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ANALYTIC STRUCTURES FOR H^∞ OF CERTAIN DOMAINS IN C^n

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Let $\Omega \subset \mathbf{C}^n$ be a bounded domain; let $H^\infty(\Omega)$ be the uniform algebra of bounded analytic functions on Ω; and let $\Sigma(\Omega)$ be the maximal ideal space of $H^\infty(\Omega)$. In the weak-$*$ topology of $(H^\infty(\Omega))^*$, $\Sigma(\Omega)$ is a compact Hausdorff space in which Ω is embedded in a natural fashion, so that to every $g \in H^\infty(\Omega)$ there corresponds the Gelfand transform $\hat{g} \in C(\Sigma(\Omega))$; $\hat{g}|\Omega = g$. Let $\mathscr{M} = \mathscr{M}(\Omega)$ be the weak-$*$ closure of Ω in $\Sigma(\Omega)$. We want to know when $\mathscr{M}\backslash\Omega$ contains analytic structures of dimension n. More precisely, if $\mathscr{U}$ is a domain in $\mathbf{C}^k$, an analytic map from $\mathscr{U}$ to $\Sigma(\Omega)$ is a function, $f : \mathscr{U} \to \Sigma(\Omega)$, with the property that for every g in $H^\infty(\Omega)$, $\hat{g} \circ f$ is holomorphic on $\mathscr{U}$. An *analytic structure of dimension k* is the range of a one-to-one analytic map from a domain in $\mathbf{C}^k$ into $\Sigma(\Omega)$. We shall prove the following theorem. (A sequence $\{z_k\}_{k=1}^\infty \subset \Omega$ is an *interpolating sequence* if the map $T : H^\infty(\Omega) \to l^\infty$ given by $Tf = \{f(z_k)\}_{k=1}^\infty$ is surjective.)

THEOREM 1. *If $\Omega \subset \mathbf{C}^n$ is a bounded homogeneous domain, $\{z_k\}_{k=1}^\infty \subset \Omega$ is an interpolating sequence, and $m \in \mathscr{M}\backslash\Omega$ is in the weak-$*$ closure of $\{z_k\}_{k=1}^\infty$, then m is contained in an analytic structure of dimension n.*

In the case $n = 1$ and Ω is the unit disk, D, K. Hoffman [**7**] showed that being in the closure of an interpolating sequence is necessary and sufficient for containment in a one dimensional analytic structure. That result is just part of the material in [**7**], where Hoffman gives a rather complete description of $\mathscr{M}(D)$. Although interpolating sequences play a central role in Hoffman's paper, the interpolation property appears to be incidental. Rather, certain characteristics of Blaschke products which vanish at the points of an interpolating sequence are crucial to the story. Similarly, in this paper, another property, that of strong separation, which is possessed by interpolating sequences, plays the central role. We will find that the notion of strong separation yields a necessary and sufficient condition for the embedding of certain types of n-dimensional analytic structures in $\mathscr{M}$.

1. Notation and preliminary material. Let $\mathscr{G}$ be the group of automorphisms (biholomorphic maps) of Ω onto itself. It is possible that $\mathscr{G}$ may be trivial. On the other hand, if $\mathscr{G}$ is transitive, *i.e.* if for every pair of points, p and q in Ω, there is an element in $\mathscr{G}$ which carries one into the other, the

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domain Ω is said to be *homogeneous*. Throughout the remainder of this paper, Ω will be a bounded homogeneous domain containing the origin. Examples of such domains include simply connected domains in $\mathbf{C}$, polydisks, balls, and Cartesian products of bounded homogeneous domains. The reader is referred to [**3**; **8**; or **9**] for descriptions of more complicated bounded homogeneous domains.

Let $\mathscr{A}$ be the set of analytic maps (not necessarily one-to-one or onto) of Ω into itself. For $z \in \Omega$, let $\mathscr{A}_z = \{f \in \mathscr{A} : f(0) = z\}$, and let $\mathscr{G}_z = \mathscr{G} \cap \mathscr{A}_z$, the set of automorphisms of Ω which carry 0 into z.

If $f = (f_1, \ldots, f_n)$ is a holomorphic $\mathbf{C}^n$-valued map defined on a neighborhood of $z \in \mathbf{C}^n$, we will let $Df(z)$ denote the Jacobian matrix,

$$(Df(z))_{ij} = \frac{\partial f_i}{\partial z_j}(z).$$

The determinant of $Df(z)$ will be denoted by $Jf(z)$. The following elementary facts are doubtlessly well known.

PROPOSITION 1. *Let Ω be a bounded homogeneous domain in $\mathbf{C}^n$.*

i) *If $z \in \Omega$, and $g \in \mathscr{G}_z$, then $\mathscr{G}_z = g \circ \mathscr{G}_0$, and $\mathscr{A}_z = g \circ \mathscr{A}_0$.*

ii) *If $f \in \mathscr{A}_0$, then $|Jf(0)| \leqq 1$.*

iii) *The function $g \to |Jg(0)|$ is constant on $\mathscr{G}_z$. If $J_z = |Jg(0)|$ for $g \in \mathscr{G}_z$, then $|Jf(0)| \leqq J_z$ for every f in $\mathscr{A}_z$.*

iv) *The function $z \to J_z$ is continuous on Ω.*

Proof. Statement (i) is evident. If $f \in \mathscr{A}_0$ and $|Jf(0)| = R > 1$, let f_n be the composition of f with itself n times, then $f_n \in \mathscr{A}_0$, and $|Jf_n(0)| = R^n$. Since Ω is bounded, a subsequence of the f_n's converges uniformly on compact sets to an element in $\mathscr{A}_0$ with an infinite Jacobian determinant. Since this is impossible, (ii) is proved. From (ii) it follows that if $g \in \mathscr{G}_0$, then $|J_g(0)| = |J_{g^{-1}}(0)|^{-1} = 1$, and $|Jf(0)| \leqq 1 = J_0$, for every f in $\mathscr{A}_0$. This gives (iii) in the case where $z = 0$. The general case follows from this and statement (i). To prove (iv), fix z_0 and $\epsilon > 0$. Suppose that for every integer N, one can find $z_N \in \Omega$ such that $|z_N - z_0| < 1/N$ and $|J_{z_N} - J_{z_0}| > \epsilon$. Let $g_N \in \mathscr{G}_{z_N}$; then there are subsequences of $\{g_N\}$ and $\{g_N^{-1}\}$ which simultaneously converge uniformly on compact sets to maps g and h, respectively. Clearly $h = g^{-1}$, and $g(0) = z_0$, so $g \in \mathscr{G}_{z_0}$. But then

$$0 = ||Jg(0)| - J_{z_0}| = |\mathrm{Lim}_N\, |Jg_N(0)| - J_{z_0}| = \mathrm{Lim}_N\, |J_{z_N} - J_{z_0}| > \epsilon.$$

This contradiction proves (iv).

Since $\mathscr{M}$ is a compact Hausdorff space, it follows that the Cartesian product $\mathscr{M}^\Omega$ (the set of all functions from Ω into $\mathscr{M}$ with the topology of pointwise convergence) is also compact. Let $\mathscr{B} \subset \mathscr{M}^\Omega$ be the set of analytic maps from Ω to $\mathscr{M}$. The set $\mathscr{B}$ is a closed, hence compact, subset of $\mathscr{M}^\Omega$. Considering $\mathscr{A}$ as a subset of $\mathscr{M}^\Omega$, its closure in $\mathscr{M}^\Omega$, which we denote by $\bar{\mathscr{A}}$, is compact.

2. Strongly separated sets and strong analytic structures. We will use the symbol $H_n^\infty(\Omega)$ to represent the space of bounded, $\mathbf{C}^n$-valued holomorphic functions on Ω. If $F \in H_n^\infty(\Omega)$, $\hat{F}$ is the continuous $\mathbf{C}^n$-valued function on $\Sigma(\Omega)$ whose components are the Gelfand transforms of the components of F.

Definition. Let $m \in \mathscr{M}$. A *strong analytic structure of dimension n, containing* m is the image of a map $\phi \in \overline{\mathscr{A}}$ with the property that $\phi(0) = m$, and $|J(\hat{G} \circ \phi)(0)| > 0$, for some $G \in H_n^\infty(\Omega)$.

Clearly if $|J(\hat{G} \circ \phi)(0)| > 0$, then $\hat{G} \circ \phi$ is invertible in a neighborhood of 0, and ϕ is a one-to-one biholomorphic map of a neighborhood of 0 into $\mathscr{M}$.

Definition. A set $S \subset \Omega$ is *strongly separated* if there is a constant $r > 0$, and a map $G \in H_n^\infty(\Omega)$, such that for every $s \in S$, $G(s) = 0$, and $|JG(s)|J_s \geqq r$.

It is clear that a strongly separated set is discrete, hence countable. One can show directly that if Ω is the unit disk in $\mathbf{C}$, then a set is strongly separated if and only if it is an interpolating sequence. This fact will also follow from Theorems 3 and 4.

Our central result is the following generalization of Theorem 3.4 of [**7**], and extension of Theorem V of [**4**].

THEOREM 2. *If Ω is a bounded homogeneous domain in $\mathbf{C}^n$, and m is a maximal ideal in $\mathscr{M}\backslash\Omega$, then m is contained in a strong n-dimensional analytic structure if and only if m is in the closure of a strongly separated set.*

Proof. Suppose first, that m is in the closure of a strongly separated set S. Then there is a map $G \in H_n^\infty(\Omega)$, and a constant $r > 0$, such that for every $s \in S$, $G(s) = 0$, and $|JG(s)|J_s > r$. For each s in S, choose $g_s \in \mathscr{G}_s$. Since m is in the closure of S, there is a net $\{s_\alpha\}_{\alpha \in A} \subset S$ which converges to m. The corresponding net $\{g_{s_\alpha}\}_{\alpha \in A}$ has a converging subnet, $\{g_{s_\nu}\}_{\nu \in N}$, in the compact set $\overline{\mathscr{A}} \subset \mathscr{M}^\Omega$. Let $\phi = \operatorname{Lim}_\nu g_{s_\nu}$. Then $\phi(0) = \operatorname{Lim}_\nu g_{s_\nu}(0) = \operatorname{Lim}_\nu s_\nu = m$, and $G \circ g_{s_\nu}$ converges uniformly on compact subsets of Ω to $\hat{G} \circ \phi \in H_n^\infty(\Omega)$. Hence $J(G \circ g_{s_\nu})(z)$ converges uniformly on compact subsets of Ω to $J(\hat{G} \circ \phi)(z)$. Since

$$|J(G \circ g_{s_\nu})(0)| = |JG(s_\nu)|J_{s_\nu} > r,$$

it follows that $|J(\hat{G} \circ \phi)(0)| \geqq r$, and m is contained in a strong analytic structure of dimension n.

Before proving the converse we need to state the following lemma from advanced calculus (c.f. [**5**, Lemma 3.2]). Let $D^n(r)$ be the polydisk of radius r in $\mathbf{C}^n$, let $\|\cdot\|$ represent the operator norm on $n \times n$ matrices, and let I be the $n \times n$ identity matrix.

LEMMA 1. *For every $\epsilon > 0$ there is a $\delta > 0$, such that whenever $F : D^n(\epsilon) \rightarrow$*

$\mathbf{C}^n$ satisfies $|F(0)| < \delta$ and $||DF(z) - I|| < \delta$, for all z in $D^n(\epsilon)$, it follows that $F(b) = 0$, for a unique point $b \in D^n(\epsilon)$.

Now suppose m is contained in a strong n-dimensional analytic structure. Then there are maps $G \in H_n^\infty(\Omega)$ and $\phi \in \overline{\mathscr{A}}$ for which $\phi(0) = m$, and $|J(\hat{G} \circ \phi)(0)| = R > 0$. We can assume $\hat{G}(m) = 0$. Since $\phi \in \overline{\mathscr{A}}$, there is a net $\{\phi_\alpha\}_{\alpha \in A} \subset \mathscr{A}$, such that $\phi = \operatorname{Lim}_\alpha \phi_\alpha$. If $t_\alpha = \phi_\alpha(0)$, then $t_\alpha \in \Omega$, and $m = \operatorname{Lim}_\alpha t_\alpha$. By the inverse function theorem, there is a compact polydisk of radius $r > 0$, $\overline{D^n(r)}$, contained in Ω, on which $\hat{G} \circ \phi$ is invertible and $|J(\hat{G} \circ \phi)| \geqq R/2$. Since $\overline{D^n(r)} \subset \Omega$, it follows from (iv) of Proposition 1 that

$$(1) \quad c_0 = \min \{J_z : z \in D^n(r)\} > 0.$$

Let F be an inverse to $\hat{G} \circ \phi$ in a neighborhood of $\overline{D^n(r)}$, so that $F \in H_n^\infty(D_n(\rho))$ for some $\rho > 0$, and $F \circ \hat{G} \circ \phi(z) = z$ for all z in $D_n(r)$. In particular, $F(0) = 0$, and $F \circ G \circ \phi_\alpha(z) \to z$ uniformly on $D^n(r)$. Finally, since $J(G \circ \phi_\alpha)$ converges to $J(\hat{G} \circ \phi)$ on $D^n(r)$ we can assume that

$$(2) \quad |J(G \circ \phi_\alpha)(z)| \geqq R/4, \quad \text{for all } z \text{ in } D^n(r) \text{ and all } \alpha \in A.$$

Now a weak-$*$ neighborhood of m is a set of the form

$$\mathscr{U} = \{\psi \in \Sigma : |\hat{f}_j(\psi)| < 1, \text{where} f_j \in H^\infty(\Omega), \hat{f}_j(m) = 0, j = 1, 2, \ldots, k\}.$$

Given such a $\mathscr{U}$, we have $\hat{f}_j \circ \phi \in H^\infty(\Omega)$, and $\hat{f}_j \circ \phi(0) = 0$. Consequently, there is a constant, $\epsilon > 0$, such that for each $j = 1, 2, \ldots, k$, $|\hat{f}_j \circ \phi| \leqq 1/2$ on the poly-disk $D^n(\epsilon) \subset \Omega$. Let $\epsilon_0 < \operatorname{Min}(\epsilon, r)$. Since $\hat{f}_j \circ \phi(z) = \operatorname{Lim}_\alpha f_j \circ \phi_\alpha(z)$, and convergence is uniform on $D^n(\epsilon_0)$, there is a net index α_1 with the property that for every $\alpha \geqq \alpha_1$, $|f_j \circ \phi_\alpha| < 1$ on $D^n(\epsilon_0)$, for $j = 1, 2, \ldots, k$. Consequently, $\phi_\alpha(D^n(\epsilon_0)) \subset \mathscr{U}$, whenever $\alpha \geqq \alpha_1$. Apply Lemma 1 in the case where $\epsilon = \epsilon_0$ to obtain $\delta > 0$. There is a net index $\alpha_2 \geqq \alpha_1$ for which $\alpha \geqq \alpha_2$ implies $|F \circ G \circ \phi_\alpha(0)| < \delta$, and $||D(F \circ G \circ \phi_\alpha)(z) - I|| < \delta$, for all z in $D^n(\epsilon_0)$. Hence, for all $\alpha \geqq \alpha_2$, there is a unique point $\zeta_\alpha \in D^n(\epsilon_0)$ for which $F \circ G \circ \phi_\alpha(\zeta_\alpha) = 0$, and, since F is univalent, $G \circ \phi_\alpha(\zeta_\alpha) = 0$. For all $\alpha \geqq \alpha_2$, let $s_\alpha = \phi_\alpha(\zeta_\alpha)$. Then $s_\alpha \in \phi_\alpha(D^n(\epsilon_0)) \subset \mathscr{U}$, and $G(s_\alpha) = 0$. If $g_\alpha \in \mathscr{G}_{\zeta_\alpha}$, let $h_\alpha = \phi_\alpha \circ g_\alpha \in \mathscr{A}_{s_\alpha}$. Since $\zeta_\alpha \in D^n(\epsilon_0) \subset D^n(r)$, it follows from Proposition 1 and inequalities (1) and (2) that

$$|JG(s_\alpha)|J_{s_\alpha} \geqq |J(G \circ h_\alpha)(0)| = |J(G \circ \phi_\alpha)(\zeta_\alpha)|J_{\zeta_\alpha} \geqq c_0 R/4.$$

Let $S_{\mathscr{U}} = \{s_\alpha : \alpha \geqq \alpha_2\}$, and let $S = \bigcup \{S_{\mathscr{U}} : \mathscr{U}$ is a weak-$*$ neighborhood of $m\}$. By construction, S is strongly separated, and m is in the closure of S.

3. Interpolating sequences and simultaneous interpolation. In this section we will show that every interpolating sequence in a bounded homogenous domain is strongly separated. Theorem 1 will then be an immediate corollary to Theorem 2. We require the following very useful result of A. Bernard [**1**].

LEMMA 2 [**1**, Theorem 2]. *If $\{z_j\}_{j=1}^{\infty}$ is an interpolating sequence in a domain, $\mathscr{D} \subset \mathbf{C}^n$, then there is a sequence $\{f_j\}_{j=1}^{\infty} \subset H^{\infty}(\mathscr{D})$, and a constant M, such that $f_j(z_k) = 0$ if $j \neq k$, $f_k(z_k) = 1$, and $\sum_{j=1}^{\infty} |f_j(z)| \leqq M$, for all $z \in \mathscr{D}$.*

THEOREM 3. *In a bounded homogeneous domain, every interpolating sequence is strongly separated.*

Proof. Let $\{z_j\}_{j=1}^{\infty} \subset \Omega$ be an interpolating sequence. Let $M, f_1, f_2, \ldots$ be the constant and functions given by Bernard's Theorem. For each j, choose $g_j \in \mathscr{G}_{z_j}$, and let $G(z) = \sum_{j=1}^{\infty} g_j^{-1}(z)\ (f_j(z))^2$. Then

$$|G(z)| \leqq \sum_{j=1}^{\infty} |g_j^{-1}(z)|\, |f_j(z)|^2 \leqq K \sum_{j=1}^{\infty} |f_j(z)|^2 \leqq KM^2,$$

where the constant K reflects the fact that Ω is bounded. Thus $G \in H_n^{\infty}(\Omega)$. Clearly $G(z_j) = 0$, for every z_j. Also,

$$DG(z) = \sum_{j=1}^{\infty} (2f_j(z)A_j(z) + (f_j(z))^2 D(g_j^{-1})(z)),$$

where the ikth entry in the matrix $A_j(z)$ is $\partial f_j/\partial z_k(z)$ multiplied by the ith component of $g_j^{-1}(z)$. In particular, for each k,

(3) $\quad DG(z_k) = D(g_k^{-1})(z_k).$

Therefore $D(G \circ g_k)(0)$ is the identity matrix. Hence $|JG(z_k)|J_{z_k} = 1$, for every z_k.

Theorem 3 has applications in balls and polydisks to simultaneous interpolation of the values of a function and its derivatives. Let $B(R)$ be the (Euclidean) ball of radius R in $\mathbf{C}^n$, and let $\Omega = B = B(1)$ be the unit ball. In this case $\mathscr{G}_0$ is the unitary group, and it follows from proposition 1, that for any $r > 0$ and $z \in B$, the set $V_z(r) = ((D_g(0))^{-1})^T B(r)$ is independent of the choice of $g \in \mathscr{G}_z$. If $f \in H^{\infty}(\Omega)$, then it follows from the Cauchy integral formula in one variable that the gradient of f at 0, $Df(0)$, is in $B(||f||)^T$, the set of row vectors of norm $||f||$. Consequently, for any $z \in B$, and any $g \in \mathscr{G}_z$, $Df(z)D_g(0) \in B(||f||)^T$, so that $(Df(z))^T \in V_z(||f||)$. On the other hand, we have the following.

PROPOSITION 2. *Suppose $\{z_k\}_{k=1}^{\infty} \subset B$ is an interpolating sequence, $\{a_k\}_{k=1}$ is a sequence in $\mathbf{C}$, and $\{v_k\}_{k=1}^{\infty}$ is a sequence of (row) vectors in $\mathbf{C}^n$. Suppose, moreover, that there is a constant M, for which $|a_k| \leqq M$ and $v_k^T \in V_{z_k}(M)$, for all k. Then there is a function $f \in H^{\infty}(B)$ such that $f(z_k) = a_k$ and $Df(z_k) = v_k$, for all k.*

Proof. Since $\{z_k\}_{k=1}$ is interpolating, there is a function $h_0 \in H^{\infty}(\Omega)$ such that $h_0(z_k) = a_k$, for all k. By the argument preceding the statement of the proposition, if $w_k = Dh_0(z_k)$, then $w_k^T \in V_{z_k}(||h_0||)$. Let $u_k = v_k - w_k$, and let $M_1 = M + ||h_0||$. Then $u_k^T \in V_{z_k}(M_1)$, so that if we choose $g_k \in \mathscr{G}_{z_k}$, there are

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vectors $\lambda_k \in B(M_1)$ such that

$$u_k{}^T = ((Dg_k(0))^{-1})^T \lambda_k = (Dg_k{}^{-1}(z_k))^T \lambda_k.$$

Define $G \in H_n{}^\infty(B)$ as in the proof of Theorem 3, and let $G_1, G_2, \ldots, G_n$ be the components of G. For each k, let $\lambda_k = (\lambda_k{}^{(1)}, \ldots, \lambda_k{}^{(n)})$. Then $\{\lambda_k{}^{(j)}\}_{k=1}^\infty$ is a bounded sequence for each $j = 1, 2, \ldots, n$, so there are functions $h_1, \ldots, h_n \in H^\infty(B)$ such that $h_j(z_k) = \lambda_k{}^{(j)}$. Let $h = \sum_{j=1}^n h_j G_j$. Then $h(z_k) = 0$, for all k. Also,

$$Dh(z) = \sum_{j=1}^{n} (h_j(z)DG_j(z) + G_j(z)Dh_j(z)).$$

Applying equality (3) we have, for each k,

$$Dh(z_k) = \sum_{j=1}^{n} \lambda_k{}^{(j)}DG_j(z_k) = ((DG(z_k))^T\lambda_k)^T = v_k - w_k.$$

The desired function is $f = h_0 + h$.

A similar, but simpler, argument yields the following result.

PROPOSITION 3. *Suppose* $\{z_k\}_{k=1}^\infty = \{(z_k{}^{(1)}, \ldots, z_k{}^{(n)})\}_{k=1}^\infty$ *is an interpolating sequence in the unit polydisk,* $D^n \subset \mathbf{C}^n$. *Suppose that* $\{a_k\}_{k=1}^\infty$, $\{b_k{}^{(1)}\}_{k=1}^\infty, \ldots, \{b_k{}^{(n)}\}_{k=1}^\infty$ *are* $n + 1$ *sequences in* $\mathbf{C}$ *satisfying* $|a_k| \leqq M$, *and* $|b_k{}^{(j)}| \leqq M/(1 - |z_k{}^{(j)}|^2)$, *for* $j = 1, \ldots, n$, *and all* k, *where* M *is some fixed constant. Then there is a function* $f \in H^\infty(D^n)$ *satisfying* $f(z_k) = a_k$, *and* $\partial f/\partial z_j(z_k) = b_k{}^{(j)}$, *for all* k *and* j.

4. Uniform separation and a partial converse to Theorem 1. A sequence $\{z_k\}_{k=1}^\infty \subset \Omega$ is *uniformly separated* if there exist functions $f_1, f_2, \ldots \in H^\infty(\Omega)$, and a constant M, such that for every j, $||f_j|| \leqq M$, $f_j(z_j) = 1$, and $f_j(z_k) = 0$ when $k \neq j$. It is well known [**2**] that uniform separation is a necessary and sufficient condition for a sequence in the unit disk of $\mathbf{C}$ to be an interpolating sequence. For balls and polydisks, strong separation implies uniform separation.

THEOREM 4. *If* Ω *is the unit ball or unit polydisk in* $\mathbf{C}^n$, *then every strongly separated set is uniformly separated.*

LEMMA 3. *If* Ω *is the unit ball or unit polydisk in* $\mathbf{C}^n$, *then every* $f \in H^\infty(\Omega)$ *which vanishes at* 0 *can be written in the form*

$$f(z) = f(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i f_i(z),$$

where $f_i \in H^\infty(\Omega)$, $||f_i|| \leqq M||f||$, *and* M *is an absolute constant. (For the polydisk,* $M \leqq 2$.)

Proof. The case of the polydisk is an elementary exercise and is well known. The case of the unit ball was proved by Z. Leibenzon (see [**6**]).

We note that if $f = \sum_{i=1}^{n} z_i f_i$, then $f_i(0) = \partial f/\partial z_i(0)$.

Proof of Theorem 4. (We are indebted to B. A. Taylor for pointing out this argument.) Suppose $S = \{z_k\}_{k=1}^{\infty}$ is strongly separated. Let $G = (G_1, G_2, \ldots, G_n) \in H_n^{\infty}(\Omega)$ and $r > 0$ be as in the definition of strong separation; let $||G|| = \text{Max}_i ||G_i||$. For each $z_k \in S$, let $g_k \in \mathcal{G}_{z_k}$. For $i = 1, \ldots, n$, $G_i \circ g_k \in H^{\infty}(\Omega)$, and $G_i \circ g_k(0) = 0$. So there are functions $h_{ijk} \in H^{\infty}(\Omega)$, $||h_{ijk}|| \leqq M||G||$, such that

$$G_i \circ g_k(z) = \sum_{j=1}^{n} z_j h_{ijk}(z), \quad \text{and} \quad h_{ijk}(0) = \frac{\partial}{\partial z_j}(G_i \circ g_k)(0),$$

the ijth component of the matrix $D(G \circ g_k)(0)$. If $m_k(z)$ is the matrix valued function whose ijth component is $(m_k(z))_{ij} = h_{ijk}(g_k^{-1}(z))$, then the vector valued function G is given by the matrix equation

(4) $\quad G(z) = m_k(z)\, g_k^{-1}(z).$

Let $F_k(z) = \det(m_k(z))$. Since $(m_k(z))_{ij} \in H^{\infty}(\Omega)$, and $|(m_k(z))_{ij}| \leqq M||G||$, it follows that $F_k \in H^{\infty}(\Omega)$, and

$$||F_k|| \leqq M_1 = n!(M||G||)^n.$$

Now $|F_k(z_k)| = |\det(h_{ijk}(0))| = |J(G \circ g_k)(0)| \geqq r$. On the other hand, if $z_i \in S$, $z_i \neq z_k$, then the left hand side of (4) vanishes while the vector on the right hand side, $g_k^{-1}(z_i)$, is not zero. Hence, the matrix $m_k(z_i)$ is singular, so its determinant $F_k(z_i) = 0$. If $\phi_k(z) = F_k(z)/F_k(z_k)$, then $\phi_k \in H^{\infty}(\Omega)$, $||\phi_k|| \leqq M_1/r$, $\phi_k(z_k) = 1$, and $\phi_k(z_i) = 0$, for $i \neq k$. Therefore S is uniformly separated.

If it were known that uniform separation implies interpolation (as is the case for $n = 1$) in either the ball or the polydisk, then Theorems 2, 3, and 4 would give a converse to Theorem 1. Some partial results for the polydisk are possible.

A *wedge* in the unit disk D is the region inside D lying between two distinct circles, γ_1 and γ_2, such that $\gamma_1 \cap \gamma_2$ is contained in the boundary of D, and both γ_1 and γ_2 intersect D. A *near wedge* in the polydisk D^n is the Cartesian product of one copy of D with $n - 1$ one dimensional wedges.

THEOREM 5. *If Ω is the unit polydisk, and $m \in \mathcal{M}\backslash\Omega$ is in the closure of a near wedge, then m is contained in a strong analytic structure of dimension n if and only if it is a limit point of an interpolating sequence.*

Proof. We have already shown that a limit point of an interpolating sequence is contained in a strong n-dimensional analytic structure.

Let W be a near wedge in D^n, and let $\rho(z, w)$ be the pseudohyperbolic distance between points z and w in D^n. ($\rho(z, w) = |||g^{-1}(w)|||$, where $g \in \mathcal{G}_z$, and $||| \cdot |||$ is the polydisk norm in $\mathbf{C}^n$.) It is easy to see that for any r, $0 < r < 1$, the set $\{z \in D^n : \rho(z, w) < r$, for some $w \in W\}$ is contained in another near wedge, W_r. One can show (see, *e.g.*, Lemmas 4.6, 5.10, and 5.11 of [**10**]) that there is a constant r and an open set $\mathcal{O}$ of Σ, such that the weak-$*$ closure of W

is contained in $\mathscr{O}$, and $D^n \cap \mathscr{O}$ is contained in the near wedge W_r. Now if m is contained in a strong analytic structure of dimension n, it is in the closure of a strongly separated, hence, uniformly separated sequence S. If also m is in the weak-* closure of W, then $m \in \mathscr{O}$, so m is a limit point of $S \cap \mathscr{O} \subset \mathscr{O} \cap D^n \subset W_r$. So $S \cap \mathscr{O}$ is a uniformly separated sequence contained in a near wedge, and hence by Theorem 5.1 of [**11**], $S \cap \mathscr{O}$ is an interpolating sequence.

5. Related results and remarks. J. P. Rosay [**13**] and M. Range [**12**] have demonstrated the existence of a particular kind of strong n-dimensional analytic structure, which Rosay calls an injective system, in $\mathscr{M}(D^n)$ and $\mathscr{M}(B)$. More specifically, an *injective system* is a pair of maps, $\phi \in \mathscr{B}$ and $G \in \mathscr{A}$, such that $\widehat{f \circ G} \circ \phi = f$ for every f in $H^\infty(\Omega)$. Hence ϕ is a homeomorphism of all of B or D^n onto its image. In both Rosay's and Range's examples, ϕ is actually in $\bar{\mathscr{A}}$, and every point in the range of ϕ is in the closure of some interpolating sequence. This type of analytic structure is indeed special, since, as Hoffman points out [**7**, p. 109], there are one dimensional analytic structures in $\mathscr{M}(D)$ which are not homeomorphic images of D.

In [**4**], using a projection from $(\mathscr{M}(D^n))$ into $(\mathscr{M}(D))^n$, W. Cutrer constructs analytic structures of dimensions 0 through n in $\mathscr{M}(D^n)$. The points contained in n-dimensional analytic structures are all limit points of interpolating sequences. The lower dimensional structures are of particular interest. For most of the examples in [**4**] of analytic structures of dimension less than n, it remains to be shown that the given structures are not merely lower dimensional subsets of analytic structures of higher dimension. More specifically, since Σ is a subset of $(H^\infty(\Omega))^*$, it inherits the norm topology as well as the weak-* topology. If $m \in \Sigma$, the Gleason part of m, $\mathscr{P}(m)$, is the connected component of m in the norm topology of Σ. It is well known that if f is an analytic map from an open set in $\mathbf{C}^k$ into Σ, then $m \in \text{Range}(f)$ implies $\text{Range}(f) \subset \mathscr{P}(m)$. The parts of $\mathscr{M}(D)$ are either points or analytic disks [**7**]. If $\Omega \subset \mathbf{C}^n$, it is an open question under what conditions, if any, $\mathscr{P}(m)$ is the image of an analytic map of a domain in $\mathbf{C}^k$.

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