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MULTIPLICITY-FREE QUOTIENT TENSOR ALGEBRAS

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To Laci Kovács on his 65th birthday

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Abstract

Let V be an infinite-dimensional vector space over a field of characteristic 0. It is well known that the tensor algebra T on V is a completely reducible module for the general linear group G on V. This paper is concerned with those quotient algebras A of T that are at the same time modules for G. A partial solution is given to the problem of determining those A in which no irreducible constituent has multiplicity greater than 1.

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1. Introduction

Attention is drawn in this paper to a simply stated, unsolved problem involving multiplicity-free representations of classical general linear groups. For the proofs of several of the results stated here, the reader is referred to the research report [6].

Let V be a vector space over a field k and $T = T(V)$ the tensor algebra on V. Denote by $E = E(V)$ the monoid formed by the endomorphisms of V under multiplication. Since every element of E extends uniquely to an endomorphism of T, the latter has the structure of a left kE-module. If an ideal J of T is at the same time a kE-submodule, then $A = T/J$ is of course both a quotient algebra, and quotient kE-module, of T : we call A a *quotient tensor algebra*. Familiar examples are the symmetric and exterior algebras on a given vector space and the algebra of $n \times n$ generic matrices over a given field.

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The present paper is concerned with a very special class of quotient tensor algebras. In the first place, it will be assumed throughout that *k has characteristic* 0 *and* V *countably infinite dimension.* Under these conditions, T is a completely reducible kE-module in which each irreducible component has finite multiplicity; and the same is evidently true of any quotient tensor algebra $A = T/J$. We say that A is a *model* for the tensor representations of E if every irreducible component of T has multiplicity *exactly* 1 in A, and that A is *multiplicity-free* if every irreducible component of T has multiplicity *at most* 1 in A.

In the following considerations, V and T are kept fixed. The first result is that there are precisely two models: these are $\mathscr{A}(1)$ and $\mathscr{A}(-1)$ in the general notation introduced below. They have simple presentations. Write $a \circ b = ab + ba$, $[a, b] = ab - ba$. Then $\mathscr{A}(1)$ is the algebra generated by a countably infinite set X subject to the defining relations that $[x \circ y, z] = 0$ for all $x, y, z \in X$, and $\mathscr{A}(-1)$ is defined similarly with $[[x, y], z]$ in place of $[x \circ y, z]$. That $\mathscr{A}(1)$ is a model is proved explicitly in [5] but is already implicit in related results for symmetric groups ([2, 3]). My thanks are due to D.-N. Verma for pointing out that $\mathscr{A}(-1)$ is a model too; his comments and suggestions have been of considerable help to the present investigation.

Let us examine in a preliminary way the conditions for a quotient tensor algebra $A = T/J$ to be multiplicity-free. Let T_n denote the subspace of T generated by the products of n elements of V. Then T_n is a kE-submodule and the isomorphism types of its irreducible constituents are parametrized by the partitions λ of n: let T_λ stand generically for an irreducible of type λ. The crucial observation here is that $T_{(21)}$ has multiplicity 2 in T_3. Therefore, in order that A be multiplicity-free, $J \frown T_3$ must contain a copy of $T_{(21)}$. Now, T_3 contains infinitely many such copies and they are in fact naturally parametrized by the points q on the projective line $\mathscr{P}_1 = k \smile \{\infty\}$: let $W(q)$ be the copy corresponding to q. We define

$$\mathscr{A}(q) = T/\mathscr{J}(q), \tag{1.1}$$

where $\mathscr{J}(q)$ is the ideal of T generated by $W(q)$. It is easy to see that $\mathscr{A}(q)$ is a quotient tensor algebra and our discussion shows that every multiplicity-free quotient tensor algebra is a homomorphic image of some $\mathscr{A}(q)$.

Let $a_\lambda(q)$ denote the multiplicity of T_λ in $\mathscr{A}(q)$. That $\mathscr{A}(1)$ and $\mathscr{A}(-1)$ are the *only* models is proved by showing that

$$a_{(2^2)}(q) = 0 \quad \text{when} \quad q^2 \neq 1. \tag{1.2}$$

Let us now consider the inequality

$$a_\lambda(q) \leq 1 \tag{1.3}$$

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for fixed λ and varying q. Whether (1.3) holds or not for a particular q depends in fact on the rank of a certain matrix whose elements are polynomials in q with rational coefficients. It is easily deduced from this that there exist polynomials

$$f_1(t), \dots, f_r(t) \in \mathbb{Q}[t] \tag{1.4}$$

such that (1.3) holds if, and only if, q is not a common root. But since (1.3) holds when $q = 1$, it follows that at least one of the polynomials (1.4) is non-zero. We conclude that there are only finitely many q for which (1.3) does *not* hold and that these exceptional q (if any) are all algebraic over $\mathbb{Q}$. An immediate corollary is that *$\mathscr{A}(q)$ is multiplicity-free whenever q is transcendental over* $\mathbb{Q}$. A refinement of this argument is used in Section 5.2 to show that the value of $a_\lambda(q)$ for transcendental q is 1 when λ has any of the forms (n), $(n-1, 1)$, (1^n), $(2, 1^{n-2})$ and 0 otherwise.

There is a striking contrast between the behaviour of the $\mathscr{A}(q)$ *as algebras* and their behaviour *as modules*. It is proved in Section 3 that quotient tensor algebras T/J, T/K are isomorphic as algebras only if $J = K$, from which it follows that $\mathscr{A}(q)$, $\mathscr{A}(q')$ are isomorphic as algebras only if $q = q'$. On the other hand, the results cited above show that $\mathscr{A}(q)$, $\mathscr{A}(q')$ are certainly isomorphic as modules whenever both q and q' are transcendental over $\mathbb{Q}$.

The question left unresolved here is whether *all $\mathscr{A}(q)$ are multiplicity-free*. There is some positive evidence. Let λ, μ be partitions of n. It is proved in Section 5.1 that (1.3) holds when either λ or its conjugate partition λ' has at most 2 parts. A direct calculation in [6] shows also that (1.3) holds when $n \le 6$. A further result proved in [6] is that, for each μ,

$$\sum_\lambda K_{\lambda\mu}(1 - a_\lambda(q)) \ge 0, \tag{1.5}$$

where the $K_{\lambda\mu}$ are the Kostka numbers (see [4]). The special case $\mu = (1^n)$ gives

$$\sum_\lambda f_\lambda(1 - a_\lambda(q)) \ge 0, \tag{1.6}$$

where f_λ is the degree of the irreducible representation of the symmetric group S_n corresponding to λ. In essence, these results are proved by reducing the elements of $\mathscr{A}(q)$ to a (not necessarily unique) normal form. One might hope to prove that $\mathscr{A}(q)$ is multiplicity-free (if this is true!) by a refinement of the method.

Section 2 is preliminary. Section 3 deals with general properties of quotient tensor algebras. The particular algebras $\mathscr{A}(q)$ are introduced in Section 4 and their properties derived in Section 5.

2. Preliminaries

Some basic results and notation are set down for reference. Definitions and assumptions already made are for the most part not repeated here.

2.1. Grading We call T_n the *nth homogeneous component* of T and say that its elements are *homogeneous of degree n*. The T_n provide a grading of the algebra T. A subspace K of T is *graded* if $K = \oplus_n K_n$, where $K_n = K \frown T_n$. A quotient $Q = K/K'$, where K, K' are graded, inherits the grading in the obvious way, and Q_n denotes its nth homogeneous component.

The choice of a basis

$$X = \{x_1, x_2, \dots\} \tag{2.1}$$

of V determines a *multigrading* of T: a *multi-index* is an infinite row $\boldsymbol{n} = (n_1, n_2, \dots)$ of integers $n_i \geq 0$ with finite sum $|\boldsymbol{n}| = \sum n_i$; the corresponding *multihomogeneous component* $T_{\boldsymbol{n}}$ is the subspace of T generated by those products of basis elements that have degree n_i in x_i for all i. Multigraded subspaces and quotient spaces are defined in the expected way, and $Q_{\boldsymbol{n}}$ denotes the $\boldsymbol{n}$th component of Q. It will sometimes be convenient to refer to $Q_{\boldsymbol{n}}$ as the $(x_1^{n_1} x_2^{n_2} \cdots)$-*component* of Q.

2.2. Module structure of T The T_n are kE-submodules and the assumption that k is infinite ensures that every submodule of T is both graded and multigraded. We need therefore only look at the individual T_n. Let ${}_nM$ denote the $(x_1 \cdots x_n)$-component of T. It has basis elements

$$x_\sigma = x_{\sigma 1} \cdots x_{\sigma n} (\sigma \in S_n), \tag{2.2}$$

where S_n is the symmetric group on $\{1, \dots, n\}$. The natural definition $\sigma x_\tau = x_{\sigma\tau} (\sigma, \tau \in S_n)$ turns ${}_nM$ into a left kS_n-module, identified with the left regular module kS_n by the isomorphism

$$\iota_n : \sum a_\sigma \sigma \mapsto \sum a_\sigma x_\sigma. \tag{2.3}$$

Thus, the decomposition of ${}_nM$ into its irreducible constituents has the form

$${}_nM \sim \sum_{|\lambda|=n} f_\lambda M_\lambda, \tag{2.4}$$

where summation is over the partitions λ of n and where the multiplicity f_λ of the irreducible constituent M_λ is its dimension as vector space.

By a *section* of a module, we shall mean a quotient module of a submodule of that module. There are natural ways of passing between sections of the kE-module T_n

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and sections of the kS_n-module ${}_nM$. If Q is a section of T_n, its $(x_1 \cdots x_n)$-component is naturally identified with a section of ${}_nM$: the latter is the *multilinear restriction* of Q. If L/L' is a section of ${}_nM$, then $(kE)L/(kE)L'$ is a section of T_n called the *extension* of L/L'. The basic result is that extension and multilinear restriction are mutually inverse, isomorphism-preserving bijections between the set of all sections of the kS_n-module ${}_nM$ and the set of all sections of the kE-module T_n. It follows that T_n is a completely reducible kE-module whose decomposition into irreducible constituents has the form

$$T_n \sim \sum_{|\lambda|=n} f_\lambda T_\lambda. \tag{2.5}$$

Let R be the multilinear restriction of a section Q of T_n. Then

$$Q \sim \sum_{|\lambda|=n} h_\lambda T_\lambda, \quad R \sim \sum_{|\lambda|=n} h_\lambda M_\lambda$$

with certain common multiplicities $h_\lambda \leq f_\lambda$. For convenience of notation, we introduce the symbol $\sum_{|\lambda|=n} h_\lambda \lambda$ as the *type* of both Q and R. The definition is extended to arbitrary kE-submodules and sections of T in the obvious way: for example, T itself has type $\sum_\lambda f_\lambda \lambda$, where summation is over all partitions λ of all integers $n \geq 0$. Analogous notation will be used for the $kE(r)$-modules considered in the next section.

It will sometimes be convenient to replace the multilinear restriction R of Q by the corresponding section $R' = \iota_n^{-1}(R)$ of the kS_n-module kS_n. We call R' the *S-restriction* of Q and Q the *extension* of R' (as well as of R).

The *Weyl module* W_λ corresponding to a partition λ of n is the particular irreducible kE-submodule of T_n of type λ defined as follows. If $z_1, \ldots, z_r$ are elements of an associative algebra, write

$$\Delta(z_1, \ldots, z_r) = \sum_{\sigma \in S_r} (\operatorname{sgn} \sigma) z_{\sigma 1} \cdots z_{\sigma r}. \tag{2.6}$$

Then W_λ is the kE-module generated by the element

$$\Delta_\lambda(x_1, \ldots, x_n) = \Delta(x_1, \ldots, x_{\mu_1}) \Delta(x_1, \ldots, x_{\mu_2}) \cdots, \tag{2.7}$$

where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s > 0$ are the parts of the conjugate partition $\mu = \lambda'$.

2.3. Module structure of $T(r)$ For each integer $r > 0$, let $V(r)$ denote the subspace, $T(r)$ the subalgebra, of T generated by $X(r) = \{x_1, \ldots, x_r\}$. The considerations of Section 2.1 carry over immediately to $T(r)$: we may identify it with the tensor algebra on $V(r)$ and its intrinsic grading and multigrading coincide with those induced by T.

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The endomorphisms of $V(r)$ form a multiplicative monoid $E(r)$, and $T(r)$ is a left $kE(r)$-module. As before, we could confine attention to the homogeneous components $T_n(r)$ but it is just as easy here to deal directly with $T(r)$ itself.

There are natural ways of passing between the sections of the kE-module T and the sections of the $kE(r)$-module $T(r)$. If Q is a section of T, let $Q(r)$ denote the sum of the multihomogeneous components of Q corresponding to multi-indices of the form $(n_1, \dots, n_r, 0, 0, \dots)$. The *r-variable restriction* of Q is the section of $T(r)$ naturally identified with $Q(r)$. If L/L' is a section of $T(r)$, then its *extension* is the section $(kE)L/(kE)L'$ of T.

We denote by $T^{(r)}$ the unique kE-submodule of T of type $\sum_{\ell(\lambda)\leq r} f_\lambda \lambda$, summation being over all partition λ into at most r parts. The basic result is that extension and r-variable restriction provide mutually inverse, isomorphism-preserving bijections between the set of all sections of the $kE(r)$-module $T(r)$ and the set of all sections of the kE-module $T^{(r)}$. Moreover, if λ has more than r parts, the r-variable restriction of a section of T of type λ is zero. It follows that, $T(r)$ is a completely reducible $kE(r)$-module and its decomposition into irreducible constituents has the form

$$T(r) \sim \sum_{\ell(\lambda)\leq r} f_\lambda T_\lambda(r). \tag{2.8}$$

By the above, if λ is a partition of n into at most r parts, then the element (2.7) generates an irreducible $kE(r)$-module $W_\lambda(r)$ of type λ.

3. General properties of quotient tensor algebras

We investigate various relations between two quotient tensor algebras

$$A = T/J, \quad B = T/K. \tag{3.1}$$

THEOREM 3.1. *The algebra B is a homomorphic image of the algebra A if, and only if, $J \subseteq K$.*

PROOF. We need only prove that the condition is necessary. Suppose then that $\theta : A \to B$ is a surjective homomorphism of algebras. Since J, K are kE-submodules and J is generated as a kE-module by its multilinear elements, it will be sufficient to show that every multilinear element $g(x_1, \dots, x_n)$ of J is in K. We shall assume in the proof that neither J nor K contains V, the proof being rather easy otherwise. This assumption ensures that the vector spaces A_1 and B_1 are both isomorphic to V and hence infinite-dimensional.

In order to show that $g(x_1, \dots, x_n) \in K$, it is enough to show that there are linearly independent elements $b_1, \dots, b_n$ of B_1 such that $g(b_1, \dots, b_n) = 0$. Indeed,

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let $w_1, \ldots, w_n$ be elements of V such that $b_i = w_i + K$. Then the w_i are linearly independent and $g(w_1, \ldots, w_n) \in K$. Since K is a kE-submodule and the full linear subgroup of E permutes the sequences of n linearly independent elements of V transitively, we have $g(x_1, \ldots, x_n) \in K$, as required.

Since θ is k-linear, there exist k-linear mappings $\theta_0 : A \to k, \theta_1 : A \to B_1$ such that

$$\theta(u) \equiv \theta_0(u)1_B + \theta_1(u) \pmod{\bar{B}} \quad \text{if } u \in A,$$

where $\bar{B} = \sum_{i \geq 2} B_i$. Similarly, if $\phi : A_1 \to B$ is the restriction of θ to A_1, there exist k-linear mappings $\phi_0 : A_1 \to k, \phi_1 : A_1 \to B_1$ such that

$$\phi(a) \equiv \phi_0(a)1_B + \phi_1(a) \pmod{\bar{B}} \quad \text{if } a \in A_1.$$

Now, $u \in A$ is a k-linear combination of products $aa'a'' \cdots$ with $a, a', a'', \ldots \in A_1$ and we have $\theta(aa'a'' \cdots) = \phi(a)\phi(a')\phi(a'') \cdots$. A simple calculation shows that $\theta_1(u) \in \operatorname{im} \phi_1$, whence $\operatorname{im} \phi_1 = \operatorname{im} \theta_1$. But θ_1 is surjective because θ is and so ϕ_1 is surjective too.

In particular, $\operatorname{im} \phi_1$ is infinite-dimensional. Evidently $\ker \phi_0$ has codimension 0 or 1 in A, and so $\phi_1(\ker \phi_0)$ is also infinite-dimensional. Hence there exist $a_1, \ldots, a_n \in A_1$ such that

$$\theta(a_i) = b_i + c_i \quad (i = 1, \ldots, n),$$

where $b_1, \ldots, b_n$ are linearly independent elements of B_1 and $c_1, \ldots, c_n \in \bar{B}$. We show that $g(b_1, \ldots, b_n) = 0$ with these b_i.

We have $a_i = v_i + J$ $(i = 1, \ldots, n)$ with certain $v_i \in V$. Since $g(x_1, \ldots, x_n) \in J$ and J is a kE-module, we have $g(v_1, \ldots, v_n) \in J$ and thus $g(a_1, \ldots, a_n) = 0$. Therefore, since θ is a homomorphism,

$$0 = g(\theta(a_1), \ldots, \theta(a_n)) = g(b_1 + c_1, \ldots, b_n + c_n).$$

However, since $g(x_1, \ldots, x_n)$ is multilinear, $g(b_1, \ldots, b_n)$ is the homogeneous component of $g(b_1 + c_1, \ldots, b_n + c_n)$ of degree n; and since B is of course graded we conclude that $g(b_1, \ldots, b_n) = 0$, as required. □

COROLLARY 3.2. *The algebras A and B are isomorphic only if $J = K$.*

The corollary justifies our calling J the *kernel* of A.

COROLLARY 3.3. *The algebras A and B are anti-isomorphic if and only if $K = \alpha(J)$, where α is the principal anti-automorphism of T.*

The principal anti-automorphism of T is by definition the unique anti-automorphism of T that fixes the elements of V. The corollary follows at once from the previous one and the existence of α. When $K = \alpha(J)$, we shall call B the *opposite* of A.

THEOREM 3.4. *Opposite algebras A, B are isomorphic as kE-modules.*

PROOF. Since A, B are opposites, we have $K = \alpha(J)$ and hence, for each n, $K'_n = \alpha(J'_n)$, where K'_n, J'_n are the multilinear restrictions of K_n, J_n. But α maps $x_\sigma = x_{\sigma 1} \cdots x_{\sigma n}$ to $x_{\sigma n} \cdots x_{\sigma 1} = x_{\sigma \tau_n}$, where τ_n is the involution $(1, n)(2, n-1) \cdots \in S_n$. Hence K'_n, J'_n are isomorphic kS_n-modules and so K_n, J_n are isomorphic kE-modules. The theorem follows. □

Let $P = \oplus_n P_n$, $Q = \oplus_n Q_n$ be kE-submodules of T. Let P''_n, Q''_n be the S-restrictions of P_n, Q_n (see Section 2.2). We call Q the *conjugate* of P, and write $Q = \omega(P)$, if, for all n, $Q''_n = \omega_n(P''_n)$, where ω_n is the sign automorphism of kS_n. If P has type $\sum a_\lambda \lambda$, then $\omega(P)$ evidently has the *conjugate type* $\sum a_\lambda \lambda'$.

We define the quotient tensor algebras A, B to be *conjugate* when their kernels J, K are conjugate. The following result is obvious from our discussion.

PROPOSITION 3.5. *Conjugate quotient tensor algebras have conjugate types.*

PROPOSITION 3.6. *The operations of forming the conjugate and the opposite of a quotient tensor algebra commute.*

PROOF. This follows from the fact that the sign automorphism ω_n and the operation $R(\tau_n)$ of right multiplication by the involution $\tau_n = (1, n)(2, n-1) \cdots$ commute up to sign: $\omega_n R(\tau_n) = (\operatorname{sgn} \tau_n) R(\tau_n) \omega_n$. □

THEOREM 3.7. *Conjugate kE-submodules G, H of T generate conjugate ideals P, Q of T.*

We omit the straightforward proof.

4. Definition of the $\mathscr{A}(q)$

Before giving the definition, we require some preliminary information about the left ideals of type (21) in kS_3. It will make things clearer to begin with some rather more general results in kS_n.

4.1. Blocks of kS_n The *blocks* of kS_n are, by definition, its minimal two-sided ideals. They are parametrized by the partitions of n, the block B_λ corresponding to λ being the unique left ideal of type $f_\lambda\lambda$. As an algebra, it is isomorphic to the algebra of $f_\lambda \times f_\lambda$ matrices over k. We shall first describe B_λ explicitly when $\lambda = (n)$ or $(n-1, 1)$ $(n \geq 2)$ and then transfer the results to the cases of the conjugate partitions (1^n) and (21^{n-2}).

The identity representation of S_n of degree 1 provides a kS_n-module of type (n) : $B_{(n)}$ is obviously the 1-dimensional subspace of kS_n with generator

$$\Omega = \sum_{\sigma \in S_n} \sigma. \tag{4.1}$$

The permutation representation of S_n on the left cosets of S_{n-1} provides a kS_n-module of type $(n) + (n-1, 1)$. Hence $f_{(n-1,1)} = n-1$ and so $\dim B_{(n-1,1)} = (n-1)^2$. Let $\Delta_{ij} = \sum_{\sigma \in D_{ij}} \sigma$ $(i, j = 1, \dots, n)$, where $D_{ij} = \{\sigma \in S_n | \sigma j = i\}$. It is straightforward to verify that the elements

$$\Delta_{ij} - \frac{1}{n}\Omega \quad (i, j = 1, \dots, n-1) \tag{4.2}$$

form a basis of $B_{(n-1,1)}$.

Let $\mathscr{L}_\lambda$ denote the set of all left ideals of kS_n of type λ, that is, the set of all minimal left ideals of the algebra B_λ. Choose any minimal *right* ideal $R_\circ$ of B_λ. Then

$$L \mapsto L \frown R_\circ \tag{4.3}$$

gives a bijection of $\mathscr{L}_\lambda$ onto the set of all 1-dimensional subspaces of $R_\circ$. Thus, the elements of $\mathscr{L}_\lambda$ are parametrized by the points of the projective space $\mathscr{P}(R_\circ)$ corresponding to $R_\circ$. The mapping (4.3) has the additional functorial property that the kS_n-module homomorphisms $L \to L'$ restrict to the linear mappings $L \frown R_\circ \to L' \frown R_\circ$.

Let us consider now the special case $\lambda = (n-1, 1)$. The subspace P_i of kS_n with the elements Δ_{ij} $(j = 1, \dots, n)$ as basis is a right ideal of type $(n) + (n-1, 1)$, and its component of type $(n-1, 1)$ is the subspace Q_i of elements

$$\sum_j \lambda_j \Delta_{ij} \quad \text{with} \quad \sum_j \lambda_j = 0. \tag{4.4}$$

A left ideal of type $(n-1, 1)$ intersects Q_i in the subspace generated by a nonzero element (4.4): it is therefore represented by the homogeneous coordinates $[\lambda_1, \dots, \lambda_n]$ (in the usual sense of projective geometry) of a point on the hyperplane $\sum \lambda_j = 0$ in the projective space $\mathscr{P}_{n-1}$ over k. Notice that, since $\sigma\left(\sum \lambda_j \Delta_{ij}\right) = \sum \lambda_j \Delta_{\sigma i, j}$ when $\sigma \in S_n$, this parametrization is independent of the choice of i.

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The sign automorphism ω_n of kS_n maps B_λ to $B_{\lambda'}$, where λ' is the partition conjugate to λ. It follows that $B_{(1^n)}$ has the single basis element

$$\Omega' = \omega_n(\Omega) \tag{4.5}$$

and $B_{(21^{n-2})}$ the $(n-1)^2$ basis elements

$$\Delta'_{ij} - \frac{1}{n}\Omega' \quad (i, j = 1, \dots, n-1), \tag{4.6}$$

where $\Delta'_{ij} = \omega_n(\Delta_{ij})$. The elements

$$\sum_j \mu_j \Delta'_{ij} \quad \text{with} \quad \sum_j \mu_j = 0 \tag{4.7}$$

form a right ideal Q'_i of type (21^{n-2}). A left ideal of type (21^{n-2}) intersects Q'_i in the subspace generated by a nonzero element (4.7), and we assign to it the homogeneous coordinates $[\mu_1, \dots, \mu_n]$.

4.2. The case $n = 3$ We have assigned homogeneous coordinates to the kE-submodules of T_n of types $(n-1, 1)$ and $(n-1, 1)' = (21^{n-2})$. Here we examine the special case $n = 3$, where the two types coincide.

Let us first consider (21) as a special case of $(n-1, 1)$. Let W be a kE-submodule of T_3 of type (21) and let W' be its S-restriction (see Section 2.2). If W' has homogeneous coordinates

$$[\lambda_1, \lambda_2, \lambda_3] \quad (\lambda_1 + \lambda_2 + \lambda_3 = 0), \tag{4.8}$$

then it is generated as a kS_3-module by any one of the 3 elements

$$\Theta_i = \sum_{j=1}^{3} \lambda_j \Delta_{ij} \quad (i = 1, 2, 3), \tag{4.9}$$

and as a vector space by any two of them. Hence W is generated as a kE-module by the element

$$\iota_3(\Theta_3) = \lambda_1(x_3x_1x_2 + x_3x_2x_1) + \lambda_2(x_2x_3x_1 + x_1x_3x_2) + \lambda_3(x_1x_2x_3 + x_2x_1x_3). \tag{4.10}$$

It will usually be more convenient to designate W by the affine parameter

$$q = -\lambda_3/\lambda_1, \tag{4.11}$$

so that the homogeneous coordinates become

$$[1, q-1, -q] \quad (q \in k) \tag{4.12}$$

(or $[0, 1, -1]$ when $q = \infty$). Accordingly, $W = W(q)$ is generated as a kE-module by

$$\Phi_q(x_1, x_2, x_3) = (x_3x_1x_2 + x_3x_2x_1) + (q-1)(x_2x_3x_1 + x_1x_3x_2) - q(x_1x_2x_3 + x_2x_1x_3) \tag{4.13}$$

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(or $\Phi_\infty(x_1, x_2, x_3) = (x_2x_3x_1 + x_1x_3x_2) - (x_1x_2x_3 + x_2x_1x_3)$ when $q = \infty$).

Let us next consider (21) as a special case of (21^{n-2}). If W has homogeneous coordinates

(4.14) $$[\mu_1, \mu_2, \mu_3] \quad (\mu_1 + \mu_2 + \mu_3 = 0)$$

in this sense, then the elements corresponding to the elements (4.9) are

(4.15) $$\Theta'_i = \sum_{j=1}^{3} \mu_j \Delta'_{ij} \quad (i = 1, 2, 3).$$

Thus, in terms of the affine parameter

(4.16) $$q' = -\mu_3/\mu_1,$$

W is generated as a kE-module by

(4.17) $$\Phi'_{q'}(x_1, x_2, x_3) = (x_3x_1x_2 - x_3x_2x_1) + (q' - 1)(x_2x_3x_1 - x_1x_3x_2) - q'(x_1x_2x_3 - x_2x_1x_3)$$

(or by $\Phi'_\infty(x_1, x_2, x_3) = (x_2x_3x_1 - x_1x_3x_2) - (x_1x_2x_3 - x_2x_1x_3)$ when $q' = \infty$).

The two sets of homogeneous coordinates are related as follows: since

$$\Theta_1 - \Theta_2 = (\lambda_2 - \lambda_3)\Delta'_{31} + (\lambda_3 - \lambda_1)\Delta'_{32} + (\lambda_1 - \lambda_2)\Delta'_{33},$$

we have

(4.18) $$[\mu_1, \mu_2, \mu_3] = [\lambda_2 - \lambda_3, \lambda_3 - \lambda_1, \lambda_1 - \lambda_2].$$

Thus, the parameters q, q' are related by the projective involution

(4.19) $$2(qq' + 1) = q + q'.$$

DEFINITION. We define the quotient tensor algebra $\mathscr{A}(q)$ to be

(4.20) $$\mathscr{A}(q) = T/\mathscr{J}(q),$$

where $\mathscr{J}(q)$ is the ideal of T generated by $W(q)$.

NOTATION. Let

(4.21) $$\mathscr{A}(q) \sim \sum a_\lambda(q) T_\lambda,$$

(4.22) $$\mathscr{J}(q) \sim \sum b_\lambda(q) T_\lambda,$$

where, in view of (4.20),

(4.23) $$a_\lambda(q) + b_\lambda(q) = f_\lambda.$$

EXAMPLES. (1) The generator (4.13) of $W(1)$ is $-[x_1 \circ x_2, x_3]$, and the generator (4.17) of $W(-1)$ is $-[[x_1, x_2], x_3]$. Thus, $\mathscr{A}(1)$ and $\mathscr{A}(-1)$ are as described in Section 1.

(2) The sum of the submodules W corresponding to two different parameter values is the unique submodule of type 2(21). If $\mathscr{J}$ is the ideal of T generated by the latter, then $\mathscr{A} = T/\mathscr{J}$ is a common homomorphic image of all $\mathscr{A}(q)$. It is easy to determine its structure. Indeed, it follows from the form of the two generators in example (1) that $\mathscr{A}$ is generated by a countably infinite set X subject to the defining relations that the product of any two elements of X is in the centre of $\mathscr{A}$. Another way of expressing the defining relations is to say that a product of any number, m, of generators is unaltered by an even permutation of the m factors. It follows easily from this that the homogeneous component $\mathscr{A}_n$ is of type $(n) + (1^n)$ when $n \geq 2$.

(3) The result just proved, together with (4.23), shows that

$$a_{(n)}(q) = a_{(1^n)}(q) = 1 \tag{4.24}$$

for all n and q.

4.3. Relations between different $\mathscr{A}(q)$ Applying the general results of Section 3 to the $\mathscr{A}(q)$, we get

PROPOSITION 4.1. *$\mathscr{A}(q)$ and $\mathscr{A}(Q)$ are*

(a) *isomorphic as algebras if and only if $q = Q$,*
(b) *opposites if and only if $qQ = 1$, and*
(c) *conjugates if and only if $2(qQ + 1) = q + Q$.*

PROOF. (a) is obvious from Corollary 3.2. In proving the other parts we use the fact that the kernel $\mathscr{J}(q)$ of $\mathscr{A}(q)$ is generated as an ideal by $W(q) = \mathscr{J}_3(q)$. Thus, in order to prove (b) we must show that $\alpha(W(q)) = W(q^{-1})$, where α is the principal antiautomorphism of T. But $W(q)$ is generated as a kE-module by the element Φ_q in (4.13) and we see by inspection that $\alpha(\Phi_q) = -q\Phi_{q^{-1}} (q \neq 0)$, $\alpha(\Phi_0) = -\Phi_\infty$. This proves (b).

By Theorem 3.7, the conjugate of $\mathscr{J}(q)$ is generated by the conjugate of $W(q)$. Thus, in order to prove (c) we must show that $\omega_3(W(q)) = W(Q)$, where $2(qQ + 1) = q + Q$. Now, by its definition, $\omega_3(W(q))$ is the kE-submodule generated by $\Phi'_q(x_1, x_2, x_3)$, where the notation is as in (4.17). Then (4.19) shows that this submodule is $W(Q)$ as claimed. □

COROLLARY 4.2. *The kE-modules $\mathscr{A}(q)$, $\mathscr{A}(Q)$ have the same type if $qQ = 1$, conjugate types if $2(qQ + 1) = q + Q$.*

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REMARKS. (1) In accordance with Proposition 3.6, the relations between q, Q in (b), (c) of Proposition 4.1 define *commuting* involutions on the projective line.

(2) $\mathscr{A}(1)$ and $\mathscr{A}(-1)$ are conjugates. Hence the proof that either one is a model implies that the other is too.

5. The main results for the $\mathscr{A}(q)$

5.1. Evaluating $a_\lambda(q)$ for small $\ell(\lambda)$ It has been mentioned several times that

$$a_\lambda(1) = a_\lambda(-1) = 1 \tag{5.1}$$

for every partition λ. Suppose now that λ is a partition of n into 1 or 2 parts:

$$\lambda = (n-r, r) \quad (0 \le r \le n/2). \tag{5.2}$$

We shall prove here that, if $q^2 \neq 1$,

$$a_\lambda(q) = \begin{cases} 1 & r < 2, \\ 0 & r \ge 2. \end{cases} \tag{5.3}$$

There are two immediate corollaries. First, $\mathscr{A}(1)$ and $\mathscr{A}(-1)$ are the only quotient tensor algebras that provide a model for the tensor representations of E. Second, for all partitions λ of n into 1 or 2 parts and all q,

$$a_{\lambda'}(q) = a_\lambda(q), \tag{5.4}$$

where λ' denotes, as usual, the conjugate of λ. This follows from the results above and Corollary 4.2.

Let $\mathscr{A}(2, q)$ denote the subalgebra of $\mathscr{A}(q)$ generated by the canonical images x, y of x_1, x_2. Then, by Section 2.3,

$$\mathscr{A}(2, q) \sim \sum_{\ell(\lambda) \le 2} a_\lambda(q) T_\lambda(2). \tag{5.5}$$

Thus, our task here is to determine the multiplicities in (5.5). *We shall exclude the case $q = \infty$ from the main argument*, returning to it at the end.

It is easily seen from the considerations in Section 4.2 that $\mathscr{A}(2, q)$ is the algebra with generators x, y and *defining* relations

$$f(x, y) = f(y, x) = 0, \tag{5.6}$$

where

$$f(x, y) = [y, x]x + qx[y, x]. \tag{5.7}$$

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It is convenient to rewrite (5.6) as

(5.8) $$zx = -qxz, \quad zy = -qyz$$

where $z = [y, x]$.

We shall now prove that the nth homogeneous component $\mathscr{A}_n(2, q)$ is generated as a vector space by the elements

(5.9) $$\langle x^l y^m \rangle z^r (l, m, r \geq 0, l + m + 2r = n),$$

where $\langle x^l y^m \rangle$ denotes the sum of the $\binom{l+m}{l}$ formally distinct monomials in x, y having respective partial degrees l, m (the latter will be referred to as (l, m)-*monomials*). The result being trivial for $n \leq 2$, we assume that $n \geq 3$. Let a, b be nonnegative integers with sum n. We will show that a given (a, b)-monomial p can be expressed as a linear combination of the elements (5.9).

The proof rests on the simple observation that the difference of two (a, b)-monomials p, p' can be expressed as a linear combination of terms uzv, where uv is an $(a-1, b-1)$-monomial. Indeed, we have $p = p_1 \cdots p_n$, where each p_i is x or y, and $p' = p_{\sigma 1} \cdots p_{\sigma n}$ for some permutation $\sigma \in S_n$. If σ is an adjacent transposition, our assertion is obvious, and it follows in general because every σ is a product of adjacent transpositions.

Since $\langle x^a y^b \rangle$ is the sum of $\binom{n}{a}$ (a, b)-monomials, it follows that

$$p - \binom{n}{a}^{-1} \langle x^a y^b \rangle$$

is likewise a linear combination of such terms uzv. But the defining relations (5.8) show that uzv is a scalar multiple of uvz, and so p is a linear combination of $\langle x^a y^b \rangle$ and terms wz, where w is an $(a-1, b-1)$-monomial. Our result now follows in an obvious way by induction on n.

The next step is to prove that, for all q,

(5.10) $$x^n \neq 0 \quad (n \geq 0),$$

(5.11) $$x^{n-2} z \neq 0 \quad (n \geq 2),$$

and that, if $q^2 \neq 1$,

(5.12) $$z^2 = 0.$$

Since (5.10) and (5.11) are obvious when $n \leq 2$, we shall assume when proving them that $n \geq 3$. It is essential here to note that $\mathscr{A}(2, q)$ is multigraded. The multihomogeneous component spanned by the (a, b)-monomials will be called its (a, b)-*component*.

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The $(n, 0)$-component is spanned by the single element x^n. The defining relations (5.8) obviously impose no linear relation on x^n and so (5.10) holds.

The $(n-1, 1)$-component—call it D—is spanned by the n formally different $(n-1, 1)$-monomials, on which the defining relations (5.6) impose the $n-2$ linear relations

$$x^c f(x, y) x^d = 0 \quad (c, d \geq 0, c + d = n - 3).$$

Hence $\dim D \geq 2$. On the other hand, since the elements (5.9) span $\mathscr{A}(n, q)$, the elements $\langle x^{n-1} y \rangle$ and $x^{n-2} z$ span D and so $\dim D \leq 2$. Hence the two elements form a basis of D and in particular (5.11) holds.

Finally, the relations (5.8) give $zxy = q^2 xyz$ and $zyx = q^2 yxz$, whence $z^2 = q^2 z^2$; therefore $z^2 = 0$ if $q^2 \neq 1$.

In order to prove (5.3), we have now only to interpret the results already proved in terms of modules. Let λ be the partition (5.2). Now the $kE(2)$-submodule $\mathscr{T}_\lambda(2)$ of $T(2)$ generated by $x_1^{n-2r}(x_2 x_1 - x_1 x_2)^r$ is irreducible of type λ and the $n - 2r + 1$ elements

$$\langle x_1^l x_2^m \rangle (x_2 x_1 - x_1 x_2)^r \quad (l, m \geq 0, l + m = n - 2r)$$

form a basis ($\mathscr{T}_\lambda(2)$ is the image of $W_\lambda(2)$ (see Section 2.3) under the principal antiautomorphism). Let $\mathscr{T}'_\lambda(2)$ denote the canonical image of $\mathscr{T}_\lambda(2)$ in $\mathscr{A}(2, q)$. Clearly, $\mathscr{T}'_\lambda(2)$ is the $kE(2)$-submodule of $\mathscr{A}_n(2, q)$ generated by $x^{n-2r} z^r$ and it is spanned by the elements

$$\langle x^l y^m \rangle z^r \quad (l, m \geq 0, \ l + m = n - 2r). \tag{5.13}$$

Since $\mathscr{T}_\lambda(2)$ is irreducible, *either* $x^{n-2r} z^r = 0$ *or* $\mathscr{T}'_\lambda(2) \cong \mathscr{T}_\lambda(2)$ and the elements (5.13) form a basis of $\mathscr{T}'_\lambda(2)$.

Now, since $\mathscr{A}_n(2, q)$ is spanned by the elements (5.9), it is the sum of the $\mathscr{T}'_\lambda(2)$ above; and since the $\mathscr{T}_\lambda(2)$ are pairwise non-isomorphic, this sum is direct. If $q^2 \neq 1$, then, by (5.10) - (5.12) and the discussion above, $\mathscr{T}'_\lambda(2) \cong \mathscr{T}_\lambda(2)$ when $r \leq 1$ and $\mathscr{T}'_\lambda(2) = \{0\}$ otherwise. Hence we have, in this case,

$$\mathscr{A}_n(2, q) = \mathscr{T}'_{(n)}(2) \oplus \mathscr{T}'_{(n-1,1)}(2) \cong \mathscr{T}_{(n)}(2) \oplus \mathscr{T}_{(n-1,1)}(2),$$

thus proving (5.3).

It remains only to deal with the omitted case $q = \infty$. Here the defining relations (5.8) must be replaced by $xz = yz = 0$ and it is no longer true that the elements (5.9) span $\mathscr{A}_n(2, \infty)$. There is a simple remedy: since the algebras $\mathscr{A}(2, \infty)$ and $\mathscr{A}(2, 0)$ are anti-isomorphic by Proposition 4.1, the reversed elements $z^r \langle x^l y^m \rangle$ span $\mathscr{A}_n(2, \infty)$ and so (5.3) holds as before. (Alternatively, $\mathscr{A}(2, \infty)$ and $\mathscr{A}(2, 0)$ are isomorphic as $kE(2)$-modules by Theorem 3.4.)

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5.2. The generic multiplicities $a_\lambda(\cdot)$ Throughout this section, λ is a fixed partition of the integer n. Write

$$a_\lambda(\cdot) = \min_q a_\lambda(q) \tag{5.14}$$

Since $a_\lambda(1) = 1$, $a_\lambda(\cdot)$ is either 0 or 1. The following result justifies the name *generic* for $a_\lambda(\cdot)$.

PROPOSITION 5.1. *$a_\lambda(q) > a_\lambda(\cdot)$ for at most f_λ values of q. Each of these exceptional values is either ∞ or algebraic over $\mathbb{Q}$.*

PROOF. Let u, v be the $-\lambda_3$, λ_1 in (4.11), so that $q = u/v$. Consider the S-restriction of $\mathscr{J}_n(q)$ (see Section 2.2 and (4.20)–(4.23)). It is a left ideal of kS_n of type $\sum_\rho b_\rho(q)\rho$ and is generated as a vector space by finitely many, say N, elements of the form $fu + gv$, where $f, g \in \mathbb{Q}S_n$. If ϵ_λ is a primitive idempotent of $\mathbb{Q}S_n$ of type λ, then the coefficient $b_\lambda(q)$ is the dimension of the intersection of the above S-restriction with the primitive right ideal $\epsilon_\lambda(kS_n)$. This intersection is generated as a vector space by the N elements $\epsilon_\lambda(fu + gv)$.

We now have all that is necessary to prove the proposition. Since $\dim \epsilon_\lambda(kS_n) = f_\lambda$, $b_\lambda(q)$ is the rank of a certain $f_\lambda \times N$ matrix of the form $X(u, v) = uY + vZ$, where the entries of Y, Z are in $\mathbb{Q}$. Let r be the rank of $X(s, t)$, where s, t are independent indeterminates over $\mathbb{Q}$, and let $F(s, t)$ be a nonzero $r \times r$ determinantal minor of $X(s, t)$. Then $b_\lambda(q) \le r$ with strict inequality only when the ratio $u : v$ is a solution of the homogeneous equation $F(u, v) = 0$ of degree $r \le f_\lambda$. When translated into terms of the coefficients $a_\lambda(q) = f_\lambda - b_\lambda(q)$, this gives the proposition. □

An immediate corollary is that $a_\lambda(q) = a_\lambda(\cdot)$ whenever q is transcendental over $\mathbb{Q}$. The remainder of this section is devoted to calculating $a_\lambda(\cdot)$.

Call λ *exceptional* if it is one of (n), $(n-1, 1)$, (1^n), (21^{n-2}) and *general* otherwise. The results of Section 5.1 show that $a_\lambda(\cdot) = 1$ when λ is exceptional. We prove here that

$$a_\lambda(\cdot) = 0 \quad \text{if } \lambda \text{ is general.} \tag{5.15}$$

A property of $\mathscr{A}(q)$ will be said to hold for *almost all* q if it holds for all q apart from finitely many exceptions. Equation (5.15) will be proved in the obviously equivalent form that, when λ is general,

$$a_\lambda(q) = 0 \quad \text{for almost all } q. \tag{5.16}$$

The following special result is required at one point in the proof. It is proved in [6] by a direct calculation.

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LEMMA 5.2. $a_{(31^2)}(q) = 0$ *if* $q^2 \neq 1$.

NOTATION. Let the parts of λ be $\lambda_1 \geq \cdots \geq \lambda_r > 0$ and let μ with parts $\mu_1 \geq \cdots \geq \mu_s > 0$ be the conjugate partition λ'. The Weyl submodule W_λ of T is defined in Section 2.2.

The proof that (5.16) holds when λ is general rests on properties of the particular algebra $A = \mathscr{A}(-1)$. Let $y_1, y_2, \ldots$ be the images of $x_1, x_2, \ldots$ under the canonical homomorphism $T \to A$ and denote by L the Lie subalgebra of A generated by the y_i. The presentation of A by generators and relations cited in Section 1 implies that L is the free nilpotent-of-class-2 Lie algebra on the generators y_i and that A is its universal enveloping algebra.

LEMMA 5.3. *A has no divisors of zero.*

PROOF. This property is shared by all universal enveloping algebras: see [1, Section 2.3]. □

For the time being, λ may be either general or exceptional. Since A is a model for the tensor representations of E, it contains a unique submodule A_λ of type λ.

COROLLARY 5.4. *A_λ is the canonical image of W_λ.*

PROOF. Since W_λ is irreducible of type λ, its canonical image is either zero or isomorphic to—and hence equal to—A_λ. Thus, we have only to prove that the canonical image of the generator (2.7) of W_λ, namely

$$\Delta_\lambda(y_1, \ldots, y_n) = \Delta(y_1, \ldots, y_{\mu_1})\Delta(y_1, \ldots, y_{\mu_2}) \cdots,$$

is nonzero.

By Lemma 5.3, it will be sufficient to prove that

$$\Delta(y_1, \ldots, y_m) \neq 0$$

for all m. Thus, the proof of the corollary has been reduced to the special case of a partition of the form (1^m). But in this case the proof is immediate: $A_{(1^m)}$ is the canonical image of *some* submodule of T type (1^m), and $W_{(1^m)}$ is the *only* such submodule. □

Write $\mathscr{J}^\lambda(q) = \mathscr{J}(q) \frown T^\lambda$, where T^λ is the unique submodule of T of type $f_\lambda \lambda$.

COROLLARY 5.5. *For almost all q,* $\mathscr{J}^\lambda(q) + W_\lambda = T^\lambda$.

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PROOF. We follow the same procedure as in the proof of Proposition 5.1: form the S-restriction of $\mathscr{J}^\lambda(q) + W_\lambda$; intersect this with the primitive right ideal $\epsilon_\lambda(kS_n)$. The linear generators of this intersection are the same N elements $\epsilon_\lambda(fu + gv)$ as in the previous proof plus one more element $w_\lambda \in \mathbb{Q}S_n$ corresponding to W_λ. Thus, in place of the previous $b_\lambda \times N$ matrix $X(u, v)$, we get an $b_\lambda \times (N+1)$ matrix $\widetilde{X}(u, v)$, where first N columns form $X(u, v)$ and whose final column represents $\epsilon_\lambda w_\lambda$ and thus has entries in $\mathbb{Q}$.

What the corollary asserts is that $\widetilde{X}(u, vc)$ has rank f_λ for almost all q. This follows at once from Corollary 5.4, which asserts that the rank is f_λ when $q = -1$. □

Before proving (5.16), we point out two alternative formulations of it. Clearly, $a_\lambda(q) = 0$ if and only if $\mathscr{J}^\lambda(q) = T^\lambda$. Hence, by Corollary 5.5, (5.16) holds if and only if

$$W_\lambda \subseteq \mathscr{J}^\lambda(q) \quad \text{for almost all } q. \tag{5.17}$$

Further, since W_λ is generated as a module by $\Delta_\lambda(x_1, \ldots, x_n)$, (5.17) in turn holds if and only if

$$\Delta_\lambda(z_1, \ldots, z_n) = 0 \quad \text{for almost all } q, \tag{5.18}$$

where $z_1, z_2, \ldots$ are the images of $x_1, x_2, \ldots$ under the canonical homomorphism $T \to \mathscr{A}(q)$.

PROOF OF (5.16). We assume that λ is general and prove that (5.16) holds. The proof rests on two simple arguments. The *conjugacy argument* is that, if (5.16) holds for λ, then it also holds for λ'. This follows directly from Corollary 4.2, which shows that $a_{\lambda'}(q) = a_\lambda(q')$, where $2(qq' + 1) = q + q'$. Let us call the partitions

$$(\mu_1)', (\mu_1, \mu_2)', \ldots, (\mu_1, \ldots, \mu_s)' = \lambda$$

the *factors* of λ. The *factor argument* is that, if (5.16) holds for a factor ρ of λ, then it holds for λ itself. This is an obvious result when we take (5.18) into account, for, by (2.7), $\Delta_\rho(z_1, z_2, \ldots)$ is a factor of $\Delta_\lambda(z_1, z_2, \ldots)$.

Consider now a general partition λ. Since $\lambda \neq (1^n)$, we have $\mu_2 > 0$. Suppose first that $\mu_2 \geq 2$. Let $\rho = (\mu_1, \mu_2)'$. By (5.3) and (5.4), $a_\rho(q) = 0$ when $q^2 \neq 1$. By the factor argument (5.16) holds for λ.

This leaves the case $\mu_2 = 1$. Here, $\lambda = (l+1, 1^m)$, where, since λ is general, $l, m \geq 2$. We shall use the symmetrical notation $(l+1, 1^m) = [l, m]$, so that $[l, m]' = [m, l]$.

Suppose next that $m = 2$. Then $(31^2) = [2, 2]$ is a factor of λ. By Lemma 5.2, $a_{(31^2)}(q) = 0$ when $q^2 \neq 1$. By the factor argument, (5.16) holds for λ.

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Consider finally the general case $m \geq 2$. By what we have just proved, (5.16) holds for $[m, 2]$. It therefore holds for $[2, m] = [m, 2]'$ by the conjugacy argument. But $[2, m]$ is a factor of $\lambda = [l, m]$, so that (5.16) holds for λ by the factor argument. □

References

[1] J. Dixmier, *Enveloping algebras* (North-Holland, Amsterdam, 1977).

[2] N. F. J. Inglis, R. W. Richardson and J. Saxl, 'An explicit model for the complex representations of S_n', *Arch. Math.* **54** (1990), 258–259.

[3] A. A. Kljačko, 'Models for complex representations of the groups $GL(n, q)$ and Weyl groups', *Soviet Math. Doklady* **24** (1981), 496–499.

[4] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd edition (Oxford University Press, Oxford, 1994).

[5] G. E. Wall, 'A note on multiplicity-free tensor representations', *J. Pure Appl. Algebra* **88** (1993), 249–263.

[6] ———, 'More on multiplicity-free tensor representations', Research report 95–29, (School of Mathematics and Statistics, University of Sydney, 1995).

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