FINDING EISENSTEIN ELEMENTS IN CYCLIC NUMBER FIELDS OF ODD PRIME DEGREE

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Let $L = \mathbb{Q}[\alpha]$ be a cyclic number field of odd prime degree $q$ over the field $\mathbb{Q}$ of rationals. In this paper we give an algorithm to compute the discriminant of $L/\mathbb{Q}$, which relies upon a fast method to find Eisenstein elements in $L$. The algorithm accepts as input the minimal polynomial of $\alpha$ over $\mathbb{Q}$ and a complete factorisation of the discriminant of $\alpha$, and computes, in time polynomial in the size of the input, a list consisting of all the ramified primes with corresponding Eisenstein elements.

1. INTRODUCTION

Let $L$ be a normal extension of degree $q$ over the rational field $\mathbb{Q}$, where $q$ is an odd prime. Without loss of generality assume that $L = \mathbb{Q}[\alpha]$, where $\alpha$ is an algebraic integer which is given by its minimal polynomial $m_\alpha(x)$ over $\mathbb{Q}$. Clearly the Galois group $\text{Gal}(L/\mathbb{Q})$ of $L$ over $\mathbb{Q}$ is cyclic.

In [1] we describe an algorithm to determine if a given $\alpha \in \mathbb{Q}$ is the norm of some $x$ in $L$. The algorithm requires one to know (i) the rational primes $p \nmid q$ which ramify in $L$; (ii) for each ramified prime $p \neq q$ a generator $\pi$ of the value group of the (unique) valuation that extends the $p$-adic valuation from $\mathbb{Q}$ to $L$. Such a $\pi$ is sometimes called a prime element or a local uniformiser.

To find the ramified primes, we need the discriminant $D_{L/\mathbb{Q}}$ of the extension $L/\mathbb{Q}$. The discriminant can be computed using a very general algorithm due to Pohst and Zassenhaus [6, 9, 2, p.297]: this algorithm indeed computes an integral basis $B = \{\omega_1, \ldots, \omega_q\}$ for the extension $L/\mathbb{Q}$, and the discriminant $D_{L/\mathbb{Q}}$.

We show in [1] that, if $p$ is a ramified prime not equal to $q$, then a corresponding local uniformiser $\pi$ can be found in the set $\{\text{Tr}_{L/\mathbb{Q}}(\omega_i) - q\omega_i \mid i = 1, \ldots, q\}$, where $\text{Tr}_{L/\mathbb{Q}}$ denotes the trace from $L$ to $\mathbb{Q}$.

In this paper we show that, if we do not need an integral basis for $L/\mathbb{Q}$ for other reasons, then the full power of the Pohst-Zassenhaus' algorithm is not required. Indeed, we give an algorithm which takes as input $m_\alpha(x)$ and a complete factorisation of the discriminant $D_{L/\mathbb{Q}}(\alpha)$ of $\alpha$, and computes in time polynomial in the size of the input a list consisting of all the ramified primes $p$ with corresponding local uniformisers $\pi$.

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1.1 Notation.

Let \( P \) be a prime ideal of the ring of algebraic integers \( \mathcal{O} \) of \( L \), and let \( p \) be a rational prime.

If \( a \in L \) and \( a \neq 0 \), we shall denote by \( \nu_P(a) \) the order of \( a \) at \( P \), that is, the power of \( P \) in the factorisation of the fractional ideal \( a\mathcal{O} \). We define \( \nu_P(0) \) to be \( \infty \).

If \( a \in \mathbb{Q} \) and \( a \neq 0 \), then \( \nu_P(a) \) will denote the order of \( a \) at \( p \), that is, the power of the ideal \( p\mathbb{Z} \) in the factorisation of the fractional ideal \( a\mathbb{Z} \). We define \( \nu_P(0) \) to be \( \infty \).

\( \mathbb{Q}_p \) will denote the field of \( p \)-adic numbers, and \( L_P \) will denote the completion of \( L \) with respect to the valuation determined by \( P \). Then \( \mathbb{Z}_p \) will denote the ring of \( p \)-adic integers, that is \( \{ x \in \mathbb{Q}_p \mid \nu_P(x) \geq 0 \} \), and \( \mathcal{O}_P \) the ring of \( P \)-adic integers, that is \( \{ x \in L_P \mid \nu_P(x) \geq 0 \} \).

Finally, \( \mathbb{F}_p \) will denote the finite field of \( p \) elements, and \( \mathbb{F}_p^* \) its multiplicative group.

2. The method

Cyclic extension of the rationals of prime power degree have been intensively studied by B.M. Urazbaev. In [7] he proved the following:

**Lemma 1.** The discriminant \( D_{L/Q} \) of a cyclic extension \( L/Q \) of odd prime degree \( q \) has the form:

\[
D_{L/Q} = q^a \prod p_i^{q-1}
\]

where the \( p_i \) are distinct rational primes of the form \( nq + 1 \), and \( a = 0 \) or \( a = 2(q - 1) \).

Clearly, \( D_{L/Q} \mid D_{L/Q}(\alpha) \). Now, let

\[
D_{L/Q}(\alpha) = q^a \prod_{p_i \in S} p_i^{k_i}
\]

be a complete factorisation of \( D_{L/Q}(\alpha) \) into primes, with \( p_i \neq p_j \) for \( i \neq j \), and \( a \geq 0 \).

For each \( p_i \in S \) we have to decide if \( p_i \) ramifies in \( L \), that is, if \( p_i \mid D_{L/Q} \).

Firstly, by Urazbaev's criterion, we can ignore those primes \( p_i \in S \) for which either \( p_i \equiv 1 \pmod{q} \) or \( k_i < q - 1 \).

Secondly, we take into account the fact that \( L/Q \) is Galois. This implies that all the ideals of \( \mathcal{O} \) lying above \( p\mathbb{Z} \) (where \( p \) is a rational prime) are conjugate under \( \text{Gal}(L/Q) \) and so they have the same ramification index \( e \) and the same inertial degree \( f \). Let \( g \) be the number of distinct prime ideals lying above \( p\mathbb{Z} \). From the formula \( efg = [L : Q] = q \) and the primality of \( q \), it follows that, either \( p \) splits completely in \( L \) \( (e = 1, \ f = 1 \) and \( g = q) \), or \( p \) is inert in \( L \) \( (e = 1, \ f = q \) and \( g = 1) \), or \( p \)
is totally ramified in $L$ ($e = q$, $f = 1$ and $g = 1$). In this section we show how to recognise when $p$ is inert.

By assumption $\alpha \in \mathcal{O}$, and therefore the coefficients of $m_{\alpha}(x)$ lie in $\mathbb{Z}$. The next lemma relates the decomposition of a prime $p$ in $L$ to the factorisation of $m_{\alpha}(x)$ over $\mathbb{F}_p$.

**Lemma 2.** Let $L$ be a cyclic extension of $\mathbb{Q}$, of odd prime degree $q$. Let $p$ be a rational prime, and $\alpha$ be an algebraic integer in $L \setminus \mathbb{Z}$. If $p$ ramifies in $L$, then the minimal polynomial $m_{\alpha}(x)$ of $\alpha$ over $\mathbb{Q}$ splits into the product of $q$ identical linear factors over $\mathbb{F}_p$.

**Proof:** Let us assume that $p$ ramifies in $L$. Then $m_{\alpha}(x)$ is irreducible over $\mathbb{Q}_p$ (see [4, Theorem 5.1.5, p.75]), and therefore by Hensel's Lemma it is either irreducible or a $q^{th}$ power over $\mathbb{F}_p$. However, it can be shown that if $m_{\alpha}(x)$ is irreducible over $\mathbb{F}_p$ then $p$ must be inert (see [3, Proposition 5.11, p.102]). Hence $m_{\alpha}(x)$ must split into the product of $q$ identical linear factors over $\mathbb{F}_p$. \]

To apply Lemma 2, we compute $l(x) = GCD(x^p - x, m_{\alpha}(x))$ over $\mathbb{F}_p$. Then $m_{\alpha}(x)$ is a $q^{th}$ power over $\mathbb{F}_p$ precisely when $\deg l(x) = 1$ and $l(x)^q \equiv m_{\alpha}(x)$ (mod $p$). In practice we compute $j(x) = x^p \mod m_{\alpha}(x)$ in $\mathbb{F}_p$, using the binary powering algorithm [2, p.8]. Then $l(x)$ is given by $GCD(j(x) - x, m_{\alpha}(x))$.

Unfortunately, the previous lemma gives only a necessary condition for a prime $p$ to ramify in $L$. In the next section we shall develop some some necessary and sufficient conditions.

### 3. Eisenstein Polynomials

Let us assume that $p$ is totally ramified, and let $\mathcal{P}$ be the unique prime ideal lying above $p\mathbb{Z}$. Since there is only one extension of the $p$-adic valuation from $\mathbb{Q}$ to $L$, if $\theta \in L$ we must have [8, Corollary 2.5.8, p.68]

\[
\nu_p(\theta) = \nu_p\left(N_{L/\mathbb{Q}}(\theta)\right).
\]

We shall use this fact often in the following.

In particular, if $\theta \in \mathcal{P} \setminus \mathcal{P}^2$, then $\nu_p\left(N_{L/\mathbb{Q}}(\theta)\right) = \nu_p(\theta) = 1$. This shows that if $p$ is ramified, then $\mathcal{O}$ contains elements whose norms have $p$-order equal to 1. On the other hand

**Lemma 3.** If a rational prime $p$ is inert in $L$ then there is no $\theta \in \mathcal{O} \setminus \mathbb{Z}$ whose norm has $p$-order 1.

**Proof:** Assume that $\theta \in \mathcal{O} \setminus \mathbb{Z}$ is an element whose norm has $p$-order 1. If $\theta_1, \theta_2, \ldots, \theta_q$ denote the conjugates of $\theta$, with $\theta = \theta_1$ say, then $N_{L/\mathbb{Q}}(\theta) = \theta_1 \theta_2 \cdots \theta_q$.

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Since \( p \) is inert, \( p \mathcal{O} \) is the only prime ideal of \( \mathcal{O} \) lying above \( p \mathbb{Z} \). By assumption \( \theta_1 \theta_2 \cdots \theta_q \in p \mathbb{Z} \subseteq p \mathcal{O} \), and hence, since \( p \mathcal{O} \) is a prime ideal, some conjugate of \( \theta \) must lie in \( p \mathcal{O} \). But then, since \( p \mathcal{O} \) is \( \sigma \)-invariant, all the conjugates of \( \theta \) must lie in \( p \mathcal{O} \), and therefore \( N_{L/Q}(\theta) \in p^q \mathcal{O} \cap \mathbb{Z} = p^q \mathbb{Z} \), against our assumption.

**Theorem 1.** Let \( p \) be a rational prime. Assume that there is an element \( \theta \in \mathcal{O} \setminus \mathbb{Z} \) whose norm has \( p \)-order 1. Then \( p \) ramifies in \( L \) if and only if \( m_\theta(x) \) is Eisenstein at \( p \).

**Proof:** By Lemma 3, the existence of \( \theta \in \mathcal{O} \setminus \mathbb{Z} \) whose norm has \( p \)-order 1 implies that \( p \) cannot be inert.

Assume first that \( m_\theta(x) \) is Eisenstein at \( p \). Then \( m_\theta(x) \) is irreducible in \( \mathbb{Q}_p[x] \), and \( \theta \) generates a totally ramified extension of \( \mathbb{Q}_p \) of degree \( q \), that is, \( p \) is totally ramified (see [5, Proposition 11, p.52]).

Conversely, assume that \( p \) ramifies in \( L \). Then \( \nu_p(\theta) = \nu_p\left(N_{L/Q}(\theta)\right) = 1 \). Since \( \text{Gal}(L/Q) \) permutes the prime ideals lying above \( p \mathbb{Z} \) transitively, and there is only one prime ideal \( P \) above \( p \mathbb{Z} \), it follows that \( \nu_p(\sigma(\theta)) = 1 \) for all \( \sigma \in \text{Gal}(L/Q) \). Let

\[
m_\theta(x) = x^q + b_{q-1}x^{q-1} + \ldots + b_1x + b_0.
\]

Then each \( b_i \) lies in \( \mathbb{Z} \) and is an elementary symmetric function of the set \( \{\theta, \sigma(\theta), \ldots, \sigma^{q-1}(\theta)\} \), where \( \sigma \) is any generator of \( \text{Gal}(L/Q) \). Hence \( b_i \in P \cap \mathbb{Z} = p \mathbb{Z} \). Moreover

\[
\nu_p(b_0) = \nu_p(\theta \sigma(\theta) \cdots \sigma^{q-1}(\theta)) = 1,
\]

which shows that \( m_\theta(x) \) is Eisenstein at \( p \).

In order to apply Theorem 1, we need an efficient algorithm to solve the following problem: find an element of \( \mathcal{O} \) whose norm has \( p \)-order 1. The next lemma shows that it is enough to find any algebraic integer whose norm has \( p \)-order not divisible by \( q \).

**Lemma 4.** Let \( p \) be a ramified prime. Given \( \gamma' \in \mathcal{O} \) with \( q \nmid \nu_p\left(N_{L/Q}(\gamma')\right) \), we can construct an element \( \gamma \in \mathcal{O} \) with \( \nu_p\left(N_{L/Q}(\gamma)\right) = 1 \).

**Proof:** Let \( r = \nu_p\left(N_{L/Q}(\gamma')\right) \). Since \( N_{L/Q}(p) = p^q \), and the norm elements form a multiplicative group, we can find an \( s \in \mathbb{N} \) which acts as a multiplicative inverse of \( r \) (mod \( q \)), that is, such that \( rs = 1 + ql \) \((l \in \mathbb{N}) \). Let \( \gamma = (\gamma')^s/p^l \). Clearly

\[
\nu_p\left(N_{L/Q}(\gamma)\right) = s \nu_p\left(N_{L/Q}(\gamma')\right) - \nu_p(p) = 1
\]

and therefore \( \nu_p(\gamma) = 1 \). It is left to prove that \( \gamma \in \mathcal{O} \). Clearly, \( (\gamma')^s \in \mathcal{O} \). Let \( P \) be the unique prime ideal of \( \mathcal{O} \) lying above \( p \mathbb{Z} \). Now, \( \nu_P((\gamma')^s/p^l) = 1 \), and \( \nu_Q((\gamma')^s/p^l) = \nu_Q((\gamma')^s) \geq 0 \) for any prime ideal \( Q \) of \( \mathcal{O} \) not equal to \( P \). Therefore \( (\gamma')^s/p^l \in \mathcal{O} \) (see [8, Corollary 4.1.8, p.125]).
4. Finding Eisenstein Elements

We shall continue to assume that $p$ is ramified. The inertia group $I_p$ of $P$ has order $e = q$ (see [4, Corollary 5.4.5, p.83]), and so it must be equal to $Gal(L/Q)$. Thus, if $\sigma \in Gal(L/Q)$ and $\beta \in O$, we must have $\sigma(\beta) - \beta \in P$. We shall use this fact often, in the following.

Let us consider the embedding $O \hookrightarrow O_P$. For this purpose, we fix, once for all, an element $\pi \in P'P^2$, and we take $R = \{0, 1, \ldots, p - 1\}$ to be a set of representatives of $O/P$ in $O$. Every $\beta \in O_P$ can be written as a convergent series (in the $P$-adic metric)

$$\beta = \sum_{i=0}^{\infty} \sum_{j=0}^{q-1} a_{i,j} p^i \pi^j \quad (a_{i,j} \in R)$$

where the coefficients $a_{i,j}$ are uniquely determined by $\beta$.

Moreover, if $\beta \in O\setminus Z$, then for some $h, k \in N$, with $0 < k < q$ we must have

(i) $a_{h,k} \neq 0$; and
(ii) $a_{i,j} = 0$ whenever $(i < h$ and $0 < j < q)$ or $(i = h$ and $0 < j < k)$.

for otherwise, using the fact that $ef = [L_P : Q_p] = q = [L : Q]$, the element $\beta$ would be a $p$-adic integer in $O$, and therefore an element of $Z$.

We define now a function $\Lambda : O \rightarrow O$ as follows: if $\beta$, $h$, $k$ are as above, then

$$\Lambda(\beta) = \sum_{j=k}^{q-1} a_{h,j} p^j \pi^j + \sum_{i=h+1}^{\infty} \sum_{j=0}^{q-1} a_{i,j} p^i \pi^j.$$

Since $\sigma$ fixes $p$ and any element of $R$, clearly we have

**Lemma 5.** Let $\beta \in O$. If $\sigma \in Gal(L/Q)$ then $\sigma(\beta) - \beta = \sigma(\Lambda(\beta)) - \Lambda(\beta)$.

4.1 $p$ is totally and tamely ramified.

In this section we assume that $p$ is ramified and $p \neq q$, and we let $P$ denote the unique ideal of $O$ above $pZ$.

**Lemma 6.** Let $\sigma$ be a generator of $Gal(L/Q)$. Then $\nu_P (\sigma(\pi) - \pi) = 1$.

**Proof:** Since $\{1, \pi, \ldots, \pi^{q-1}\}$ is a local basis at $p$, we must have (see [8, Proposition 4.8.18, p.164])

$$\nu_P \left( D_{L/Q}(\pi) \right) = \nu_P \left( D_{L/Q} \right) = q - 1.$$

But $D_{L/Q}(\pi) = N_{L/Q}(m_\pi^L(\pi))$, and

$$\nu_P \left( N_{L/Q}(m_\pi^L(\pi)) \right) = \nu_P (m_\pi^L(\pi)) = \nu_P (\sigma(\pi) - \pi) \cdots (\sigma^{q-1}(\pi) - \pi)).$$
Each factor on the right hand side has \( P \)-order greater than zero, there are \( q - 1 \) factors, and so by the pigeon hole principle \( \nu_P(\sigma(\pi) - \pi) \) must be 1.

**Lemma 7.** Let \( \sigma \) be a generator of Gal(\( L/Q \)). If \( 0 < r < q \) then \( \nu_P(\sigma(\pi^r) - \pi^r) = r \).

**Proof:** Since \( P \) and all its powers are \( \sigma \)-invariant, it follows that \( \sigma(\pi) \equiv a\pi \pmod{P^2} \), with \( 0 < a < p \). Then \( \sigma^2(\pi) \equiv a\sigma(\pi) \pmod{P^2} \), that is, \( \sigma^2(\pi) \equiv a^2\pi \pmod{P^2} \), and more generally \( \sigma^i(\pi) \equiv a^i\pi \pmod{P^2} \). But \( \sigma^q(\pi) = \pi \), and so \( a^q \equiv 1 \pmod{p} \). Therefore the order of \( a \) in \( \mathbb{F}_p^* \) must divide \( q \). Since \( q \) is prime and \( a \not\equiv 1 \pmod{p} \) by Lemma 6, the order of \( a \) in \( \mathbb{F}_p^* \) must be equal to \( q \). If \( 0 < r < q \), then

\[
\sigma(\pi^r) - \pi^r = \sigma(\pi)^r - \pi^r \equiv a^r\pi - \pi^r \pmod{P^r+1}
\]

with \( a^r \not\equiv 1 \pmod{p} \), which proves the assertion.

**Corollary 1.** Let \( \sigma \) be a generator of Gal(\( L/Q \)). If \( \beta \in \mathbb{O} \setminus \mathbb{Z} \), then

\[
\nu_P(\sigma(\Lambda(\beta)) - \Lambda(\beta)) = \nu_P(\Lambda(\beta)).
\]

In particular, \( q \not\mid \nu_P(\sigma(\Lambda(\beta)) - \Lambda(\beta)) \).

**Proof:** Define a function \( F : L \to L \) by \( F(x) = \sigma(x) - x \). Since \( F \) is \( \mathbb{Z} \)-linear, we have

\[
F(\Lambda(\beta)) = F \left( \sum_{j=k}^{q-1} a_{h,j}p^h\pi^j + \sum_{i=h+1}^{\infty} \sum_{j=0}^{q-1} a_{i,j}p^i\pi^j \right)
\]

\[
= \sum_{j=k}^{q-1} F(a_{h,j}p^h\pi^j) + F \left( \sum_{i=h+1}^{\infty} \sum_{j=0}^{q-1} a_{i,j}p^i\pi^j \right)
\]

\[
= \sum_{j=k}^{q-1} F(a_{h,j}p^h\pi^j) + F(t)
\]

with \( t = \sum_{i=h+1}^{\infty} \sum_{j=0}^{q-1} a_{i,j}p^i\pi^j \). Now, \( \nu_P(t) \geq (h + 1)q \), and so \( \nu_P(F(t)) \geq (h + 1)q \).

Note that \( \nu_P(F(a_{h,j}p^h\pi^j)) = qh + j \) \((j = k, \ldots, q - 1)\) if \( 0 < a_{h,j} < p \), and \( \nu_P(F(a_{h,j}p^h\pi^j)) = \infty \) if \( a_{h,j} = 0 \). Clearly \( 0 < a_{h,k} < p \), by the definition of the function \( \Lambda \), and so \( \nu_P \left( \sum_{j=k}^{q-1} F(a_{h,j}p^h\pi^j) \right) = hq + k \). Therefore \( \nu_P(F(\Lambda(\beta))) = hq + k = \nu_P(\Lambda(\beta)) \).
THEOREM 2. If \( \beta \in \mathcal{O}\setminus\mathbb{Z} \) then \( q \nmid \nu_p\left(m'_p(\beta)\right) \).

PROOF: By Lemma 5, if \( \sigma \) denotes a generator of \( \text{Gal}(L/\mathbb{Q}) \), we have

\[
m'_p(\beta) = (\sigma(\beta) - \beta) \cdots (\sigma^{q-1}(\beta) - \beta) \\
= (\sigma(\Lambda(\beta)) - \Lambda(\beta)) \cdots (\sigma^{q-1}(\Lambda(\beta)) - \Lambda(\beta))
\]

By Corollary 1, then \( \nu_p\left(m'_p(\beta)\right) = (q - 1)\nu_p(\Lambda(\beta)) \). Since \( q \nmid \nu_p(\Lambda(\beta)) \), it follows that \( q \nmid \nu_p\left(m'_p(\beta)\right) \).

\[
\]

4.2 \( p \) IS TOTALLY AND WILDLY RAMIFIED.

In this section we assume that \( p \) is ramified and \( p = q \), and we let \( \mathcal{P} \) denote the unique ideal of \( \mathcal{O} \) above \( q\mathbb{Z} \). Define a function \( G : L \to L \) by \( G(x) = Tr_{L/Q}(x) - qx \).

CLEARLY, \( G \) IS \( \mathbb{Z} \)-LINEAR and it vanishes on \( \mathbb{Q} \).

LEMMA 8. Let \( 0 < r < q \). Then \( G(\pi^r) \equiv aq - q\pi^r \pmod{\mathcal{P}^2q} \), with \( 0 \leq a < q \).

PROOF: Since \( Tr_{L/Q}(\pi^r) \in q\mathbb{Z} \), we can write \( Tr_{L/Q}(\pi^r) \equiv aq \pmod{q^2} \), with \( 0 \leq a < q \). This proves the assertion.

THEOREM 3. If \( \beta \in \mathcal{O}\setminus\mathbb{Z} \), then \( G(\beta) = G(\Lambda(\beta)) \) and

\[
G(\beta) \equiv bq^{h+1} - cq^{h+1}\pi^k \pmod{\mathcal{P}^{(h+1)q+k+1}}
\]

with \( 0 \leq b < q \) and \( 0 < c < q \).

PROOF: Since the function \( G \) is \( \mathbb{Z} \)-linear, and it vanishes on \( \mathbb{Q} \), we have

\[
G(\beta) = G(\Lambda(\beta)) \\
= G\left(\sum_{j=k}^{q-1} a_{h,j}q^h\pi^j + \sum_{i=h+1}^{\infty} \sum_{j=0}^{q-1} a_{i,j}q^i\pi^j\right) \\
= \sum_{j=k}^{q-1} G(a_{h,j}q^h\pi^j) + G\left(\sum_{i=h+1}^{\infty} \sum_{j=0}^{q-1} a_{i,j}q^i\pi^j\right) \\
= \sum_{j=k}^{q-1} G(a_{h,j}q^h\pi^j) + \sum_{j=0}^{q-1} G(a_{h+1,j}q^{h+1}\pi^j) + G(t)
\]

with \( t = \sum_{i=h+2}^{\infty} \sum_{j=0}^{q-1} a_{i,j}q^i\pi^j \). Now, \( \nu_p(t) \geq (h+2)q \), and so \( \nu_p(G(t)) \geq (h+2)q \). Also, by Lemma 8, \( \nu_p\left(G(a_{h+1,j}q^{h+1}\pi^j)\right) \geq q(h+2) \) \( (j = 0, \ldots, q-1) \), and

\[
G(a_{h,k}q^h\pi^k) \equiv bq^{h+1} - cq^{h+1}\pi^k \pmod{\mathcal{P}^{(h+2)q}}
\]
with \( c_k \not\equiv 0 \pmod{q} \), since \( a_{h,k} \not\equiv 0 \pmod{q} \) by the definition of the function \( \Lambda \). Moreover, if \( a_{h,s} \not\equiv 0 \pmod{q} \) (\( s = k + 1, \ldots, q - 1 \)) then

\[
G(a_{h,s}q^h\pi^s) \equiv b_s q^{h+1} - c_s q^{h+1} \pi^s \pmod{\mathcal{P}^{(h+2)q}}.
\]

This shows that

\[
G(\beta) \equiv q^{h+1} \left( \sum_{i=k}^{q-1} b_i \right) - q^{h+1} c_k \pi^k \pmod{\mathcal{P}^{(h+1)q+k+1}}
\]

with \( c_k \not\equiv 0 \pmod{q} \). To prove our assertion, let \( b = \sum_{i=k}^{q-1} b_i \pmod{q} \), and \( c = c_k \).

We show next how Theorem 3 can be used to obtain an algebraic integer whose norm has \( q \)-order not divisible by \( q \). Let \( w = \nu_q \left( N_{L/Q}(G(\beta)) \right) / q \).

If \( w \in \mathbb{Z} \) then \( b \equiv 0 \pmod{q} \), and \( G(\beta) \) is the desired element.

Otherwise, \( w = h + 1 \), and \( G(\beta)/q^w \equiv b - c \pi^k \pmod{\mathcal{P}^{k+1}} \). Note that \( G(\beta)/q^w \in \mathcal{O} \), since \( \nu_P(G(\beta)/q^w) = 0 \) and \( \nu_Q(G(\beta)/q^w) = \nu_Q(G(\beta)) \geq 0 \), when \( Q \) is any prime ideal of \( \mathcal{O} \) not equal to \( \mathcal{P} \) (use again [8, Corollary 4.1.8, p.125]). Let \( \rho = G(\beta)/q^w \). It is easily seen that, if

\[
m_{G(\beta)}(x) = x^q + b_{q-1}x^{q-1} + \ldots + b_1 x + b_0
\]

then

\[
m_{\rho}(x) = x^q + \left( b_{q-1}/q^w \right) x^{q-1} + \ldots + \left( b_1/q^{w(q-1)} \right) x + \left( b_0/q^{qw} \right)
\]

Since \( q \) is assumed to be ramified, \( m_{\rho}(x) \equiv (x - \hat{s})^q \pmod{q} \). Let \( s \) be a representative of the residue class of \( \hat{s} \). Then \( \rho - s \equiv -c \pi^k \pmod{\mathcal{P}^{k+1}} \), and so \( \rho - s \) is the desired element.

The pseudo code for the algorithm is sketched in Figures 1 and 2. The algorithm EISENSTEIN takes as input \( m_\alpha(x) \) and returns a list consisting of the ramified primes and corresponding local uniformisers. If the factorisation of \( D_{L/Q}(\alpha) \) is given as part the input, the entire algorithm runs in polynomial time.

procedure CONSTRUCT(\( \gamma, p \)):
let \( r = \nu_p(N_{L/Q}(\gamma)) \);
find \( l, s \in \mathbb{N} \) such that \( rs = 1 + qt \);
let \( \epsilon = (\gamma)^s/p^l \);
if \( m_\epsilon(x) \) is Eisenstein at \( p \) then return(\( \epsilon \)); endif;
return(0);

Figure 1: Pseudo Code for the Algorithm CONSTRUCT.
procedure EISENSTEIN($m_\alpha(x)$):
let $List = \emptyset$;
let $D_{L/Q}(\alpha) = q^a \prod_{p \in S} p^{k_i}$;
for all the $p \in S$ do
  if $(p_i \equiv 1 \pmod{q})$ and $k_i \geq q - 1$
    and $m_\alpha(x)$ is a $q^{th}$ power over $F_p$
    then let $\gamma = m^{\prime}_\alpha(\alpha)$;
    let $\pi = \text{CONSTRUCT}(\gamma, p)$;
    if $\pi \neq 0$ then add $\{p, \pi\}$ to $List$; endif;
  endif;
endfor;
if $a < 2(q - 1)$ then return($List$); endif;
if $m_\alpha(x)$ is not a $q^{th}$ power over $F_q$ then return($List$); endif;
let $\delta = \text{Tr}_{L/Q}(\alpha) - qa$;
let $w = \nu_q(N_{L/Q}(\delta))/q$;
if $w \notin \mathbb{Z}$ then
  let $\gamma = \delta$;
else
  let $\rho = \delta/q^w$, and compute $m_\rho(x)$;
  if $m_\rho(x) \notin \mathbb{Z}[x]$ then return($List$); endif;
  compute $c(x) = \text{GCD}(x^q - x, m_\rho(x))$ over $F_q$.
  if $c(x) \neq x - s$ then return($List$); endif;
  let $\gamma = \rho - s$;
  if $q \mid \nu_q(N_{L/Q}(\gamma))$ then return($List$); endif;
  endif;
let $\pi = \text{CONSTRUCT}(\gamma, q)$;
if $\pi \neq 0$ then add $\{q, \pi\}$ to $List$; endif;
return($List$);

Figure 2: Pseudo Code for the Algorithm EISENSTEIN.

REFERENCES


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