting these expressions into the integrand, we get

$$\sqrt{\frac{\pi}{8b}}\exp(-b)\int_{|\xi|\leqslant 10|t|^{1/2}|b|^{1/2}}\left(1+O\left(\frac{1+|\xi|^2}{|b|}\right)\right)\,d\mu(\xi),$$

and now the integral evaluates to $1+O(\frac{1}{|b|})$. Thus it will suffice to establish the tail bound

$$\int_{|\xi|>10|t|^{1/2}|b|^{1/2}}\exp(-(b+i\xi)\log b)\Gamma(b+i\xi)\,d\mu(\xi) \ll \exp(-\mathrm{Re}(b))|b|^{-3/2}.$$

By applying the triangle inequality and bounding the integrand with

$$|\exp(-(b+i\xi)\log b)| \leqslant \exp\left(-\mathrm{Re}(b)\log|b| + \frac{\pi}{2}(|b|+|\xi|)\right)$$

and

$$|\Gamma(b+i\xi)| \leqslant \Gamma(\mathrm{Re}(b)) \leqslant \exp(\mathrm{Re}(b)\log|b| - \mathrm{Re}(b)),$$

we get the following upper bound:

$$\exp\left(-\mathrm{Re}(b)+\frac{\pi}{2}|b|\right)\frac{2}{\sqrt{\pi|t|}}\int_{|\xi|>10|t|^{1/2}|b|^{1/2}}\exp\left(\frac{\pi}{2}|\xi| - \frac{4\xi^2}{|t|}\right).$$

Now again we assume $|b|$ is large enough so that we have $\frac{\pi}{2}|\xi| - \frac{4\xi^2}{|t|} \leqslant -\frac{\xi^2}{|t|}$ for all ξ in the given region, and hence the integral is bounded above by

$$\exp\left(-\mathrm{Re}(b)+\frac{\pi}{2}|b|\right)\frac{2}{\sqrt{\pi|t|}}\int_{|\xi|>10|t|^{1/2}|b|^{1/2}}\exp\left(-\frac{\xi^2}{|t|}\right) \ll \exp(-\mathrm{Re}(b)-10|b|)$$

(say), and the claim follows. □

From the above two lemmas and (22), we have the asymptotic

$$I_t(b,\zeta) = \sqrt{\frac{\pi}{8}}\exp(tw_0^2 - be^{4w_0} + \zeta w_0)\left(\frac{1}{\sqrt{be^{4w_0}}} + O\left(\frac{1}{|be^{4w_0}|^{3/2}}\right)\right) \quad (31)$$

for any $b \geqslant 1$ and $\zeta \in \Omega$, where $w_0 = w_0(b,\zeta)$ is the quantity in Lemma 6.

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Now we can control sum (18). As before, we assume that $z = x - iy$, where $y = \kappa \log_+ x$ for some $C' \leqslant \kappa \leqslant C$. From (18), one has

$$H_t(x - iy) = \frac{1}{2}\sum_{n=1}^{\infty} Q_{t,n}, \tag{32}$$

where $Q_{t,n}$ is the quantity

$$Q_{t,n} := 2\pi^2 n^4 I_t(\pi n^2, 9 + y + ix) - 3\pi n^2 I_t(\pi n^2, 5 + y + ix).$$

We first consider the estimation of Q_n in the main case when n is not too huge, in the sense that

$$n \leqslant x \exp\left(100\frac{x^{1/2}}{|t|}\right). \tag{33}$$

In this case, if we apply Lemma 6(ii) with $\zeta = 9 + y + ix$ and $b = \pi n^2$, we have that the quantity $w_0 = w_{0,t,n}$ arising in that lemma obeys the asymptotics

$$w_0 = \frac{1}{4}\log\frac{x}{4\pi n^2} + O^{\mathbb{R}}\left(\frac{1}{x}\right) + i\left(\frac{\pi}{8} - \frac{9+y}{4x} - \frac{t\log\frac{x}{4\pi n^2}}{8x} + O_C^{\mathbb{R}}\left(\frac{\log_+^2 x}{x^{3/2}}\right)\right), \tag{34}$$

which when combined with (23) gives

$$4be^{4w_0} = ix + O_C(x^{1/2}).$$

In particular, the factor $\frac{1}{\sqrt{be^{4w_0}}} + O\left(\frac{1}{|be^{4w_0}|^{3/2}}\right)$ in (31) can be expressed as

$$\frac{1}{\sqrt{ix/4}}(1 + O_C(x^{-1/2})),$$

and thus by (31),

$$|I_t(\pi n^2, 9+y+ix)| = \sqrt{\frac{\pi}{2x}}\exp\left(\mathrm{Re}\left(tw_0^2 - \frac{\zeta}{4} - \frac{tw_0}{2} + \zeta w_0\right) + O_C\left(x^{-1/2}\right)\right),$$

where we have again used (23). From (34) (and using $t = O(1)$ and $y = O_C(\log_+ x)$ to bound some small error terms), we can calculate the quantity $\mathrm{Re}\left(tw_0^2 - \frac{\zeta}{4} - \frac{tw_0}{2} + \zeta w_0\right)$ to be

$$\frac{t}{16}\log^2\frac{x}{4\pi n^2} - \frac{t\pi^2}{64} - \frac{9+y}{4} - \frac{t}{8}\log\frac{x}{4\pi n^2} + \frac{9+y}{4}\log\frac{x}{4\pi n^2} - \frac{\pi x}{8}$$
$$+ \frac{9+y}{4} + \frac{t\log\frac{x}{4\pi n^2}}{8} + O_C\left(x^{-1/2}\right),$$

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and thus on cancelling and gathering terms, we obtain

$$|I_t(\pi n^2, 9+y+ix)| = \left(\frac{x}{4\pi n^2}\right)^{\frac{9+y}{4}} J_t K_{t,n} \exp(O_C(x^{-1/2},))$$

where $J_t = J_t(x)$ and $K_{t,n} = K_{t,n}(x)$ are the positive quantities

$$J_t := \sqrt{\frac{\pi}{2x}} \exp\left(\frac{t}{16}\log^2\frac{x}{4\pi} - \frac{t\pi^2}{64} - \frac{\pi x}{8}\right) \tag{35}$$

and

$$K_{t,n} := \exp\left(-\frac{t}{4}\left(\log\frac{x}{4\pi}\right)\log n + \frac{t}{4}\log^2 n\right).$$

A similar computation gives

$$|I_t(\pi n^2, 5+y+ix)| = \left(\frac{x}{4\pi n^2}\right)^{\frac{5+y}{4}} J_t K_{t,n} \exp\left(O_C\left(x^{-1/2}\right)\right).$$

In particular, we have the upper bound

$$Q_{t,n} \ll n^4 \left(\frac{x}{4\pi n^2}\right)^{\frac{9+y}{4}} J_t K_{t,n}$$

for $1 \leqslant n \leqslant x\exp(100\frac{x^{1/2}}{|t|})$, and for $n = 1$, we have the refinement

$$|Q_{t,1}| = (2\pi^2 + O_C(x^{-1/2}))\left(\frac{x}{4\pi}\right)^{\frac{9+y}{4}} J_t. \tag{36}$$

Using the crude bound

$$K_{t,b} \leqslant \exp\left(-\frac{t}{4}\left(\log\frac{x}{4\pi}\right)\log n\right) \leqslant n^{-\frac{t}{4}\log x},$$

we conclude that

$$Q_{t,n} \ll n^{-\frac{1+y}{2} - \frac{t}{4}\log x}|Q_{t,1}|.$$

Since $y \geqslant C'\log_+ x$, the $2 \leqslant n \leqslant x\exp(100\frac{x^{1/2}}{|t|})$ terms sum to $O(|Q_{t,1}|/x)$; thus

$$\sum_{n \leqslant x\exp(100\frac{x^{1/2}}{|t|})} Q_{t,n} = \left(1 + O_C\left(\frac{1}{x}\right)\right) Q_{t,1}.$$

Also, from (35), we have

$$|Q_{t,1}| \asymp \left(\frac{x}{4\pi}\right)^{\frac{9+y}{4}} J_t = \exp\left(-\frac{\pi x}{8} + O_C(\log_+^2 x)\right).$$

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Thus, to finish the proof of (8) (or (16)), one just needs to show that the tail $\sum_{n > x\exp(100\frac{x^{1/2}}{|t|})} Q_{t,n}$ is negligible compared with the main term $Q_{t,1} = \exp(-\frac{\pi x}{8} + O_C(\log_+^2 x))$. Suppose now that $n > x\exp(100\frac{x^{1/2}}{|t|})$. If we now apply Lemma 6(iii) with $\zeta = 9 + y + ix$ and $b = \pi n^2$, and write $w_0 = \alpha + i\beta$ with $0 < \beta < \pi/8$, we have that α is negative with

$$-\alpha \geqslant \tfrac{1}{8}\log n,$$

while from (31) and (23) (and Lemma 6(i)), we have

$$\begin{aligned} I_t(\pi n^2, 9 + y + ix) &\ll \exp\left(\operatorname{Re}\left(tw_0^2 - \frac{\zeta}{4} - \frac{tw_0}{2} + \zeta w_0\right)\right) \\ &\ll \exp\left(-|t||\alpha|^2 - \frac{|t||\alpha|}{2} + O_C(\log_+^2 x)\right), \end{aligned}$$

and similarly for $I_t(\pi n^2, 5+y+ix)$. Since $\log n \geqslant 100\frac{x^{1/2}}{|t|}$, we have $|\alpha| \geqslant 10\frac{x^{1/2}}{|t|}$ and thus

$$|t||\alpha|^2 \geqslant 10x^{1/2}\log n.$$

In particular, $n^4\exp(-|t||\alpha|^2) \ll \exp(-9x^{1/2}\log n)$ and thus

$$Q_{t,n} \ll \exp(-8x^{1/2}\log n + O_C(\log_+^2 x))$$

(say). Summing, we conclude that

$$\sum_{n > x\exp(100\frac{x^{1/2}}{|t|})} Q_{t,n} \ll \exp(-100x/|t|)$$

(say), which is certainly $O(|Q_1|/x)$. Inserting these bounds into (18), we conclude that

$$H_t(x - iy) = \left(\frac{1}{2} + O_C\left(\frac{\log_+^2 x}{x}\right)\right) Q_{t,1},$$

which already gives (7). Sending t to 0, taking absolute values, and then dividing using (36) and (35), we obtain after cancelling all the t-independent terms that

$$\left|\frac{H_t}{H_0}\right|(x - iy) = \left(1 + O_C\left(\frac{\log_+^2 x}{x}\right)\right)\exp\left(\frac{t}{16}\log^2\frac{x}{4\pi} - \frac{t\pi^2}{64}\right).$$

Since the ratio $\frac{H_t}{H_0}$ is holomorphic in the region of interest, we can thus find a holomorphic branch of $\log\frac{H_t}{H_0}$ for which

$$\operatorname{Re}\log\frac{H_t}{H_0}(z) - \frac{t}{16}\log^2\frac{z}{4\pi i} = -\frac{t\pi^2}{64} + O_C\left(\frac{\log_+^2 x}{x}\right)$$

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for all $z = x - iy$ in this region. Varying x, y by $O(\log_+ x)$ (adjusting the constants C, C', C'' slightly as necessary) and using the Borel–Carathéodory theorem and the Cauchy integral formula, we conclude that

$$\frac{d}{dz}\left(\log \frac{H_t}{H_0}(z) - \frac{t}{16}\log^2 \frac{z}{4\pi i}\right) = O_C\left(\frac{\log_+ x}{x}\right),$$

which gives (17) after a brief calculation.

3. Riemann–von Mangoldt type formulae

For any $\Lambda < t \leqslant 0$, the zeros of H_t are all real and simple [**11**, Corollary 1]. For any interval $I \subset \mathbb{R}$, let $N_t(I)$ denote the number of zeros of H_t in I. The classical Riemann–von Mangoldt formula (see for example, [**29**, Theorem 9.4]), combined with (1), gives the asymptotic

$$N_0([0, T]) = \Psi(T) + O(\log_+ T) \tag{37}$$

for all $T \geqslant 0$, where we use $\Psi : \mathbb{R}^+ \to \mathbb{R}$ to denote the function

$$\Psi(T) := \frac{T}{4\pi}\log \frac{T}{4\pi} - \frac{T}{4\pi}. \tag{38}$$

(It is traditional to also insert the lower order term $-\frac{7}{8}$ here, but this term will not be of use in our analysis and will therefore be discarded. The factors of 4π are not of particular significance and may be ignored by the reader on a first read.) For future reference, we record the derivative of Ψ as

$$\Psi'(T) = \frac{1}{4\pi}\log \frac{T}{4\pi}; \tag{39}$$

in particular, Ψ is increasing for $T > 4\pi$. Applying (37) with T replaced by $T + \alpha$ and subtracting, we conclude from the mean value theorem that

$$N_0([T, T + \alpha]) = \frac{\alpha \log_+ T}{4\pi} + O(\log_+ T) \tag{40}$$

for all $T \geqslant 0$ and $0 \leqslant \alpha \leqslant C$ for any fixed C, where the implied constants in the asymptotic notation are allowed to depend on C. Because we are assuming the Riemann hypothesis (and hence the Lindelöf hypothesis), one can improve this latter bound to

$$N_0([T, T + \alpha]) = \frac{\alpha \log_+ T}{4\pi} + o_{T\to\infty}(\log_+ T), \tag{41}$$

a result of Littlewood (see [**29**, Theorem 13.6]). (Indeed, on the Riemann hypothesis, one can improve the error term to $O\left(\frac{\log_+ T}{\log_+ \log_+ T}\right)$; see [**29**, Theorem

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14.13]. However, we will not need this further refinement in this paper.) A key input in these bounds is a lower bound on $|\zeta(s)|$ when $\mathrm{Re}(s)$ is somewhat large, for example, between 2 and 3; this is easily obtained through the Dirichlet series identity $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ that is valid in this region.

Define the *classical location* ξ_j of the jth zero for $j \geqslant 1$ to be the unique quantity in $(1, +\infty)$ solving the equation

$$\Psi(\xi_j) = j, \tag{42}$$

and extend this to negative j by setting $\xi_{-j} := -\xi_j$. (As with the quantity w_0 introduced in Lemma 6, one could express ξ_j explicitly in terms of the Lambert W function if desired as $\xi_j = 4\pi e \exp(W(j/e))$, but we will not use this relation in this paper.) Clearly, ξ_j are increasing in j. For future reference, we record the following bounds on ξ_j.

LEMMA 8 (Spacing of the classical locations).

(i) *For any* $j \geqslant 1$*, one has*

$$\xi_j = (1 + o_{|j|\to\infty}(1)) \frac{4\pi j}{\log_+ j}. \tag{43}$$

In particular, $\xi_j \asymp \frac{j}{\log_+ j}$ *and* $\log_+ \xi_j \asymp \log_+ j$.

(ii) *For any* $j, k \in \mathbb{Z}^*$*, one has*

$$|\xi_k - \xi_j| \asymp \frac{|k-j|}{\log_+(|\xi_j| + |\xi_k|)}. \tag{44}$$

(iii) *If* $1 \leqslant j \asymp k$*, then one has the more precise approximation*

$$\xi_k - \xi_j = \frac{4\pi(k-j)}{\log \xi_j} + O\left(\frac{|k-j|^2}{j \log^2 \xi_j}\right). \tag{45}$$

Of course, the implied constant in the error term in (45) can depend on the implied constants in the hypothesis $j \asymp k$.

Proof. If $j \geqslant 1$, then from (38), one has

$$\xi_j \log_+ \xi_j = (1 + o_{j\to\infty}(1)) 4\pi j, \tag{46}$$

which implies that $j^{1/2} \ll \xi_j \ll j$ (say), which implies that $\log_+ \xi_j \asymp \log_+ j$; substituting this back into (46) yields

$$\xi_j \asymp \frac{j}{\log_+ j}.$$

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This in turn implies that $\log_+ \xi_j = (1 + o_{j\to\infty}(1)) \log_+ j$, and using (46) one last time gives (42).

Now we obtain (ii). If j, k have opposing sign, then (44) follows from (43), so by symmetry we may assume that j, k are both positive. If j is much larger than k or *vice versa*, then bound (44) follows from (43) and the triangle inequality, so we may now restrict attention to the case $1 \leqslant j \asymp k$. Estimates (44) and (45) are trivial for $j = O(1)$, so we may assume j to be large.

From (42), we have

$$\Psi(\xi_k) - \Psi(\xi_j) = k - j$$

and hence by the mean value theorem and (39), we have

$$\frac{1}{4\pi} \log \frac{T}{4\pi} (\xi_k - \xi_j) = k - j$$

for some T between ξ_k and ξ_j. From (43), we see that $T \asymp \xi_j$, and so (44) follows. Furthermore, we can conclude that

$$T = \xi_j + O(|\xi_k - \xi_j|) = \xi_j + O\left(\frac{|k-j|}{\log \xi_j}\right)$$

and hence

$$\log T = \log \xi_j + O\left(\frac{|k-j|}{\xi_j \log \xi_j}\right) = \log \xi_j + O\left(\frac{|k-j|}{j}\right)$$

and

$$\frac{1}{\log T} = \frac{1}{\log \xi_j} + O\left(\frac{|k-j|}{j \log^2 \xi_j}\right),$$

giving (45). □

Applying (37) to $T = x_j(0)$ for some $j \geqslant 1$, we conclude in particular that

$$\Psi(x_j(0)) - \Psi(\xi_j) = O(\log_+ x_j(0)).$$

From (39) and the mean value theorem, we conclude that

$$x_j(0) = \xi_j + O(1) \tag{47}$$

for all $j \geqslant 1$, and hence for all $j \in \mathbb{Z}^*$ by symmetry. (One may wish to treat the bounded case $j = O(1)$ separately to avoid the minor issue that $\Psi(T)$ becomes decreasing for $T < 1$.) In particular, from (43) and the fact that $x_1(0) > 0$, we conclude that

$$x_j(0) \asymp \frac{j}{\log_+ \xi_j} \asymp \frac{j}{\log_+ j}$$

for all $j \geqslant 1$.

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In a similar vein, if $1 \leqslant j < k \leqslant j + \log_+ j$, then from applying (41) with $T = x_j(0)$ and α equal to (or slightly less than) $x_k(0) - x_j(0)$, we have

$$k - j = \frac{x_k(0) - x_j(0)}{4\pi} \log_+ \xi_j + o_{j\to\infty}(\log_+ \xi_j)$$

and hence

$$x_k(0) - x_j(0) = \frac{4\pi(k-j)}{\log_+ \xi_j} + o_{j\to\infty}(1).$$

Informally, this asserts that the zeros $x_j(0)$ behave like an arithmetic progression of spacing $\frac{4\pi}{\log_+ \xi_j}$ at spatial scales between $o(1)$ and 1. (In fact, when combined with (47) and (45), we see that this behaviour persists for all scales between $o(1)$ and $o(\xi_j)$.)

In this section, we use the asymptotics on H_t obtained in the previous section to establish analogous, but weaker, bounds for the zeros $x_j(t)$ of the functions H_t, in which we lose an additional logarithm factor in the error estimates.

THEOREM 9 (Riemann–von Mangoldt type formulae). *Let $\Lambda < t \leqslant 0$, $T > 0$, and let $0 \leqslant \alpha \leqslant C$ for some $C > 0$. Then one has*

$$N_t([0, T]) = \Psi(T) + O(\log_+^2 T) \tag{48}$$

and

$$N_t([T, T + \alpha \log_+ T]) = \frac{\alpha \log_+^2 T}{4\pi} + o_{T\to\infty}(\log_+^2 T). \tag{49}$$

The decay rate in the $o_{T\to\infty}()$ error term is permitted to depend on C but is otherwise uniform in α.

Repeating the previous analysis, we conclude the following.

COROLLARY 10 (Macroscopic structure of zeros). *Let $\Lambda < t \leqslant 0$. Then one has*

$$x_j(t) = \xi_j + O(\log_+ \xi_j) \tag{50}$$

for all $j \in \mathbb{Z}^*$; in particular,

$$x_j(t) \asymp \frac{j}{\log_+ \xi_j} \asymp \frac{j}{\log_+ j} \tag{51}$$

for all $j \geqslant 1$. We also have

$$x_k(t) - x_j(t) = \frac{4\pi(k-j)}{\log_+ \xi_j} + o_{j\to\infty}(\log_+ \xi_j) \tag{52}$$

whenever $1 \leqslant j < k \leqslant j + \log_+^2 \xi_j$.

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Informally, this corollary asserts that the zeros $x_j(t)$ behave like an arithmetic progression of spacing $\frac{4\pi}{\log_+ \xi_j}$ at spatial scales between $o(\log_+ \xi_j)$ and $o(\xi_j)$. This level of spatial resolution is worse by a factor of $\log_+ \xi_j$ than what one can achieve for $x_j(0)$, but will still (barely) be enough for our applications. We remark that a significantly sharper estimate (with an error term of just $O(1)$ in the analogue of (48)) is available for any fixed $t > 0$; see [**13**, Theorem 1.4]. (In press: even sharper estimates have recently been obtained in [**21**, Theorem 1.5].)

We now turn to the proof of the two bounds in Theorem 9.

Proof of (48). We make use of the argument principle in exactly the same manner as in the classical proof of the Riemann–von Mangoldt formula. By perturbing T slightly if necessary, we may assume that T is not a zero of H_t. Let $\kappa > 0$ be a sufficiently large absolute constant. Then the argument principle yields

$$N_t([0, T]) = \frac{1}{2\pi i} \int_{\Gamma} \frac{H_t'}{H_t}(z)\, dz,$$

where Γ is the counterclockwise contour carved out by a straight line from $i\kappa \log_+ 0 = i\kappa \log 2$ to $-i\kappa \log_+ 0 = -i\kappa \log 2$, then along the curve Γ_I parameterized by $x - i\kappa \log_+ x$ for $x \in [0, T]$, then along the line Γ_{II} from $T - i\kappa \log_+ T$ to T, then along the vertical line conjugate to Γ_{II} and the curve conjugate to Γ_I, leading back to $i \log 2$. As the integrand is odd, the integral along the line from $i\kappa \log_+ 0$ to $-i\kappa \log_+ 0$ vanishes. Using the symmetry $H_t(\overline{z}) = \overline{H_t(z)}$, we thus have

$$N_t([0, T]) = \frac{1}{\pi} \mathrm{Im} \left(\int_{\Gamma_I} + \int_{\Gamma_{II}} \right) \frac{H_t'}{H_t}(z)\, dz.$$

From (9) and (39), one sees that

$$\frac{1}{\pi} \frac{H_t'}{H_t}(z) = \frac{d}{dz}(\Psi(iz)) + O\left(\frac{\log_+ x}{x}\right)$$

for $z = x - i\kappa \log_+ x$ on Γ_I (extending Ψ to the right half-plane using the standard branch of the logarithm), and hence by the fundamental theorem of calculus

$$\begin{aligned}\frac{1}{\pi} \mathrm{Im} \int_{\Gamma_I} \frac{H_t'}{H_t}(z)\, dz &= \mathrm{Im}\Psi(iT + \kappa \log_+ T) - \Psi(\log_+ 0) + O(\log_+^2 T) \\ &= \Psi(T) + O(\log_+^2 T).\end{aligned}$$

On the other hand, if we let θ be a phase so that $e^{i\theta} H_t(T - i\kappa \log_+ T)$ is real and positive, then

$$\left| \mathrm{Im} \int_{\Gamma_{II}} \frac{H_t'}{H_t}(z)\, dz \right| \leqslant \pi(m + 1),$$

where m is the number of zeros of $\operatorname{Re} e^{i\theta} H_t(z)$ along the contour Γ_{II}, since the left-hand side is the change in $\arg e^{i\theta} H_t(z)$ as z varies over this contour, and for each increment of π in the value of $\arg e^{i\theta} H_t(z)$, we must have that $\operatorname{Re} e^{i\theta} H_t(z)$ is zero for some z. Note that the number of zeros of $\operatorname{Re} H_t(z)$ along this contour is the same as the number of zeros of

$$g(s) := \tfrac{1}{2}(e^{-i\theta} H_t(is + T) + e^{i\theta} H_t(-is + T))$$

as s ranges along the line from 0 to $\kappa \log_+ T$. Hence m is no more than the number of zeros m' of $g(s)$ in the disc of radius $\kappa \log_+ T$ centred at $\kappa \log_+ T$.

We can estimate the count m' with the Jensen formula as follows. Let $\mathcal{M}$ be the maximum of $g(s)$ in a disc centred at $\kappa \log_+ T$ of radius $2\kappa \log_+ T$. Using (7) and the conjugate symmetry of $H_t(z)$, we have

$$\mathcal{M} \ll e^{-\frac{\pi}{8} T + O(\log_+^2 T)}.$$

Since from (8) we have $g(\kappa \log_+ T) = e^{i\theta} H_t(T - i\kappa \log_+ T) = e^{-\frac{\pi}{8} T + O(\log_+^2 T)}$, it therefore follows from the Jensen formula (see for example, [16, Lemma 6.1]) that

$$m' \ll \log_+^2 T.$$

This induces a corresponding bound on the integral of $\frac{H_t'}{H_t}$ over Γ_{II} and therefore establishes the claimed estimate for $N_t([0, T])$. $\square$

Proof of (49). We will use a 'limiting profile argument' (also known as a 'compactness argument' or 'normal families argument'), in which one extracts and then studies a limit of suitably rescaled versions of a family of analytic functions to conclude asymptotic information about these functions. We remark that this sort of argument can also be used in a similar fashion to deduce the Lindelöf hypothesis from the Riemann hypothesis: see Theorem 1 of `terrytao.wordpress.com/2015/03/01`.

Suppose for contradiction that this claim failed; then there exists a sequence $T_n \to \infty$, and bounded sequences $\Lambda < t_n \leqslant 0$ and $0 \leqslant \alpha_n \leqslant C$, as well as an $\varepsilon > 0$, such that

$$\left| N_{t_n}([T_n, T_n + \alpha_n \log_+ T_n]) - \frac{\alpha_n \log_+^2 T_n}{4\pi} \right| > \varepsilon \log_+^2 T_n \tag{53}$$

for all n. By perturbing T_n slightly, we may assume that H_{t_n} does not vanish at T_n or $T_n + \alpha_n$.

Let $\kappa > 0$ be a sufficiently large absolute constant. By the hypothesis $\Lambda < t_n$, the function H_{t_n} has no zeros in the lower half-plane. Thus we can define holomorphic functions F_n on the lower half-plane by the formula

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$$F_n(z) := \frac{1}{\log_+^2 T_n} \log \frac{H_{t_n}(T_n + z \log_+ T_n)}{H_{t_n}(T_n - i\kappa \log_+ T_n)}$$

with the branch of the logarithm chosen so that $F_n(-i\kappa) = 0$. From (7), we see that F_n are uniformly bounded on any compact subset of the lower half-plane. Thus, by Montel's theorem (see [**24**, Section 3.2]), we may pass to a subsequence and assume that F_n converge locally uniformly to a holomorphic function F on the lower half-plane; since F_n all vanish on $-i\kappa$, F does also. Then by the Cauchy integral formula, the derivatives

$$F_n'(z) = \frac{1}{\log_+ T_n} \frac{H_{t_n}'}{H_{t_n}}(T_n + z \log_+ T_n)$$

converge locally uniformly to F'. Comparing this with (9), we conclude that

$$F'(z) = \tfrac{1}{4}$$

whenever the imaginary part of z is sufficiently large and negative. By unique continuation, we thus have $F'(z) = \frac{1}{4}$ for all z in the lower half-plane; as F vanishes on $-i\kappa$, we thus have

$$F(z) = \frac{z + i\kappa}{4}$$

on the lower half-plane. Since F_n converges locally uniformly to F, we conclude that

$$H_{t_n}(T_n + z \log_+ T_n) = H_{t_n}(T_n - i\kappa \log_+ T_n) \exp\left(\frac{z + i\kappa + o_{n\to\infty}(1)}{4} \log_+^2 T_n\right) \tag{54}$$

uniformly for z in a compact subset of the lower half-plane. Similarly, since F_n' converges locally to F, we have

$$\frac{H_{t_n}'}{H_{t_n}}(T_n + z \log_+ T_n) = \frac{1 + o_{n\to\infty}(1)}{4} \log_+ T_n \tag{55}$$

uniformly for z in a compact subset of the lower half-plane.

Let $\delta > 0$ be a small constant. As in the proof of (48), we can use the argument principle (and a rescaling) to write

$$\begin{aligned} &N_{t_n}([T_n, T_n + \alpha_n \log_+ T_n]) \\ &\quad = \frac{\log_+ T_n}{\pi} \mathrm{Im}\left(\int_{\Gamma_{I,n}} + \int_{\Gamma_{II,n}} + \int_{\Gamma_{III,n}}\right) \frac{H_{t_n}'}{H_{t_n}}(T_n + z \log_+ T_n)\, dz, \end{aligned}$$

where $\Gamma_{I,n}$, $\Gamma_{II,n}$, $\Gamma_{III,n}$ trace the line segments from 0 to $-i\delta$, from $-i\delta$ to $\alpha_n - i\delta$ and from $\alpha_n - i\delta$ to α_n, respectively. By (55), the contribution of the $\Gamma_{II,n}$ integral is $\frac{\alpha + o_{n\to\infty}(1) + O(\delta)}{4\pi} \log_+^2 T_n$ (we allow the decay rate in the $o_{n\to\infty}(1)$ errors to depend on δ). Applying the Jensen formula argument used to prove (48), we see that the contribution of the $\Gamma_{I,n}$ integral is bounded in magnitude by

$$\ll \int_0^1 \log|g_n(\delta + 2\delta e^{2\pi i\alpha})| - \log|g_n(\delta)|\, d\alpha,$$

where

$$g_n(s) := \tfrac{1}{2}\left(e^{-i\theta_n} H_{t_n}(T_n + is\log_+ T_n) + e^{i\theta_n} H_{t_n}(T_n - is\log_+ T_n)\right),$$

and the phase θ_n is chosen so that $e^{i\theta_n} H_{t_n}(T_n - i\delta \log_+ T_n)$ is real and positive. Applying (54) (and the functional equation $H_{t_n}(\overline{z}) = \overline{H_{t_n}(z)}$) when $|\mathrm{Im}(z)| \geqslant \sqrt{\delta}$ (say), and (7) (and the functional equation) otherwise, we conclude that the $\Gamma_{I,n}$ integral is equal to $(o_{n\to\infty}(1) + O(\sqrt{\delta}))\log_+^2 T_n$. We have this similarly for the $\Gamma_{III,n}$ integral. Taking δ to be sufficiently small and n sufficiently large, we contradict (53). □

4. Dynamics of zeros

As remarked in Section 1, the functions H_t solve a backward heat equation. As worked out in [11], this induces a corresponding dynamics on the zeros x_j of H_t.

THEOREM 11 (Dynamics of zeros). *For $\Lambda < t \leqslant 0$, the zeros $x_j(t)$ depend in a continuously differentiable fashion on t for each j, with the equations of motion*

$$\partial_t x_k(t) = 2 \sideset{}{'}\sum_{j:\, j\neq k} \frac{1}{x_k(t) - x_j(t)} \tag{56}$$

for $k \in \mathbb{Z}^*$ and $\Lambda < t \leqslant 0$, where recall that the tick denotes principal value summation over $j \in \mathbb{Z}^*$ (which will converge thanks to (50) and (43)).

Proof. This follows from [11, Lemma 2.4] (the continuity of the derivative following for instance from [11, Lemma 2.1]). □

Informally, ODE (56) indicates that the zeros $x_k(t)$ will repel each other as one goes forward in time. On the other hand, if $x_k(t)$ are arranged (locally, at least) in an arithmetic progression, then ODE (56) suggests that the zeros will be

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in equilibrium. If x_k are not arranged in an arithmetic progression, and instead have some fluctuation in the spacing between zeros, then heuristically ODE (56) suggests that the zeros would move away from the more densely spaced regions and towards more sparsely spaced regions, thus converging towards the equilibrium of an arithmetic progression. This is the intuition behind the convergence to local equilibrium mentioned in Section 1.

One can estimate the speed of this local convergence to equilibrium by the following heuristic calculation. Consider the zeros in a region $[T, T+\alpha]$ of space, where $T > 0$ is large and α is reasonably small (for example, $\alpha = O(\log_+ T)$). From Theorem 9 (or (43) and (50)), we see that we expect about $\frac{\alpha}{4\pi}\log T$ zeros in this interval, with an average spacing of $\frac{4\pi}{\log_+ T}$. Suppose for the sake of informal discussion that there is some moderate fluctuation in this spacing, for instance, suppose that the left half of the interval contains about $1.5\frac{\alpha}{8\pi}\log T$ zeros and the right half contains only about $0.5\frac{\alpha}{8\pi}\log_+ T$ zeros. Then a back of the envelope calculation suggests that for $x_k(t)$ near the middle of this interval, the right-hand side of (56) would be positive and have magnitude $\asymp \frac{\alpha \log_+ T}{\alpha} = \log_+ T$. Since the length of the interval is α, one may then predict that the time needed to relax to equilibrium is about $\alpha/\log_+ T$. Since we can evolve the flow for time $|\Lambda| \asymp 1$, one would expect to attain equilibrium at the final time $t = 0$ if the initial length scale α of the fluctuation obeys the bound $\alpha = o_{T\to\infty}(\log_+ T)$. Happily, this upper bound is precisely what asymptotic (52) gives, so we heuristically expect to (barely) be able to establish local equilibrium at time $t = 0$.

Of course, one has to make this intuition more precise. Our strategy for doing so involves exploiting the formal gradient flow structure of ODE (56). (This strategy was loosely inspired by the work of Erdős, Schlein and Yau [12] exploiting the Hamiltonian structure of Dyson Brownian motion to obtain local convergence to equilibrium since the equations for Dyson Brownian motion resemble that in (56) but with an additional Brownian motion term. Indeed, Dyson Brownian motion is the diffusion related to the Gibbs measure $\frac{1}{Z}e^{-\beta \mathrm{H}}$ for the Hamiltonian studied here.) Indeed, one may formally write (56) as the gradient flow

$$\partial_t x_k(t) = -\partial_{x_k}\mathrm{H}((x_j(t))_{j\in\mathbb{Z}^*}),$$

where H is the formal 'Hamiltonian'

$$\mathrm{H}((x_j)_{j\in\mathbb{Z}^*}) := \sum_{j,k\in\mathbb{Z}^*:\, j\neq k} \log\frac{1}{|x_k - x_j|},$$

where we ignore for this nonrigorous discussion the fact that the series defining $\mathcal{H}$ is not absolutely convergent. The Hamiltonian is convex, so one expects the

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quantity

$$\mathcal{H}(t) := \mathrm{H}((x_j(t))_{j\in\mathbb{Z}^*}) = \sum_{j,k\in\mathbb{Z}^*:\, j\neq k} H_{jk}(t)$$

to be decreasing and convex in time, and for the state $(x_j(t))_{j\in\mathbb{Z}^*}$ to converge to a critical point of the Hamiltonian, where

$$H_{jk}(t) := \log \frac{1}{|x_j(t) - x_k(t)|} \tag{57}$$

denotes the Hamiltonian interaction between $x_j(t)$ and $x_k(t)$. Indeed, a formal calculation using (56) yields the identity

$$\partial_t \mathcal{H}(t) = -4E(t),$$

where E is the 'energy'

$$E(t) := \sum_{k,k'\in\mathbb{Z}^*:\, k\neq k'} E_{kk'}(t)$$

and

$$E_{kk'}(t) := \frac{1}{|x_k(t) - x_{k'}(t)|^2} \tag{58}$$

denotes the 'interaction energy' between $x_k(t)$ and $x_{k'}(t)$, and we once again ignore the issue that the series is not absolutely convergent. A further formal calculation using (56) again eventually yields

$$\partial_t E(t) = -2 \sum_{k,k'\in\mathbb{Z}^*:\, k\neq k'} \left(\frac{2}{|x_k(t) - x_{k'}(t)|^2} \right.$$
$$\left. - \sum_{k''\in\mathbb{Z}^*:\, k''\neq k,k'} \frac{1}{(x_{k''}(t) - x_k(t))(x_{k''}(t) - x_{k'}(t))} \right)^2,$$

suggesting that $\mathcal{H}(t)$ and $E(t)$ are decreasing and that $\mathcal{H}(t)$ is convex, as claimed.

In order to deal with the divergence of the infinite series appearing above, we will need to truncate the Hamiltonian and energy before differentiating them. The following lemma records some of the identities that arise when doing such truncations.

LEMMA 12 (Identities). *For brevity, we suppress explicit dependence on the time parameter $t \in (\Lambda, 0]$. Let $K \subset \mathbb{Z}^*$ be a finite set of some cardinality $|K|$. All summation indices such as i, j, k are assumed to lie in $\mathbb{Z}^*$.*

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(i) *(Dynamics of a gap, cf. [11, Lemma 2.4]) If $j, k \in \mathbb{Z}^*$ are distinct, then*

$$\partial_t (x_k - x_j) = \frac{4}{x_k - x_j} - 2(x_k - x_j) \sum_{i: i \neq k, j} \frac{1}{(x_i - x_k)(x_i - x_j)}.$$

(ii) *(Cross-energy inequality, cf. [11, Lemma 2.5]) One has*

$$\partial_t \sum_{k \in K; j \notin K} E_{jk} \geqslant - \sum_{k \in K; j \notin K} \frac{8}{(x_k - x_j)^4}$$

in the weak sense that

$$\sum_{k \in K; j \notin K} E_{jk}(t_2) - E_{jk}(t_1) \geqslant - \int_{t_1}^{t_2} \sum_{k \in K; j \notin K} \frac{8}{(x_k - x_j)^4}(t)\, dt$$

whenever $\Lambda < t_1 < t_2 \leqslant 0$.

(iii) *(Energy identity) One has*

$$\partial_t \sum_{k,k' \in K: k \neq k'} E_{kk'} = \sum_{\substack{j \notin K \\ k,k' \in K: k \neq k'}} \frac{4}{(x_k - x_{k'})^2 (x_k - x_j)(x_{k'} - x_j)}$$
$$- 2 \sum_{k,k' \in K: k \neq k'} \left(\frac{2}{(x_k - x_{k'})^2} - \sum_{k'' \in K: k'' \neq k,k'} \frac{1}{(x_{k''} - x_k)(x_{k''} - x_{k'})} \right)^2.$$

(iv) *(Virial identity) One has*

$$\partial_t \sum_{k,k' \in K: k \neq k'} (x_k - x_{k'})^2 = 4|K|^2(|K| - 1)$$
$$- \sum_{k,k' \in K: k \neq k'} (x_k - x_{k'})^2 \sum_{j \notin K} \frac{4}{(x_k - x_j)(x_{k'} - x_j)}.$$

(The terminology here is in analogy with the virial identity in N-body classical gravitational physics; see for example, [27, Exercise 1.48].)

(v) *(Hamiltonian identity) One has*

$$\partial_t \sum_{k,k' \in K: k \neq k'} H_{kk'} = -4 \sum_{k,k' \in K: k \neq k'} E_{kk'} + 2 \sum_{\substack{j \notin K \\ k,k' \in K: k \neq k'}} \frac{1}{(x_j - x_k)(x_j - x_{k'})}.$$

A key point in identities (iii), (iv) and (v) is that if one ignores the ‘cross terms’ involving interactions between indices in K (representing some ‘local subsystem’ of particles) and indices outside of K (representing the ‘environment’ that subsystem interacts with), the right-hand side has a definite sign (negative in the case of (iii) and (v) and positive in the case of (iv)). This gives a number of useful ‘monotonicity formulae’ as long as cross terms are under control. As discussed above, many of these various monotonicity formulae reflect the formal convexity properties of the Hamiltonian $\mathcal{H}$. With more effort, one can obtain a precise formula for the defect in the inequality in (ii); see [11, Lemma 2.5].

Proof. From (56), one has

$$\partial_t x_k - \partial_t x_j = \frac{2}{x_k - x_j} - \frac{2}{x_j - x_k} + \sum_{i:i\neq k,j} \frac{2}{x_k - x_i} - \frac{2}{x_k - x_j},$$

which gives (i). Note that the series is now absolutely convergent thanks to (43) and (50).

Now we prove (ii). By monotone convergence, it suffices to show that

$$\sum_{\substack{k\in K\\ j\in[-R,R]_{\mathbb{Z}^*}\setminus K}} E_{jk}(t_2) - E_{jk}(t_1) \geqslant -\int_{t_1}^{t_2} \sum_{\substack{k\in K\\ j\in[-R,R]_{\mathbb{Z}^*}\setminus K}} \frac{8}{(x_k - x_j)^4}(t)\, dt$$

for all $\Lambda < t_1 \leqslant t_2 \leqslant 0$ and all sufficiently large R. By the fundamental theorem of calculus, it suffices to show that

$$\partial_t \sum_{\substack{k\in K\\ j\in[-R,R]_{\mathbb{Z}^*}\setminus K}} E_{jk} \geqslant - \sum_{\substack{k\in K\\ j\in[-R,R]_{\mathbb{Z}^*}\setminus K}} \frac{8}{(x_k - x_j)^4}.$$

We can expand the left-hand side as

$$-2 \sum_{\substack{k\in K\\ j\in[-R,R]_{\mathbb{Z}^*}\setminus K}} \frac{\partial_t (x_k - x_j)}{(x_k - x_j)^3},$$

which by (i) becomes

$$- \sum_{\substack{k\in K\\ j\in[-R,R]_{\mathbb{Z}^*}\setminus K}} \frac{8}{(x_k - x_j)^4} + 4 \sum_{\substack{k\in K\\ j\in[-R,R]_{\mathbb{Z}^*}\setminus K\\ i:i\neq j,k}} \frac{1}{(x_k - x_j)^2 (x_i - x_k)(x_i - x_j)}$$

and so it will suffice to show that

$$\sum_{\substack{k\in K\\ j\in[-R,R]_{\mathbb{Z}^*}\setminus K\\ i:i\neq j,k}} \frac{1}{(x_k - x_j)^2 (x_i - x_k)(x_i - x_j)} \geqslant 0.$$

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If R is large enough that $[-R, R]_{\mathbb{Z}^*}$ contains k, we can split this sum into three parts, depending on whether $i \in K$, $i \in [-R, R]_{\mathbb{Z}^*} \backslash K$ or $i \notin [-R, R]_{\mathbb{Z}^*}$. The contribution of the case $i \in K$ can be rewritten as

$$\sum_{\substack{j \notin K \\ k,k' \in K: k \neq k'}} \frac{4(x_{k'} - x_j)}{(x_k - x_j)^2 (x_{k'} - x_j)^2 (x_k - x_{k'})},$$

which equals

$$\sum_{\substack{j \in [-R,R]_{\mathbb{Z}^*} \\ k,k' \in K: k \neq k'}} \frac{2}{(x_k - x_j)^2 (x_{k'} - x_j)^2}$$

after symmetrizing in k and k', which is clearly non-negative. Similarly the contribution of the case $i \in [-R, R]_{\mathbb{Z}^*} \backslash K$ is

$$\sum_{\substack{k \in K \\ j,j' \in [-R,R]_{\mathbb{Z}^*} \backslash K: j \neq j'}} \frac{2}{(x_k - x_j)^2 (x_k - x_{j'})^2},$$

which is also clearly non-negative. Finally, for $i \notin [-R, R]_{\mathbb{Z}^*}$, all summands are already non-negative. This gives (ii).

For (iii), we can similarly expand the left-hand side as

$$-2 \sum_{k,k' \in K: k \neq k'} \frac{\partial_t (x_k - x_{k'})}{(x_k - x_{k'})^3},$$

which by (i) becomes

$$-\sum_{k,k' \in K: k \neq k'} \frac{8}{(x_k - x_{k'})^4} + 4 \sum_{\substack{k,k' \in K: k \neq k' \\ i: i \neq k,k'}} \frac{1}{(x_k - x_{k'})^2 (x_i - x_k)(x_i - x_{k'})}.$$

To prove (iii), it thus suffices to establish the identity

$$\begin{aligned} &\sum_{k,k' \in K: k \neq k'} \left(\frac{2}{(x_k - x_{k'})^2} - \sum_{k'' \in K: k'' \neq k,k'} \frac{1}{(x_{k''} - x_k)(x_{k''} - x_{k'})} \right)^2 \\ &\quad = \sum_{k,k' \in K: k \neq k'} \frac{4}{(x_k - x_{k'})^4} - 2 \sum_{k,k',k'' \in K: k,k',k'' \text{ distinct}} \\ &\qquad \times \frac{1}{(x_k - x_{k'})^2 (x_{k''} - x_k)(x_{k''} - x_{k'})}. \end{aligned}$$

The left-hand side expands as

$$\sum_{k,k'\in K:k\neq k'} \frac{4}{(x_k - x_{k'})^4} - \sum_{k,k'\in K:k\neq k'} \frac{4}{(x_k - x_{k'})^2} \sum_{k''\in K:k''\neq k,k'} \frac{1}{(x_{k''} - x_k)(x_{k''} - x_{k'})}$$
$$+ \sum_{k,k'\in K:k\neq k'} \sum_{k''\in K:k''\neq k,k'} \frac{1}{(x_{k''} - x_k)^2(x_{k''} - x_{k'})^2}$$
$$+ \sum_{k,k'\in K:k\neq k'} \sum_{k'',k'''\in K:k''\neq k,k'} \frac{1}{(x_{k''} - x_k)(x_{k''} - x_{k'})(x_{k'''} - x_k)(x_{k'''} - x_{k'})}.$$

The final sum can be rewritten as

$$\sum_{k,k',k'',k'''\in K:k,k',k'',k'''\ \text{distinct}} \frac{(x_k - x_{k'})(x_{k''} - x_{k'''})}{(x_{k''} - x_k)(x_{k''} - x_{k'})(x_{k'''} - x_k)(x_{k'''} - x_{k'})(x_k - x_{k'})(x_{k''} - x_{k'''})}.$$

The denominator is a Vandermonde determinant and is totally antisymmetric in k, k', k'', k'''. All the monomials appearing in the numerator disappear upon antisymmetrization, so the final sum vanishes. To conclude the proof of (iii), it suffices to show that

$$\sum_{k,k'\in K:k\neq k'} \sum_{k''\in K:k''\neq k,k'} \frac{1}{(x_{k''} - x_k)^2(x_{k''} - x_{k'})^2}$$
$$= \sum_{k,k'\in K:k\neq k'} \frac{2}{(x_k - x_{k'})^2} \sum_{k''\in K:k''\neq k,k'} \frac{1}{(x_{k''} - x_k)(x_{k''} - x_{k'})}.$$

The difference between the left-hand and right-hand sides can be written as

$$\sum_{k,k',k''\in K:k,k',k''\ \text{distinct}} \frac{(x_k - x_{k'})^2 - 2(x_{k''} - x_k)(x_{k''} - x_k)}{(x_{k''} - x_k)^2(x_{k''} - x_{k'})^2(x_k - x_{k'})^2}.$$

The denominator is totally symmetric in k, k', k'', while the numerator symmetrizes to zero, giving the claim.

Now we prove (iv). The left-hand side expands as

$$2 \sum_{k,k'\in K:k\neq k'} (x_k - x_{k'})\partial_t(x_k - x_{k'}),$$

which by (i) becomes

$$8|K|(|K| - 1) - 4 \sum_{k,k'\in K:k\neq k'} (x_k - x_{k'})^2 \sum_{i\neq k,k'} \frac{1}{(x_i - x_k)(x_i - x_{k'})}.$$

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It will thus suffice to show that

$$\sum_{k,k',k''\in K:\, k,k',k''\text{ distinct}} \frac{(x_k - x_{k'})^2}{(x_{k''} - x_k)(x_{k''} - x_{k'})} = -|K|(|K|-1)(|K|-2).$$

But the left-hand side can be written as

$$\sum_{k,k',k''\in K:\, k,k',k''\text{ distinct}} \frac{(x_k - x_{k'})^3}{(x_{k''} - x_k)(x_{k''} - x_{k'})(x_k - x_{k'})} = -|K|(|K|-1)(|K|-2).$$

The denominator is totally antisymmetric in k, k', k''. The numerator antisymmetrizes to $-(x_{k''} - x_k)(x_{k''} - x_{k'})(x_k - x_{k'})$, giving the claim.

Finally we prove (v). The left-hand side expands as

$$-\sum_{k,k'\in K:\, k\neq k'} \frac{\partial_t (x_k - x_{k'})}{x_k - x_{k'}},$$

which by (i) becomes

$$-\sum_{k,k'\in K:\, k\neq k'} \frac{4}{(x_k - x_{k'})^2} + 2\sum_{k,k'\in K:\, k\neq k'} \sum_{i:\, i\neq k,k'} \frac{1}{(x_i - x_k)(x_i - x_{k'})}.$$

It thus suffices to show that the expression

$$\sum_{k,k',k''\in K:k,k',k''\text{ distinct}} \frac{1}{(x_{k''} - x_k)(x_{k''} - x_{k'})}$$

vanishes. But the summand antisymmetrizes to zero, giving the claim. □

5. A weak bound on gaps

In order to analyse (truncated versions of) the Hamiltonian $\mathcal{H}(t) = \sum_{j\neq k} H_{jk}(t)$, we will need some upper bounds on the individual terms $H_{jk}(t)$. It was shown in [11, Corollary 1] that these quantities are finite (that is, the zeros are simple) when $\Lambda < t \leqslant 0$. It turns out that by refining the analysis in [11] (and by narrowing the range of times t to the region $\Lambda/2 \leqslant t \leqslant 0$), one can establish a more quantitative lower bound.

PROPOSITION 13 (Lower bound on gaps). *For any $j \in \mathbb{Z}^*$ and any $\Lambda/2 \leqslant t \leqslant 0$, one has*

$$\max_{k\in\mathbb{Z}^*:\, k\neq j} H_{jk}(t) \ll (\log_+^2 j) \log_+ \log_+ j. \tag{59}$$

The bound in (59) is probably not optimal, but for our application any bound that grows more slowly than (say) $|j|^{0.1}$ as $j \to \infty$ would suffice.

To prove this proposition, we first need the following variant of a result in [11].

LEMMA 14. *Let K be a finite subset of $\mathbb{Z}^*$ of cardinality $|K| \geqslant 2$, and let $\Lambda/2 \leqslant t \leqslant 0$. Then*

$$\sum_{k,k' \in K : k \neq k'} (x_k(t) - x_{k'}(t))^2 \gg \frac{|K|^3}{1 + \sum_{\substack{k \in K \\ j \notin K}} E_{jk}(t)}.$$

Informally, this lemma asserts that the gaps within K cannot be too small, unless there is also a small gap between an element of K and an element outside of K. The strategy will be to iterate this observation to show that a very small gap will therefore propagate until it contradicts (52).

Proof. Let $A = A(t)$ and $B = B(t)$ denote the functions

$$A(t) := \sum_{k,k' \in K : k \neq k'} (x_k(t) - x_{k'}(t))^2$$

$$B(t) := \sum_{\substack{k \in K \\ j \notin K}} E_{kj}(t).$$

The function $A(t)$ is continuously differentiable. The corresponding claim for $B(t)$ is not obvious; however, the sum defining $B(t)$ is uniformly convergent (thanks to (51)) and hence $B(t)$ is at least continuous. From Lemma 12(ii), we have the lower bound

$$\partial_{t'} B(t') \geqslant -8B(t')^2$$

(cf. [11, Lemma 2.5]) in the weak sense for $\Lambda < t' \leqslant 0$. In particular, if there exists a time $\Lambda < t_- < t$ such that

$$\sup_{t_- \leqslant t' \leqslant t} B(t') = B(t_-) = 2B(t),$$

then we have

$$B(t) - B(t_-) \geqslant -8B(t)^2(t - t_-),$$

which rearranges as

$$t - t_- \geqslant \frac{1}{8B(t)}.$$

By continuity, we conclude that $B(t')$ cannot attain or exceed the value $2B(t)$ anywhere in the interval $(-\Lambda, t] \cap (t - \frac{1}{8B(t)}, t)$, that is to say that

$$B'(t) < 2B(t)$$

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whenever

$$t - \frac{1}{8B(t)}, \Lambda < t' \leqslant t.$$

By hypothesis, this is a range of size at least

$$\min\left(\frac{\Lambda}{2}, \frac{1}{16B(t)}\right) \gg \frac{1}{1+B(t)}.$$

On the other hand, for t' in the above range, we see from Lemma 12(iv) that

$$\begin{aligned}\partial_{t'} A(t') &= 4|K|^2(|K|-1) + O(B(t')A(t')) \\ &= 4|K|^2(|K|-1) + O(B(t)A(t'))\end{aligned}$$

and hence by Gronwall's inequality, one has

$$A(t) \gg \frac{4|K|^2(|K|-1)}{1+B(t)},$$

giving the claim. □

Now we fix a time $\Lambda/2 \leqslant t \leqslant 0$ and drop the dependence on t. For any finite set $K \subset \mathbb{Z}^*$ with $|K| \geqslant 2$, set $\delta(K) := \max_{k,k' \in K} |x_k - x_{k'}|$ to be the largest gap in K. Then

$$\sum_{k,k' \in K : k \neq k'} (x_k - x_{k'})^2 \leqslant |K|^2 \delta(K)^2,$$

and so from the above lemma, we have

$$1 + \sum_{\substack{k \in K \\ j \notin K}} E_{kj} \geqslant |K|^{-5} \delta(K)^{-2}.$$

In particular, if $\delta(K) \leqslant c|K|^{-5/2}$ for a sufficiently small absolute constant $c > 0$, then we have

$$\sum_{\substack{k \in K \\ j \notin K}} E_{kj} \geqslant |K|^{-5} \delta(K)^{-2},$$

and hence by the pigeonhole principle, there exists $k \in K$ such that

$$\sum_{j \notin K} E_{kj} \gg |K|^{-6} \delta(K)^{-2}.$$

From (52) and (58), we have

$$\sum_{j \notin K} E_{kj} \ll 1 + (\log_+^2 \xi_k) \min_{j \notin K} |x_k - x_j|^{-2}.$$

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We conclude that if $\delta(K) \leqslant c|K|^{-3}$ for a sufficiently small $c > 0$, then there exists $k \in K$ such that

$$(\log_+^2 \xi_k) \min_{j \notin K} |x_k - x_j|^{-2} \gg |K|^{-6} \delta(K)^{-2}$$

or equivalently

$$\min_{j \notin K} |x_k - x_j| \ll |K|^3 \delta(K) \log_+ \xi_k.$$

Now suppose that K is a discrete interval $[k_-, k_+]_{\mathbb{Z}^*}$ for some $1 < k_- < k_+$. Then

$$\min_{j \notin K} |x_k - x_j| \geqslant \min(|x_{k_-} - x_{k_- - 1}|, |x_{k_+} - x_{k_+ + 1}|),$$

and thus (assuming that $\delta(K) \leqslant c|K|^{-3}$) we have

$$\min(|x_{k_-} - x_{k_- - 1}|, |x_{k_+} - x_{k_+ + 1}|) \ll |K|^3 \delta(K) \log_+ k_+,$$

which implies that

$$\delta(K') \ll |K|^3 \log(k_+) \delta(K) \tag{60}$$

whenever $\delta(K) \leqslant c|K|^{-3}$, where K' is either the interval $K' = [k_- - 1, k_+]_{\mathbb{Z}^*}$ or $K' = [k_-, k_+ + 1]_{\mathbb{Z}^*}$. In either case, we call K' an *enlargement* of K.

Now we can prove Proposition 13. By symmetry, we may assume j is positive. We can also assume j is large, as the claim follows from compactness for bounded j. As before, we suppress the dependence on t. It thus suffices to show that

$$\log \frac{1}{|x_{j+1} - x_j|} \ll (\log^2 j) \log \log j$$

for large positive j.

By iterating (60) at most $\log j$ times starting from the interval $K_1 := [j, j + 1]_{\mathbb{Z}^*}$, we can find a sequence

$$[j, j+1]_{\mathbb{Z}^*} = K_1 \subset K_2 \subset \cdots \subset K_r$$

of discrete intervals $K_i = [k_{-,i}, k_{+,i}]_{\mathbb{Z}^*}$ for some $1 \leqslant r \leqslant \log_+^2 \xi_j$ with the following properties:

(i) For each $1 \leqslant i < r$, K_{i+1} an enlargement of K_i with $\delta(K_{i+1}) \ll |K_i|^3 \delta(K_i) \log_+ k_{+,i}$.

(ii) Either $\delta(K_r) > c|K_r|^{-3}$ or $r + 1 > \log_+^2 \xi_j$.

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Since $|K_i| \leqslant r + 1 \ll \log_+^2 \xi_j \ll \log^2 j$ and $k_{+,i} \leqslant j + r \ll j$, we have from property (i) that

$$\delta(K_{i+1}) \ll j \log^2 j \delta(K_i)$$

for all $1 \leqslant i < r$, and hence

$$\delta(K_r) \ll \exp(O(\log^2 j \log\log j))\delta(K_1).$$

On the other hand, from property (ii), using the bound $|K_r| \leqslant r+1 \ll \log^2 \xi_j$ in the first case and (52) and the pigeonhole principle in the second case, we have

$$\delta(K_r) \gg \log^{-6} \xi_j \gg \log^{-6} j.$$

Combining the two estimates, we obtain the claim.

6. A weak bound on integrated energy

In addition to truncations of the Hamiltonian, we will also need to control truncations of the energy $\sum_{j \neq k} E_{jk}(t)$. Although Proposition 13 provides some control on the summands here, it is too weak for our purposes (being of worse than polynomial growth in j, k), and we will need the following integrated bound that, while still weak, is at least of polynomial growth.

PROPOSITION 15 (Weak bound on integrated energy). *Let $J > 0$. Then*

$$\int_{\Lambda/2}^0 \sum_{J \leqslant j < k \leqslant 2J} E_{jk}(t)\, dt \ll J^2 \log_+^{O(1)} J.$$

We will use this bound to justify an interchange of a derivative and an infinite series summation in the next section.

Proof. We may take J to be large, as the claim is trivial for J in the compact region $J = O(1)$. For any discrete interval I, let Q_I denote the quantity

$$Q_I := \int_{\Lambda/2}^0 \sum_{j,k \in I : j \neq k} E_{jk}(t)\, dt.$$

From (52), we have a crude lower bound

$$Q_{[J,2J]_{\mathbb{Z}^*}} \gg J \log^{-O(1)} J$$

while from Proposition 13 we have an extremely crude upper bound

$$Q_{[0.5J,3J]_{\mathbb{Z}^*}} \ll \exp(O(\log^2 J \log\log J)).$$

The ratio between $Q_{[0.5J,3J]_{\mathbb{Z}^*}}$ and $Q_{[J,2J]_{\mathbb{Z}^*}}$ is thus less than $(1+J^{-0.1})^{0.5J/J^{0.1}}$. By the pigeonhole principle, we can then therefore find an interval $K := [J_-, J_+]_{\mathbb{Z}^*}$ containing $[J, 2J]_{\mathbb{Z}^*}$ and contained in $[0.5J + J^{0.1}, 3J - J^{0.1}]_{\mathbb{Z}^*}$ such that

$$Q_{K'} \leqslant (1 + J^{-0.1}) Q_K, \tag{61}$$

where $K' := [J_- - J^{0.1}, J_+ + J^{0.1}]_{\mathbb{Z}^*}$ is a slight enlargement of K. Next, we apply Lemma 12(v) and use the fundamental theorem of calculus to obtain the identity

$$\begin{aligned}\sum_{k,k'\in K: k\neq k'} & H_{kk'}(\Lambda/2) - H_{kk'}(0)\\ &= 4Q_K - 2\int_{\Lambda/2}^{0} \sum_{\substack{j\notin K\\ k,k'\in K: k\neq k'}} \frac{1}{(x_j(t) - x_k(t))(x_j(t) - x_{k'}(t))}\, dt.\end{aligned}$$

From Proposition 13, the left-hand side is $O(J^2 \log^{O(1)} J)$; thus

$$Q_K \ll J^2 \log^{O(1)} J + \int_{\Lambda/2}^{0} \sum_{\substack{j\notin K\\ k,k'\in [J_-,J_+]_{\mathbb{Z}^*}: k\neq k'}} \frac{1}{(x_j(t) - x_k(t))(x_j(t) - x_{k'}(t))}\, dt.$$

Using $ab \ll a^2 + b^2$, we thus have

$$Q_K \ll J^2 \log^{O(1)} J + \int_{\Lambda/2}^{0} \sum_{\substack{j\notin K\\ k\in K}} \frac{1}{(x_j(t) - x_k(t))^2}\, dt.$$

Using (52), the contribution to the integral of those j outside of K' may be crudely bounded by $O(J^2 \log^{O(1)} J)$ (in fact, one can improve this bound to $O(J \log^{O(1)} J)$ if desired, although this will not help us significantly here). The contribution of those j inside K' may be bounded by

$$Q_{K'} - Q_K \leqslant J^{-0.1} Q_K,$$

thanks to (61). We conclude that

$$Q_K \ll J^2 \log^{O(1)} J$$

, and the claim follows. □

7. Strong control on integrated energy

As discussed previously, the strategy to establish convergence to local equilibrium is to study (a suitable variant of) the formal Hamiltonian

$$\mathcal{H}(t) = \sum_{j,k\in\mathbb{Z}^*: j\neq k} H_{jk}(t)$$

and its derivatives, with the intention of controlling (suitable variants of) integrated energies such as

$$\int_{\Lambda/4}^{0} \sum_{j,k\in\mathbb{Z}^*:j\neq k} E_{jk}(t)\,dt.$$

Unfortunately, even with the bound just obtained in Proposition 13, the above expression is far from being absolutely convergent. To address this issue, we need to mollify and renormalize the Hamiltonian and the energy in a number of ways. We renormalize the inverse square function $x \mapsto \frac{1}{|x|^2}$ for $x \neq 0$ that appears in the definition of the energy interactions $E_{jk}(t)$ by introducing the modified potential

$$V(x) := \frac{1}{|x|^2} - 1 + 2(|x| - 1),$$

which (for positive x) is $\frac{1}{x^2}$ minus the linearization $1 - 2(x - 1)$ of that function at $x = 1$. As $\frac{1}{x^2}$ is convex, V is non-negative, and one can verify the asymptotics

$$\begin{aligned} V(x) &\asymp \frac{1}{|x|^2} \quad \text{for } |x| \leqslant 1/2 \\ V(x) &\asymp (|x| - 1)^2 \quad \text{for } 1/2 < |x| \leqslant 2 \\ V(x) &\asymp |x| \quad \text{for } |x| > 2. \end{aligned} \tag{62}$$

For any distinct j, k and any $\Lambda/2 \leqslant t \leqslant 0$, we define the renormalization

$$\tilde{E}_{jk}(t) := \frac{1}{|\xi_k - \xi_j|^2} V\left(\frac{x_k(t) - x_j(t)}{\xi_k - \xi_j}\right)$$

of the interaction energy $E_{jk}(t)$; we observe that

$$\tilde{E}_{jk}(t) = E_{jk}(t) - \frac{1}{|\xi_k - \xi_j|^2} + 2\frac{(x_k(t) - \xi_k) - (x_j(t) - \xi_j)}{(\xi_k - \xi_j)^3}. \tag{63}$$

For any discrete interval $I \subset \mathbb{Z}^*$, we define the renormalized energy

$$\tilde{E}^I(t) := \sum_{j,k\in I:j\neq k} \tilde{E}_{jk}(t);$$

this is clearly a non-negative quantity that is nondecreasing in I. It can also be simplified up to negligible error as follows.

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LEMMA 16. *If* $I = [I_-, I_+]_{\mathbb{Z}^*}$ *is a discrete interval and* $\Lambda/2 \leqslant t \leqslant 0$, *then*

$$\tilde{E}^I(t) = \left(\sum_{j,k \in I:\, j \neq k} E_{jk}(t) - \frac{1}{|\xi_k - \xi_j|^2} \right) + O(\log_+^{O(1)}(|I_-| + |I_+|)).$$

Proof. By symmetry and the triangle inequality, we may assume without loss of generality that $0 \leqslant I_- \leqslant I_+$; we may then assume that I_+ is large, as the claim is trivial for I_+ in the compact region $I_+ = O(1)$. By (63), it suffices to show that

$$\sum_{j,k \in I:\, j \neq k} \frac{(x_k(t) - \xi_k) - (x_j(t) - \xi_j)}{(\xi_k - \xi_j)^3} \ll \log^{O(1)} I_+.$$

We may desymmetrize the left-hand side as

$$2 \sum_{j \in I} (x_j(t) - \xi_j) \sum_{k \in I:\, k \neq j} \frac{1}{(\xi_k - \xi_j)^3}.$$

By (50), it thus suffices to show that

$$\sum_{j \in I} \left| \sum_{k \in I:\, k \neq j} \frac{1}{(\xi_k - \xi_j)^3} \right| \ll \log^{O(1)} I_+. \tag{64}$$

Consider the inner sum $\sum_{k \in I:\, k \neq j} \frac{1}{(\xi_k - \xi_j)^3}$. From (44), we see that the contribution to this inner sum of those k with $|k - j| \geqslant \frac{1}{2} j$ (say) is $O\left(\frac{\log^{O(1)} I_+}{j^2}\right)$. For the remaining range $|k - j| < \frac{1}{2} j$, we can use (45) to estimate

$$\frac{1}{(\xi_k - \xi_j)^3} = \frac{\log^3 \xi_j}{(4\pi)^3} \frac{1}{(k - j)^3} + O\left(\frac{\log^{O(1)} I_+}{j(k - j)^2} \right),$$

and so on summing we obtain

$$\sum_{k \in I:\, k \neq j} \frac{1}{(\xi_k - \xi_j)^3} = \frac{\log^3 \xi_j}{(4\pi)^3} \sum_{k \in I:\, 0 < |k-j| < \frac{1}{2} j} \frac{1}{(k - j)^3} + O\left(\frac{\log^{O(1)} I_+}{j} \right).$$

As $k \mapsto \frac{1}{k-j}$ is odd around j, the sum on the right-hand side can be estimated as $O(\frac{1}{\max(|j - I_-|, |I_+ - j|)^2})$. Using this bound, we obtain (64). □

In this section, we will establish the following significant improvement to Proposition 15.

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THEOREM 17. *For any $T > 0$, one has*

$$\int_{\Lambda/4}^{0} \tilde{E}^{[0.5T\log T, 3T\log T]_{\mathbb{Z}}}(t)\,dt = o_{T\to\infty}(T\log_+^3 T). \tag{65}$$

The remainder of this section is devoted to a proof of Theorem 17. The claim is trivial for T in any compact region $T = O(1)$, so we may assume without loss of generality that T is large. Recall the notation $X \lessapprox Y$ or $X = \tilde{O}(Y)$ for $X \ll Y \log^{O(1)} T$ introduced in the notation section of the paper; this will be convenient to use in the argument that follows. (Typically, when we use this notation, we will also have some sort of power gain T^{-c} that will safely absorb all the $\log^{O(1)} T$ factors.) Let $\psi_T : \mathbb{Z}^* \to \mathbb{R}^+$ be the weight function

$$\psi_T(j) := \left(1 + \frac{|j|}{T\log T}\right)^{-100}. \tag{66}$$

This is a smooth positive weight that is mostly localized to the region $j = O(T\log T)$ and fairly rapidly decaying away from this region.

We introduce the smoothly truncated renormalized energy

$$\tilde{E}_T(t) := \sum_{j,k\in\mathbb{Z}^*:\, j\neq k} \psi_T(j)\psi_T(k)\tilde{E}_{jk}(t) \tag{67}$$

for $\Lambda/2 \leqslant t \leqslant 0$. This is clearly non-negative, and from Proposition 15, (66), (62) and Fubini's theorem, we see that $\tilde{E}_T$ is absolutely integrable in time (in particular, it is finite for almost every $\Lambda/2 \leqslant t \leqslant 0$). Since E_{jk} are non-negative, we see from (66) that to prove (65) it will suffice to show that

$$\int_{\Lambda/4}^{0} \tilde{E}_T(t)\,dt = o_{T\to\infty}(T\log_+^3 T). \tag{68}$$

We have an analogue of Lemma 16.

LEMMA 18. *For almost every $\Lambda/2 \leqslant t \leqslant 0$, one has*

$$\tilde{E}_T(t) = \left(\sum_{j,k\in\mathbb{Z}^*:\, j\neq k} \psi_T(j)\psi_T(k)\left(E_{jk}(t) - \frac{1}{|\xi_k - \xi_j|^2}\right)\right) + \tilde{O}(1).$$

Proof. For almost every t, one sees from Proposition 15, (66) and Fubini's theorem (and (51)) that the series

$$\sum_{j,k:\, j\neq k} \psi_T(j)\psi_T(k)\left(E_{jk}(t) + \frac{1}{|\xi_k - \xi_j|^2} + \frac{|x_j(t)| + |x_k(t)|}{|\xi_k - \xi_j|^3}\right)$$

is absolutely convergent. Thus, by Fubini's theorem and (63), it will suffice to show that

$$\sum_{j,k\in\mathbb{Z}^*:\, j\neq k} \psi_T(j)\psi_T(k)\frac{(x_k(t)-\xi_k)-(x_j(t)-\xi_j)}{(\xi_k-\xi_j)^3} \lessapprox 1.$$

We may desymmetrize the left-hand side (again using Fubini's theorem) as

$$2\sum_{j\in\mathbb{Z}^*}\psi_T(j)(x_j(t)-\xi_j)\sum_{k\in\mathbb{Z}^*:\, k\neq j}\frac{\psi_T(k)}{(\xi_k-\xi_j)^3},$$

and so it will suffice to establish the bound

$$\sum_{k\in\mathbb{Z}^*:\, k\neq j}\frac{\psi_T(k)}{(\xi_k-\xi_j)^3} \lessapprox \frac{1}{|j|}+\frac{1}{T}$$

for all $j\in\mathbb{Z}^*$.

As in the proof of Lemma 16, we see from (44) that the contribution of those k with $|k-j|\geqslant \frac{1}{2}j$ is acceptable. For the remaining range $|k-j|<\frac{1}{2}j$, we again use (45) to estimate

$$\frac{1}{(\xi_k-\xi_j)^3} = \frac{\log^3\xi_j}{(4\pi)^3}\frac{1}{(k-j)^3} + \tilde{O}\left(\frac{1}{j(k-j)^2}\right)$$

and similarly

$$\psi_T(k) = \psi_T(j) + \tilde{O}\left(\frac{|k-j|}{T}\right),$$

and the claim follows by direct computation using the fact that $k\mapsto \frac{1}{k-j}$ is odd around j. $\square$

Recall that two indices $j,k\in\mathbb{Z}^*$ are said to be *nearby*, and we write $j\sim_T k$, if one has

$$0<|j-k|<(T^2+|j|+|k|)^{0.1}.$$

This is clearly a symmetric relation.

Next, for $\Lambda/2\leqslant t\leqslant 0$, we define the smoothly truncated renormalized Hamiltonian

$$\tilde{\mathcal{H}}_T(t) := \sum_{j,k\in\mathbb{Z}^*:\, j\sim_T k}\psi_T(j)\psi_T(k)\left(H_{jk}(t)-\log\frac{1}{|\xi_j-\xi_k|}\right). \tag{69}$$

From (66), Proposition 13 and (45), we see that the sum here is absolutely convergent for every $\Lambda/2 \leqslant t \leqslant 0$. We can also express it in terms of non-negative quantities plus a small error, in a manner similar to Lemma 18, as follows. We first introduce the renormalization

$$L(x) := \log \frac{1}{|x|} + |x| - 1$$

of the logarithm function $x \mapsto \log \frac{1}{|x|}$; this is a convex non-negative function on $\mathbb{R}\backslash\{0\}$ that vanishes precisely when $|x| = 1$ and obeys the asymptotics

$$\begin{aligned} &L(x) \asymp \log_+ \frac{1}{|x|} \quad \text{for } 0 < |x| \leqslant 1/2 \\ &L(x) \asymp (|x|-1)^2 \quad \text{for } 1/2 < |x| \leqslant 2 \\ &L(x) \asymp |x| \quad \text{for } |x| > 2. \end{aligned} \tag{70}$$

For any $\Lambda/2 \leqslant t \leqslant 0$ and distinct $j, k \in \mathbb{Z}^*$, we define the normalization

$$\tilde{H}_{jk}(t) := L\left(\frac{x_j(t) - x_k(t)}{\xi_j - \xi_k}\right) \tag{71}$$

of the Hamiltonian interaction $H_{jk}(t)$; this is symmetric in j, k and non-negative, vanishing precisely when $x_k(t) - x_j(t) = \xi_k - \xi_j$.

Lemma 19. *For every $\Lambda/2 \leqslant t \leqslant 0$, one has*

$$\tilde{\mathcal{H}}_T(t) = \sum_{j,k \in \mathbb{Z}^*:\, j \sim_T k} \psi_T(j)\psi_T(k)\tilde{H}_{jk}(t) + o_{T\to\infty}(T \log_+^3 T).$$

Proof. From (71), one has

$$\tilde{H}_{jk}(t) = H_{jk}(t) - \log \frac{1}{|\xi_j - \xi_k|} - \frac{(x_j(t) - \xi_j) - (x_k(t) - \xi_k)}{\xi_j - \xi_k},$$

so by (69) it suffices to show that

$$\sum_{j,k \in \mathbb{Z}^*:\, j \sim_T k} \psi_T(j)\psi_T(k) \frac{(x_j(t) - \xi_j) - (x_k(t) - \xi_k)}{\xi_j - \xi_k} = o_{T\to\infty}(T \log_+^3 T).$$

Note from (43), (45), (51) and (66) that the sum here is absolutely convergent. Desymmetrizing, it suffices to show that

$$\sum_{j \in \mathbb{Z}^*} \psi_T(j)|x_j(t) - \xi_j| \left| \sum_{k \in \mathbb{Z}^*: j \sim_T k} \frac{\psi_T(k)}{\xi_j - \xi_k} \right| = o_{T\to\infty}(T \log^3 T).$$

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The inner sum can be crudely bounded by $\tilde{O}(1)$ for all j thanks to (66) and (45). By (43), (50) and (66), it thus suffices to show that

$$\sum_{k\in\mathbb{Z}^*:\, j\sim_T k} \frac{\psi_T(k)}{\xi_j - \xi_k} = o_{T\to\infty}(\log T) \tag{72}$$

whenever $T^{0.5} \leqslant |j| \leqslant T^{1.5}$ (say). For $j \sim_T k$, one has $\psi_T(k) = \psi_T(j) + \tilde{O}(T^{-0.8})$, and the contribution of the error term is acceptable by (45), so it suffices to show that

$$\sum_{k\in\mathbb{Z}^*:\, j\sim_T k} \frac{1}{\xi_j - \xi_k} = o_{T\to\infty}(\log T) \tag{73}$$

whenever $|j| \geqslant T^{0.5}$. But from (45), we have

$$\xi_j - \xi_k = \frac{4\pi}{\log \xi_j}(j-k) + O\left(\frac{|j-k|^2}{|j|\log_+^2 \xi_j}\right),$$

and hence

$$\frac{1}{\xi_j - \xi_k} = \frac{\log \xi_j}{4\pi}\frac{1}{j-k} + O\left(\frac{1}{|j|}\right). \tag{74}$$

As $k \mapsto \frac{\log \xi_j}{4\pi}\frac{1}{j-k}$ is odd around j and the set $\{k : j \sim_T k\}$ is very nearly symmetric around j, it is then easy to establish (73) as required. □

In contrast to the non-normalized interaction $H_{jk}(t)$, the quantity $\tilde{H}_{jk}(t)$ is well controlled when k and j are far apart.

LEMMA 20 (Long-range decay of $\tilde{H}_{jk}$). *Let j, k be distinct elements of $\mathbb{Z}^*$, and let t be in the range $\Lambda/2 \leqslant t \leqslant 0$. There exists a quantity $\varepsilon(j)$ that goes to zero as $|j| \to \infty$, such that if $|k-j| \geqslant \varepsilon(j)^{-1}\log_+^2 \xi_j$, then*

$$\tilde{H}_{jk}(t) \ll \frac{\log_+^4(|j|+|k|)}{|k-j|^2},$$

and if $\varepsilon(j)\log_+^2 \xi_j \leqslant |k-j| \leqslant \varepsilon(j)^{-1}\log_+^2 \xi_j$, one has the refinement

$$\tilde{H}_{jk}(t) \ll \varepsilon(j)^2 \frac{\log_+^4 j}{|k-j|^2}.$$

Finally, in the remaining region $|k-j| < \varepsilon(j)\log_+^2 \xi_j$, one has the crude bound

$$\tilde{H}_{jk}(t) \ll (\log_+^2 j)\log_+\log_+ j.$$

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Proof. First, suppose that $|k-j| \geqslant \frac{1}{2}|j|$ (so in particular $|k-j| \asymp |j|+|k|$). From (50), one has

$$x_k(t) - x_j(t) = \xi_k - \xi_j + O(\log_+(|j|+|k|))$$

while from (44), one has

$$|\xi_k - \xi_j| \gg \frac{|j|+|k|}{\log_+(|j|+|k|)}$$

and thus

$$\frac{x_k(t)-x_j(t)}{\xi_k-\xi_j} - 1 \ll \frac{\log_+^2(|j|+|k|)}{|j|+|k|},$$

and the claim then follows from (70) (noting that the case $|j|+|k| = O(1)$ can be treated by compactness).

Now suppose that $\varepsilon(j)^{-1}\log_+^2 \xi_j \leqslant |k-j| < \frac{1}{2}|j|$. By symmetry, we can take j positive; we may also assume j to be large, as the bounded case $j = O(1)$ may be treated by compactness. From (50), one then has

$$x_k(t) - x_j(t) = \xi_k - \xi_j + O(\log j)$$

and from (44), one has

$$|\xi_k - \xi_j| \asymp \frac{|k-j|}{\log j} \tag{75}$$

and hence

$$\frac{x_k(t)-x_j(t)}{\xi_k-\xi_j} - 1 \ll \frac{\log^2(j)}{|k-j|} \leqslant \varepsilon(j).$$

The claim then follows from (70).

Next, suppose that $\varepsilon(j)\log_+^2 \xi_j \leqslant |k-j| \leqslant \varepsilon(j)^{-1}\log_+^2 \xi_j$. In this case, from (52) (iterated $O(\varepsilon(j)^{-1})$ times) and (45), we have

$$x_k(t) - x_j(t) = \xi_k - \xi_j + o_{j\to\infty}(\varepsilon(j)^{-1}\log j)$$

while from (44) we continue to have (75), and hence

$$\frac{x_k(t)-x_j(t)}{\xi_k-\xi_j} - 1 = o_{j\to\infty}\left(\varepsilon(j)^{-1}\frac{\log^2(j)}{|k-j|}\right),$$

with the decay rate in the $o_{j\to\infty}$ notation independent of the choice of function $\varepsilon()$. For $\varepsilon(j)$ going to zero sufficiently slowly, the claim once again follows from (70).

Finally, for the remaining case $|k-j| < \varepsilon(j)\log_+^2 \xi_j$ (which implies $x_k(t) - x_j(t) \ll \log_+^2 j$ thanks to (52)), the claim follows from Proposition 13 and (70). $\square$

We call a (time-dependent) quantity *moderately sized* if it is of the form $O(T\log_+^3 T + \tilde{E}_T(t))$, and *negligible* if it is of the form $o_{T\to\infty}(T\log_+^3 T + \tilde{E}_T(t))$. The following lemma gives some examples of moderately sized and negligible quantities.

LEMMA 21. *Let t be in the range $\Lambda/2 \leqslant t \leqslant 0$.*

(i) *The quantity*

$$\sum_{j,k\in\mathbb{Z}^*:\, j\neq k} \frac{\psi_T(j)\psi_T(k)}{|x_j(t)-x_k(t)|^2}$$

is moderately sized.

(ii) *The quantity*

$$(\log_+ T) \sum_{j,k\in\mathbb{Z}^*:\, j\neq k} \frac{\psi_T(j)\psi_T(k)}{|x_j(t)-x_k(t)|}$$

is moderately sized.

(iii) *For any absolute constants $C, c > 0$, the expression*

$$(\log_+^C T) \sum_{j,k\in\mathbb{Z}^*:\, |j|,|k|\leqslant T^{1-c}} \frac{\psi_T(j)\psi_T(k)}{|x_j(t)-x_k(t)|}$$

is negligible.

(iv) *For any absolute constants $C, c > 0$, the expression*

$$(\log_+^C T) \sum_{j,k\in\mathbb{Z}^*:\, |j|,|k|\geqslant T^{1+c}} \frac{\psi_T(j)\psi_T(k)}{|x_j(t)-x_k(t)|}$$

is negligible.

We have this similarly if $x_i(t)$ are replaced by ξ_i throughout.

Proof. For brevity, we omit the explicit dependence on the time t. Also, all summation indices i, j, k are understood to range in $\mathbb{Z}^*$.

From (44), we see that

$$\sum_{k:\, k\neq j} \frac{1}{|\xi_j-\xi_k|^2} \ll \log_+^2 j$$

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for all $j \in \mathbb{Z}^*$, and hence

$$\sum_{j,k:\, j\neq k} \psi_T(j)\psi_T(k)\frac{1}{|\xi_j - \xi_k|^2} \ll T\log_+^3 T.$$

From this and Lemma 16, we conclude (i). Using

$$\frac{\log_+ T}{|x_j(t) - x_k(t)|} \leqslant \frac{1}{|x_j(t) - x_k(t)|^2} + \log_+^2 T,$$

we then obtain (ii). If instead we use

$$\frac{\log_+^C T}{|x_j(t) - x_k(t)|} \leqslant \frac{1}{\log_+ T}\frac{1}{|x_j(t) - x_k(t)|^2} + \log_+^{2C+1} T,$$

we obtain (iii) and (iv). We have this similarly if x_i are replaced by ξ_i throughout. □

We now have the following crucial derivative computation.

PROPOSITION 22. *In the range $\Lambda/2 \leqslant t \leqslant 0$, the function $\mathcal{H}_T$ is absolutely continuous, and the derivative $\partial_t \tilde{\mathcal{H}}_T(t)$ is equal to $-4\tilde{E}_T(t)$ plus negligible terms for almost all t. In other words, one has*

$$\partial_t \tilde{\mathcal{H}}_T(t) = -4\tilde{E}_T(t) + o_{T\to\infty}\Big(T\log^3 T + \tilde{E}_T(t)\Big) \tag{76}$$

for almost every t.

REMARK 23. This may be compared with Lemma 12(v) or indeed the formal identity (57). That the right-hand side is approximated in terms the renormalized energy, rather than just the energy, may be thought of heuristically as being a result of $\partial_t \mathcal{H}$ vanishing when the zeros x_j settle on an equilibrium, being spaced like the points ξ_j.

Proof. As before, we omit the explicit dependence on t, and all summation indices are understood to lie in $\mathbb{Z}^*$. By (56), we have

$$\partial_t H_{jk}(t) = -\frac{2}{x_k - x_j}\left(\sideset{}{'}\sum_{i:\, i\neq k}\frac{1}{x_k - x_i} - \sideset{}{'}\sum_{i:\, i\neq j}\frac{1}{x_j - x_i}\right). \tag{77}$$

If we *formally* insert this into (69) and desymmetrize in j and k, we would obtain the identity

$$\partial_t \tilde{\mathcal{H}}_T = -4\sum_{j,k:\, j\sim_T k}\psi_T(j)\psi_T(k)\frac{1}{x_k - x_j}\sideset{}{'}\sum_{i:\, i\neq k}\frac{1}{x_k - x_i}. \tag{78}$$

However, we need to justify the interchange of the derivative and the infinite summation. First, we use the fundamental theorem of calculus to rewrite (77) in integral form as

$$\begin{aligned}
&H_{jk}(0) - H_{jk}(t_0)\\
&\quad = -2\int_{t_0}^{0} \frac{1}{x_k(t) - x_j(t)} \left({\sum_{i:i\neq k}}' \frac{1}{x_k(t) - x_i(t)} - {\sum_{i:i\neq j}}' \frac{1}{x_j(t) - x_i(t)} \right) dt
\end{aligned}$$

for any $\Lambda/2 \leqslant t_0 \leqslant 0$. Multiplying by $\psi_T(j)\psi_T(k)$, we conclude that

$$\begin{aligned}
\mathcal{H}_T(0) - \mathcal{H}_T(t_0) = -2 \sum_{j,k:j\sim_T k} \psi_T(j)\psi_T(k) \int_{t_0}^{0} \frac{1}{x_k(t) - x_j(t)}\\
\times \left({\sum_{i:i\neq k}}' \frac{1}{x_k(t) - x_i(t)} - {\sum_{i:i\neq j}}' \frac{1}{x_j(t) - x_i(t)} \right) dt.
\end{aligned}$$

By the dominated convergence theorem, we can interchange the outer sum and the integral as soon as we can show that the expression

$$\begin{aligned}
&\sum_{j,k:j\sim_T k} \psi_T(j)\psi_T(k)\\
&\quad \times \int_{t_0}^{0} \frac{1}{|x_k(t) - x_j(t)|} \left(\left| {\sum_{i:i\neq k}}' \frac{1}{x_k(t) - x_i(t)} \right| + \left| {\sum_{i:i\neq j}}' \frac{1}{x_j(t) - x_i(t)} \right| \right) dt
\end{aligned}$$

is finite. By symmetry in j and k, it suffices to show that

$$\sum_{j,k:j\sim_T k} \psi_T(j)\psi_T(k) \int_{t_0}^{0} \frac{1}{|x_k(t) - x_j(t)|} \left| {\sum_{i:i\neq k}}' \frac{1}{x_k(t) - x_i(t)} \right| dt \tag{79}$$

is finite. But using (50) and (52), we can crudely bound

$$\begin{aligned}
\left| {\sum_{i:i\neq k}}' \frac{1}{x_k(t) - x_i(t)} \right|,\\
\frac{1}{|x_k(t) - x_j(t)|} \ll \log_+^{O(1)}(k) \left(\frac{1}{|x_k(t) - x_{k-1}(t)|} + \frac{1}{|x_k(t) - x_{k+1}(t)|} \right)
\end{aligned}$$

(using the convention $x_0(t) = 0$), so expression (79) may in turn be crudely bounded by

$$\sum_k \psi_T^2(k)(T + |k|)^{0.1} \log_+^{O(1)}(k) \int_{t_0}^{0} \frac{1}{|x_k(t) - x_{k-1}(t)|^2} + \frac{1}{|x_k(t) - x_{k+1}(t)|^2}\, dt,$$

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and this will be finite thanks to Proposition 15 and (66). We conclude (after desymmetrizing in j and k) that

$$\mathcal{H}_T(0)-\mathcal{H}_T(t_0) = -4\int_{t_0}^{0} \sum_{j,k:j\sim_T k} \psi_T(j)\psi_T(k)\frac{1}{x_k(t)-x_j(t)}\sum_{i:i\neq k}{}' \frac{1}{x_k(t)-x_i(t)}\,dt.$$

The above analysis also shows that the integrand is absolutely integrable in time. From the Lebesgue differentiation theorem, we conclude that $\tilde{\mathcal{H}}_T$ is absolutely continuous and that (78) holds at almost every time t.

To conclude the proof of the proposition, it will thus suffice to show that

$$\sum_{j,k:j\sim_T k} \psi_T(j)\psi_T(k)\frac{1}{x_k-x_j}\sum_{i\neq k}{}' \frac{1}{x_k-x_i} \tag{80}$$

is equal to $\tilde{E}_T$ plus negligible terms. We can split this expression as $X_1 + X_2 + X_3 + X_4$, where

$$\begin{aligned}
X_1 &:= \sum_{j,k:j\sim_T k} \psi_T(j)\psi_T(k)\frac{1}{(x_k-x_j)^2}\\
X_2 &:= \sum_{j,k:j\sim_T k} \psi_T(j)\psi_T(k)\frac{1}{x_k-x_j}\sum_{i:i\sim_T j,k} \frac{1}{x_k-x_i}\\
X_3 &:= \sum_{j,k:j\sim_T k} \psi_T(j)\psi_T(k)\frac{1}{x_k-x_j}\sum_{i:i\sim_T k;\, i\not\sim_T j;\, i\neq j} \frac{1}{x_k-x_i}\\
X_4 &:= \sum_{j,k:j\sim_T k} \psi_T(j)\psi_T(k)\frac{1}{x_k-x_j}\sum_{i:i\not\sim_T k;\, i\neq k}{}' \frac{1}{x_k-x_i}.
\end{aligned}$$

We first claim that X_4 is negligible. From (50), we have

$$x_k - x_i = \xi_k - \xi_i + O(\log_+(|i|+|k|))$$

and hence (by (44)),

$$\frac{1}{x_k-x_i} = \frac{1}{\xi_k-\xi_i} + O\left(\frac{\log_+^2(|i|+|k|)}{|k-i|^2}\right),$$

which implies that

$$\sum_{i:i\not\sim_T k;\, i\neq k}{}' \frac{1}{x_k-x_i} = \sum_{i:i\not\sim_T k;\, i\neq k}{}' \frac{1}{\xi_k-\xi_i} + \tilde{O}(T^{-0.1}).$$

From (44), we may crudely bound this sum by $\tilde{O}(1)$. By Lemma 21(iii), this shows that the contribution to X_4 of those k for which $|k| \leqslant T^{0.9}$ or $|k| \geqslant T^{1.1}$ (say) is negligible, so we may assume $T^{0.9} \leqslant |k| \leqslant T^{1.1}$. Let $A \geqslant 2$ be a large constant. Using (44), we may write

$$\sum_{i: i \not\sim_T k;\, i \neq k}' \frac{1}{\xi_k - \xi_i} = \sum_{i: T^{0.2} \leqslant |k-i| \leqslant A|k|} \frac{1}{\xi_k - \xi_i} + \sum_{i: |i| \geqslant A|k|} \frac{1}{\xi_k - \xi_i} + O\left(\frac{\log T}{A}\right).$$

For the first sum on the right-hand side, we use (45) (as in the proof of (74)) as well as (43) to conclude that

$$\frac{1}{\xi_k - \xi_i} = \frac{\log \xi_k}{4\pi} \frac{1}{k-i} + O_A\left(\frac{1}{|k|}\right),$$

where the subscript in the O_A notation means that the implied constant can depend on A. As $i \mapsto \frac{\log \xi_k}{4\pi} \frac{1}{k-i}$ is odd around k, we conclude that

$$\sum_{i: T^{0.2} \leqslant |k-i| \leqslant A|k|} \frac{1}{\xi_k - \xi_i} = O_A(1).$$

Meanwhile, combining the i and $-i$ terms and using (43) and (44), we have

$$\sum_{i: |i| \geqslant A|k|} \frac{1}{\xi_k - \xi_i} = -2\xi_k \sum_{i: i \geqslant A|k|} \frac{1}{\xi_i^2 - \xi_k^2} = O\left(\frac{\log T}{A}\right).$$

Sending A slowly to infinity, we conclude that

$$\sum_{i: i \not\sim_T k;\, i \neq k}' \frac{1}{x_k - x_i} = o_{T \to \infty}(\log T),$$

and the negligibility of X_4 then follows from Lemma 21(ii).

Now we claim that X_2 is negligible. Thanks to the restrictions on i, j, k, we see that

$$\psi_T(i), \psi_T(j) = (1 + \tilde{O}((T + |k|)^{-0.8}))\psi_T(k),$$

and hence

$$\psi_T(j)\psi_T(k) = \psi_T(i)^{2/3}\psi_T(j)^{2/3}\psi_T(k)^{2/3} + \tilde{O}((T + |k|)^{-0.8}\psi_T(j)\psi_T(k)).$$

The sum

$$\sum_{i,j,k:\, j \sim_T k;\, i \sim_T j,k} \frac{\psi_T(i)^{2/3}\psi_T(j)^{2/3}\psi_T(k)^{2/3}}{(x_k - x_j)(x_k - x_i)}$$

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symmetrizes to zero, and hence

$$X_2 \lessapprox \sum_{i,j,k:\, j\sim_T k;\, i\sim_T j,k} (T+|k|)^{-0.8} \frac{\psi_T(j)\psi_T(k)}{|x_k-x_j||x_k-x_i|}.$$

Estimating $\frac{1}{|x_k-x_j||x_k-x_i|} \ll \frac{1}{|x_k-x_j|^2} + \frac{1}{|x_k-x_i|^2}$ and performing the i or j summation respectively, we conclude that

$$X_2 \lessapprox \sum_{j,k:\, j\sim_T k} (T+|k|)^{-0.6} \frac{\psi_T(j)\psi_T(k)}{|x_k-x_j|^2},$$

and so X_2 is negligible thanks to Lemma 21(i).

We have shown that expression (80) is equal to $X_1 + X_3$ plus negligible terms. A similar argument (replacing x_i with ξ_i throughout) shows that the expression

$$\sum_{j,k:\, j\sim_T k} \psi_T(j)\psi_T(k) \frac{1}{\xi_k-\xi_j} \sum_{i:\, i\neq k}{}^{'} \frac{1}{\xi_k-\xi_i} \tag{81}$$

is equal to $X_1' + X_3'$ plus negligible terms, where

$$X_1' := \sum_{j,k:\, j\sim_T k} \psi_T(j)\psi_T(k) \frac{1}{(\xi_k-\xi_j)^2}$$

$$X_3' := \sum_{j,k:\, j\sim_T k} \psi_T(j)\psi_T(k) \frac{1}{\xi_k-\xi_j} \sum_{i:\, i\sim_T k;\, i\not\sim_T j;\, i\neq j} \frac{1}{\xi_k-\xi_i}.$$

From Lemma 18, we see that $\tilde{E}_T$ is equal to

$$X_1 - X_1' + \sum_{j,k:j\not\sim_T k;\, j\neq k} \psi_T(j)\psi_T(k) \left(\frac{1}{(x_k-x_j)^2} - \frac{1}{(\xi_k-\xi_j)^2} \right) \tag{82}$$

up to negligible terms. From (43) and (44), we have

$$\frac{1}{(x_k-x_j)^2} - \frac{1}{(\xi_k-\xi_j)^2} \lessapprox \frac{\log_+^{O(1)}(|j|+|k|)}{|k-j|^3}$$

when $j \neq k$ and $j \not\sim_T k$, so the final term in (82) is negligible. Thus, to complete the proof of the proposition, it will suffice to show that expression (81) and the difference $X_3 - X_3'$ are both negligible.

Expression (81) may be rearranged as

$$\sum_k \psi_T(k) \left(\sum_{j:\, j\sim_T k} \frac{\psi_T(j)}{\xi_k-\xi_j} \right) \left(\sum_{i:\, i\neq k}{}^{'} \frac{1}{\xi_k-\xi_i} \right).$$

By (44), both inner sums are $\tilde{O}(1)$, so the contribution of those $|k| \leqslant T^{0.5}$ or $|k| \geqslant T^{1.5}$ (say) is negligible. For $T^{0.5} < |k| \leqslant T^{1.5}$, we see from (72) that the factor $\sum_{j: j \sim_T k} \frac{\psi_T(j)}{\xi_k - \xi_j}$ is $o_{T\to\infty}(\log T)$, and from (44) and (73) and the triangle inequality, we also see that $\sum'_{i: i\neq k} \frac{1}{\xi_k - \xi_i} = O(\log T)$. Thus (81) is negligible as required.

Finally, we show that $X_3 - X_3'$ is negligible. This quantity may be written as

$$\sum_{i,j,k: i,j \sim_T k;\ |i-j| > (T^2+|i|+|j|)^{0.1}} \psi_T(j)\psi_T(k)\left(\frac{1}{(x_k - x_j)(x_k - x_i)} - \frac{1}{(\xi_k - \xi_j)(\xi_k - \xi_i)}\right).$$

Observe that if $|k - j|$ and $|k - i|$ are both larger than or equal to $T^{0.1}$, then from (44) and (50), one has

$$\begin{aligned} \frac{1}{(x_k - x_j)(x_k - x_i)} - \frac{1}{(\xi_k - \xi_j)(\xi_k - \xi_i)} &\ll \frac{\log_+^{O(1)}(|i| + |j| + |k|)}{T^{0.1}|\xi_k - \xi_j||\xi_k - \xi_i|} \\ &\ll \frac{\log_+^{O(1)}(|i| + |j| + |k|)}{T^{0.1}|k - j||k - i|}, \end{aligned}$$

and so the contribution of this case is negligible. From the triangle inequality, we see that it is not possible for $|k - j|$ and $|k - i|$ to both be less than $T^{0.1}$, so it remains to treat the components

$$\sum_{\substack{i,j,k: 0<|j-k|<(T^2+|j|+|k|)^{0.1} \\ 0<|i-k|<T^{0.1};\ |i-j|>(T^2+|i|+|j|)^{0.1}}} \times \psi_T(j)\psi_T(k)\left(\frac{1}{(x_k - x_j)(x_k - x_i)} - \frac{1}{(\xi_k - \xi_j)(\xi_k - \xi_i)}\right) \tag{83}$$

and

$$\sum_{\substack{i,j,k: 0<|j-k|<T^{0.1} \\ 0<|i-k|<(T^2+|i|+|k|)^{0.1};\ |i-j|>(T^2+|i|+|j|)^{0.1}}} \times \psi_T(j)\psi_T(k)\left(\frac{1}{(x_k - x_j)(x_k - x_i)} - \frac{1}{(\xi_k - \xi_j)(\xi_k - \xi_i)}\right). \tag{84}$$

Consider first (83). From the triangle inequality, we have $|j - k| \gg T^{0.2}$, and hence by (50),

$$\frac{1}{x_k - x_j} = (1 + \tilde{O}(T^{-0.2}))\frac{1}{\xi_k - \xi_j}.$$

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By Lemma 21(ii) and (44), we may thus replace $\frac{1}{x_k - x_j}$ by $\frac{1}{\xi_k - \xi_j}$ at negligible cost in (83), leaving us with

$$\sum_{\substack{i,j,k: 0<|j-k|<(T^2+|j|+|k|)^{0.1} \\ 0<|i-k|<T^{0.1};\ |i-j|>(T^2+|i|+|j|)^{0.1}}} \psi_T(j)\psi_T(k)\left(\frac{1}{x_k - x_i} - \frac{1}{\xi_k - \xi_i}\right)\frac{1}{\xi_k - \xi_j}$$

up to negligible errors. But by (45) and the hypothesis $|i-k| \leqslant T^{0.1}$, one may bound

$$\sum_{\substack{j: 0<|j-k|<(T^2+|j|+|k|)^{0.1} \\ |i-j|>(T^2+|i|+|j|)^{0.1}}} \frac{\psi_T(j)}{|\xi_k - \xi_j|} \lessapprox T^{-0.1}\psi_T(k)$$

when $T^{0.9} \leqslant |k| \leqslant T^{1.1}$, and use the weaker bound

$$\sum_{\substack{j: 0<|j-k|<(T^2+|j|+|k|)^{0.1} \\ |i-j|>(T^2+|i|+|j|)^{0.1}}} \frac{\psi_T(j)}{|\xi_k - \xi_j|} \lessapprox \psi_T(k)$$

for all other k, so this expression is also negligible by Lemma 21(ii), (iii), (iv) (noting that $\psi_T(k)$ and $\psi_T(i)$ are comparable). A similar argument also handles (84). $\square$

To use Proposition 22, we need estimates that ensure $\tilde{E}_T$ is large when $\tilde{\mathcal{H}}_T$ is large. To this end, we have the following lemma.

LEMMA 24. *Let m be a natural number, and let $\Lambda/2 \leqslant t \leqslant 0$. Let $T > 0$, and let $\delta = \delta(T)$ go to zero as $T \to \infty$ sufficiently slowly. If $\tilde{\mathcal{H}}_T(t) \geqslant \delta m T \log_+^3 T$, then $\tilde{E}_T(t) \gg \delta 2^{2m} T \log_+^3 T$, where the implied constant is absolute.*

Proof. As before, we suppress explicit dependence on t, and we may assume T to be large as the claim is trivial from compactness for $T = O(1)$. From Lemma 19, we have (for δ decaying sufficiently slowly) that

$$\sum_{j,k \in \mathbb{Z}^*:\ j \sim_T k} \psi_T(j)\psi_T(k)\tilde{H}_{jk}(t) \geqslant \frac{99}{100}\delta m T \log^3 T.$$

From Lemma 20, we see that

$$\sum_{k:\ j\sim_T k;\ |k-j| \geqslant \varepsilon(j)\log_+^2 \xi_j} \tilde{H}_{jk}(t) \ll \varepsilon(j)\log_+^2 j$$

for any $j \in \mathbb{Z}^*$, which implies that

$$\sum_{j,k:\, j\sim_T k;\, |k-j|\geqslant \varepsilon(j)\log_+^2 \xi_j} \psi_T(j)\psi_T(k)\tilde{H}_{jk}(t) \leqslant \frac{1}{2}\delta T \log^3 T$$

if $\delta(T)$ goes to zero slowly enough. By (69), we conclude that

$$\sum_{j,k:\, j\sim_T k;\, |k-j|<\varepsilon(j)\log_+^2 \xi_j} \psi_T(j)\psi_T(k)\tilde{H}_{jk}(t) \gg \delta m T \log_+^3 T. \tag{85}$$

We now claim that

$$\sum_{\substack{j,k:\, j\sim_T k;\, |k-j|<\varepsilon(j)\log_+^2 \xi_j \\ |x_j-x_k|\geqslant 2^{-m}|\xi_j-\xi_k|}} \psi_T(j)\psi_T(k)\tilde{H}_{jk}(t) \ll \delta^2 m T \log^3 T \tag{86}$$

(say). To see this, we use (45) and (70) to bound

$$L_{jk} \ll m + \frac{|x_j - x_k|}{|\xi_j - \xi_k|} \ll m + \frac{|x_j - x_k|}{|j-k|}\log T$$

and also $\psi_T(j) \asymp \psi_T(k)$ for j, k in the sum. Thus we may bound (86) by

$$m \sum_{j,k:\, |k-j|<\varepsilon(j)\log_+^2 \xi_j} \psi_T(j)^2 + \sum_{j,k:\, 0<|k-j|<\varepsilon(j)\log_+^2 \xi_j} \psi_T(j)^2 \frac{|x_j - x_k|}{|j-k|}\log T.$$

We may directly compute

$$\sum_{j,k:\, j\sim_T k;\, |k-j|<\varepsilon(j)\log_+^2 \xi_j} \psi_T(j)^2 \ll \delta^2 T \log^3 T$$

if $\delta = \delta(T)$ goes to zero slowly enough. Thus it will suffice to show that

$$\sum_{j,k:\, 0<|k-j|<\varepsilon(j)\log_+^2 \xi_j} \psi_T(j)^2 \frac{|x_j - x_k|}{|j-k|} \ll \delta^2 T \log^2 T. \tag{87}$$

But for any natural number n, we see from telescoping series and (50) that

$$\sum_{j:\, 2^n\leqslant |j|<2^{n+1}} |x_j - x_{j+h}| \ll |h|\frac{2^n}{n}$$

whenever $|h| \ll 2^n$; summing over $|h| < \varepsilon(j)\log_+^2 \xi_j$, we conclude that

$$\sum_{\substack{j,k:\, 2^n\leqslant |j|<2^{n+1} \\ 0<|k-j|<\varepsilon(j)\log_+^2 \xi_j}} \frac{|x_j - x_k|}{|j-k|} \ll \varepsilon(2^n)2^n n,$$

which gives (87) if δ goes to zero slowly enough.

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From (85) and (86), we have

$$\sum_{\substack{j,k: j \sim_T k;\ |k-j| < \varepsilon(j) \log_+^2 \xi_j \\ |x_j - x_k| \leqslant 2^{-m} |\xi_j - \xi_k|}} \psi_T(j)\psi_T(k)\tilde{H}_{jk}(t) \gg \delta m T \log^3 T.$$

But for j, k in this sum, we see from (62) and (70) that

$$\tilde{H}_{jk}(t) \ll \log \frac{|\xi_j - \xi_k|}{|x_j - x_k|} \ll \frac{m 2^{-2m} |\xi_j - \xi_k|^2}{|x_j - x_k|^2} \ll m 2^{-2m} \tilde{E}_{jk},$$

and the claim follows. □

We can now shrink $\tilde{\mathcal{H}}_T$ down to a reasonable size in finite time.

COROLLARY 25. *One has* $\tilde{\mathcal{H}}_T(t) = O(\delta T \log_+^3 T)$ *for* $\Lambda/4 \leqslant t \leqslant 0$.

Proof. We may take T to be large. From Proposition 22 and Lemma 24, we see that for any natural number m, and for almost every time t for which one has

$$\tilde{\mathcal{H}}_T(t) \geqslant \delta m T \log^3 T,$$

one has

$$\partial_t \tilde{\mathcal{H}}_T(t) \leqslant -c\delta 2^{2m} T \log^3 T$$

for some absolute constant $c > 0$. In particular, if m is larger than some large absolute constant m_0, and $\Lambda/2 \leqslant t \leqslant \Lambda/4$ is such that

$$\delta m T \log^3 T \leqslant \tilde{\mathcal{H}}_T(t) \leqslant \delta(m+1) T \log^3 T, \tag{88}$$

then it is not possible (for m_0 large enough) to have $\tilde{\mathcal{H}}_T(t') \geqslant \delta m T \log^3 T$ for all $t \leqslant t' \leqslant t + c^{-1}2^{-2m}$, as this would violate the fundamental theorem of calculus for absolutely continuous functions. Thus, by the intermediate value theorem, there exists $t \leqslant t' \leqslant t + c^{-1}2^{-2m}$ such that

$$\delta(m-1) T \log^3 T \leqslant \tilde{\mathcal{H}}_T(t') \leqslant \delta m T \log^3 T,$$

and on iterating this we conclude (for m_0 large enough) that there exists $t \leqslant t'' \leqslant t + 2c^{-1}2^{-2m_0}$ such that

$$\tilde{\mathcal{H}}_T(t'') \leqslant \delta m_0 T \log^3 T. \tag{89}$$

We run this argument with t set equal to $\Lambda/2$, and m the unique integer obeying (88), to conclude (for m_0 large enough) that there exists $\Lambda/2 \leqslant t'' \leqslant \Lambda/4$

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obeying (89). (Note that this conclusion is immediate if the initial value of m was already less than m_0.) On the other hand, from Proposition 22, we have $\partial_t \tilde{\mathcal{H}}_T(t) \leqslant O(\delta T \log^3 T)$ for almost every $t'' \leqslant t \leqslant 0$ if δ decays sufficiently slowly. The claim now follows from the fundamental theorem of calculus (absorbing m_0 into the implied constants), recalling that $\tilde{\mathcal{H}}_T$ is non-negative. □

From Proposition 22 and the fundamental theorem of calculus for absolutely continuous functions, one has

$$\tilde{\mathcal{H}}_T(\Lambda/4) - \tilde{\mathcal{H}}_T(0) = (4 + o_{T\to\infty}(1)) \int_{\Lambda/4}^{0} \tilde{E}_T(t)\, dt + o_{T\to\infty}(T \log_+^3 T),$$

and claim (68) now follows from Corollary 25. This concludes the proof of Theorem 17.

8. Controlling the energy at time 0

In the previous section, we controlled a time average of the energy. Now, using monotonicity properties of the energy, we can in fact control energy at time zero.

PROPOSITION 26 (Energy bound at time zero). *Let T be large. Then*

$$\tilde{E}^{[T\log T, 2T\log T]}(0) = o_{T\to\infty}(T \log^3 T).$$

Proposition 26 will be proven by iterating the following claim.

PROPOSITION 27 (Energy propagation inequality). *Let T be large, let $I = [I_-, I_+]$ be an interval containing $[T\log T, 2T\log T]$ and contained in $[0.5T\log T, 3T\log T]$, and let $\Lambda/4 \leqslant t_1 \leqslant t_2 \leqslant 0$ be such that $t_2 \leqslant t_1 + \frac{1}{100\log^2 T}$. Then*

$$\tilde{E}^{I'}(t_2) \leqslant \tilde{E}^{I}(t_1) + \tilde{O}(1),$$

where $I' := [I_- + \log^3 T, I_+ - \log^3 T]$ is a slightly shrunken version of I.

Recall that $\tilde{O}(1)$ is any quantity that is $O(\log^{O(1)} T)$.

Let us assume Proposition 27 for the moment and finish the proof of Proposition 26. From Theorem 17, we have

$$\int_{\Lambda/4}^{0} \tilde{E}^{[0.5T\log T, 3T\log T]}(t)\, dt = o_{T\to\infty}(T \log_+^3 T),$$

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and so by the pigeonhole principle, we may find $\Lambda/4 \leqslant t_0 \leqslant 0$ such that

$$\tilde{E}^{[0.5T \log T, 3T \log T]}(t_0) = o_{T\to\infty}(T \log_+^3 T).$$

Applying Proposition 27 $O(\log^2 T)$ times to get from t_0 to 0, we conclude that

$$\tilde{E}^{[I_-, I_+]}(0) \leqslant o_{T\to\infty}(T \log_+^3 T)$$

for some interval $[I_-, I_+]$ containing $[T \log T, 2T \log T]$ and contained in $[0.5T \log T, 3T \log T]$ (in fact, we have $I_- = 0.5T \log T + O(\log^5 T)$ and $I_+ = 3T \log T - O(\log^5 T)$). Since $\tilde{E}^I(0)$ is monotone in I, Proposition 26 follows.

It remains to establish Proposition 27. We use an argument due to Bourgain [2, Section 4] that combines local conservation laws (or, in this case, local monotonicity formulae) with the pigeonhole principle.

The first step is to locate a good subset of particles indexed by an interval close to $[I_-, I_+]$ that does not gain too much energy due to interactions with its environment because of the separation between these particles and the environment. From (50) and the pigeonhole principle, one can find natural numbers

$$I_- \leqslant j_- - 1 < j_- \leqslant I_- + \log^3 T \leqslant I_+ - \log^3 T \leqslant j_+ < j_+ + 1 \leqslant I_+$$

such that

$$x_{j_-}(t_2) - x_{j_- - 1}(t_2) \geqslant \frac{1}{\log T} \tag{90}$$

(say) and similarly

$$x_{j_+ + 1}(t_2) - x_{j_+}(t_2) \geqslant \frac{1}{\log T}.$$

From Lemma 12(iv) applied to $K = \{j_- - 1, j_-\}$, we have

$$\partial_t (x_{j_-}(t) - x_{j_- - 1}(t))^2 \leqslant 8$$

for all $t_1 \leqslant t \leqslant t_2$. Since $t_2 - t_1 \leqslant \frac{1}{100 \log^2 T}$, we conclude from the fundamental theorem of calculus and (90) that

$$x_{j_-}(t) - x_{j_- - 1}(t) \gg \frac{1}{\log T} \tag{91}$$

for all $t_1 \leqslant t \leqslant t_2$. Similarly

$$x_{j_+ + 1}(t) - x_{j_+}(t) \gg \frac{1}{\log T}. \tag{92}$$

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The basic point is that because the particles $x_{j_-}, \dots, x_{j_+}$ never get too close to the remaining particles x_j, $j < j_-$ and x_j, $j > j_+$ in the system, the total energy of the former set of particles will remain approximately conserved over short periods of time thanks to Lemma 12. More precisely, let K denote the discrete interval $K := [j_-, j_+]$, and define the un-normalized energy

$$E^K(t) := \sum_{k,k' \in K : k \neq k'} E_{kk'}(t).$$

From Lemma 12, we have

$$\partial_t E^K(t) \leqslant \sum_{\substack{j \notin K \\ k,k' \in K : k \neq k'}} \frac{4}{(x_k - x_{k'})^2 (x_k - x_j)(x_{k'} - x_j)}$$

for $t_1 \leqslant t \leqslant t_2$. But from (50), (91) and (92), we have

$$\sum_{\substack{j \notin K \\ k,k' \in K : k \neq k'}} \frac{4}{(x_k - x_{k'})^2 (x_k - x_j)(x_{k'} - x_j)} \lessapprox 1.$$

From the fundamental theorem of calculus, we conclude that

$$E^K(t_2) \leqslant E^K(t_1) + \tilde{O}(1),$$

which by monotonicity of E^K in K implies that

$$E^{I'}(t_2) \leqslant E^I(t_1) + \tilde{O}(1).$$

Applying Lemma 16, we conclude that

$$\tilde{E}^{I'}(t_2) \leqslant \tilde{E}^I(t_1) + \tilde{O}(1) + 2 \sum_{\substack{j \in I \backslash I' \\ k \in I : j \neq k}} \frac{1}{(\xi_j - \xi_k)^2}.$$

But from (44), one has

$$\sum_{\substack{j \in I \backslash I' \\ k \in I : j \neq k}} \frac{1}{(\xi_j - \xi_k)^2} \lessapprox 1,$$

and Proposition 27 follows.

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9. Contradicting pair correlation

It remains to see that Proposition 26 is in contradiction with results that are known to be the case for the points $x_j(0)$. Note in particular that

$$\sum_{T\log T \leqslant j, j+1 \leqslant 2T\log T} \frac{1}{|\xi_{j+1}-\xi_j|^2} V\left(\frac{x_{j+1}(0)-x_j(0)}{\xi_{j+1}-\xi_j}\right) \leqslant \tilde{E}^{[T\log T, 2T\log T]}(0).$$

In this range using (43) and (45), we have $\xi_{j+1}-\xi_j \sim 4\pi/\log_+ T$, and so Proposition 26 implies that

$$\log^2 T \sum_{T\log T \leqslant j, j+1 \leqslant 2T\log T} V\left(\frac{x_{j+1}(0)-x_j(0)}{\xi_{j+1}-\xi_j}\right) = o_{T\to\infty}(T\log^3 T).$$

By Markov's inequality (see [28, Ch. 1]), this implies that

$$V\left(\frac{x_{j+1}(0)-x_j(0)}{\xi_{j+1}-\xi_j}\right) = o_{T\to\infty}(1)$$

for a fraction $1-o_{T\to\infty}(1)$ of $j \in [T\log T, 2T\log T]$. But using the properties (62) of the function V, this implies that

$$\frac{x_{j+1}(0)-x_j(0)}{\xi_{j+1}-\xi_j} = 1 + o_{T\to\infty}(1)$$

or

$$x_{j+1}(0)-x_j(0) = \frac{4\pi + o_{T\to\infty}(1)}{\log T} \tag{93}$$

for a fraction $1-o_{T\to\infty}(1)$ of $j \in [T\log T, 2T\log T]$.

In particular, since the points $x_j(0)$ are twice the imaginary ordinates of nontrivial zeros of the Riemann zeta function, this implies that the gaps between the zeros of the zeta function are rarely much larger or smaller than the mean spacing. But this contradicts perhaps most strikingly the results of Montgomery [15], who determined on the Riemann hypothesis the pair correlation measure for the zeros, measured against a class of band-limited functions. As noted by Montgomery, his result implies that a positive proportion of zeros have a spacing between them strictly smaller than the mean spacing. The proof of this claim is not written down in [15], but Conrey *et al.* prove as their main result of [6] (using different ideas) that for any $\lambda > .77$, there exists a constant $c(\lambda) > 0$ such that at least a proportion $c(\lambda)$ of $j \leqslant T\log T$ satisfy

$$x_{j+1}(0)-x_j(0) \leqslant \lambda \frac{4\pi}{\log T}.$$

This contradicts (93) and therefore the assumption that $\Lambda < 0$.

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