WALLMAN COMPACTIFICATIONS AND WALLMAN REALCOMPACTIFICATIONS

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1. Introduction

In 1964 Frink [8] generalized Wallman's method [17] of compactification and asked the question: "Is every Hausdorff compactification of a Tychonoff space a Wallman compactification?". This problem, which is as yet unsolved, has led to the discovery of a number of necessary and/or sufficient conditions for a Hausdorff compactification to be Wallman; (see Alò and Shapiro [2], [3], Banasĉhewski [6], Njåstad [13] and Steiner [15]). Recently Alò and Shapiro [4], [5] have generalized the Wallman procedure to discuss what they call $\boldsymbol{Z}^*$-realcompact spaces and $\boldsymbol{Z}^*$-realcompactification $\eta(\boldsymbol{Z})$ corresponding to a countably productive (c.p.) normal base $\boldsymbol{Z}$ on X. Some further work has been done by Steiner and Steiner [14].

In this paper we exploit the known results of proximity spaces (called EF-proximity in this paper) introduced by Efremovič [7] and symmetric generalized proximity spaces (called LO-proximity spaces in this paper) introduced by Lodato [11]. We show how a normal base $\boldsymbol{Z}$ on a Tychonoff space X induces an EF-proximity $\delta(\boldsymbol{Z})$ on X and that the Wallman compactification $W(\boldsymbol{Z})$ is homeomorphic to the Smirnov compactification of X corresponding to $\delta(\boldsymbol{Z})$. This result is implicitly contained in the work of Aarts [1] and Njåstad [13], but our proof is more direct. We then use this result to derive a necessary and sufficient condition for a Hausdorff compactification to be Wallman and show that our result includes the ones derived by Alò and Shapiro [2], Banasĉhewski [6], Njåstad [13] and Steiner [15].

We next consider a c.p. normal base $\boldsymbol{Z}$ on a Tychonoff space X and the space $\eta(\boldsymbol{Z})$ of all real $\boldsymbol{Z}$-ultrafilters on X (i.e. $\boldsymbol{Z}$-ultrafilters with the c.i.p.) with the Wallman topology (Alò and Shapiro [4]). We find necessary and sufficient conditions for (a) $\eta(\boldsymbol{Z})$ to be realcompact and (b) a $\boldsymbol{Z}^*$-realcompactification Y of X to be homeomorphic to $\eta(\boldsymbol{Z})$. These results are motivated by the known results concerning Wallman compactifications. We show that our results are an improvement over the sufficient conditions for (i) and (ii) obtained by Alò and Shapiro [5], Steiner and Steiner [14].

2. Preliminaries

In this section we review the main definitions and results which are required in the sequel. For terms not defined here see Gagrat and Naimpally [9].

Let X be a T_1-space and let $\boldsymbol{Z}$ be a ring of closed subsets of X. We say that $\boldsymbol{Z}$ is a *separating base* iff (i) $\boldsymbol{Z}$ is a base for closed subsets of X and (*ii*) $x \notin E$, E a closed subset of X implies the existence of $L_1, L_2 \in \boldsymbol{Z}$ such that $x \in L_1$, $E \subset L_2$ and $L_1 \cap L_2 = \varnothing$. $\boldsymbol{Z}$ is called a *normal base* iff $\boldsymbol{Z}$ is a separating base and $L_1, L_2 \in \boldsymbol{Z}$ such that $L_1 \cap L_2 = \varnothing$ implies the existence of $L'_1, L'_2 \in \boldsymbol{Z}$ such that $L_1 \subset X - L'_1, L_2 \subset X - L'_2$ and $L'_1 \cup L'_2 = X$. $\boldsymbol{Z}$ is called *countably productive* (c.p.) iff it is closed under countable intersections. It was proved by Frink [8] that X is Tychonoff iff it has a normal base. The above definition of a normal base reminds us of the "Strong Axiom" of an EF-proximity δ viz. $A \not\delta B$ implies the existence of C, D such that $A \not\delta C$, $B \not\delta D$ and $C \cup D = X$, where $\not\delta$ replaces "void intersection". This provides motivation for the following lemma.

LEMMA 2.1. *If $\boldsymbol{Z}$ is a separating base on a T_1-space (X, τ), then $\delta = \delta(\boldsymbol{Z})$ defined by "$A \not\delta B$ iff there are $L_1, L_2 \in \boldsymbol{Z}$ such that $A \subset L_1$, $B \subset L_2$ and $L_1 \cap L_2 = \varnothing$", is a compatible LO-proximity on X. Further, if $\boldsymbol{Z}$ is a normal base then $\delta(\boldsymbol{Z})$ is an EF-proximity.*

PROOF. We verify only the axiom "$A \ \delta\ B$, $b \ \delta\ C$ for each $b \in B$ implies $A \ \delta\ C$", since the other axioms of a LO-proximity follow easily. If L_C is an arbitrary element of $\boldsymbol{Z}$ containing C, then $b \ \delta\ C$ implies $b \in L_C$ for each $b \in B$ or $B \subset L_C$. But since $A \ \delta\ B$, $A \subset L_A$ for $L_A \in \boldsymbol{Z}$ implies that $L_A \cap L_C \neq \varnothing$ for every $L_C \supset C$. Thus $A \ \delta\ C$. To show that $\tau = \tau(\delta)$, we note that $x \ \delta\ A$ iff $x \in L_x \in \boldsymbol{Z}$, $A \subset L_A \in \boldsymbol{Z}$ implies $L_x \cap L_A \neq \varnothing$ iff $x \in L_A$ for each $L_A \supset A$ iff $x \in A^-$ (where $^-$ denotes the τ-closure). Finally if $\boldsymbol{Z}$ is a normal base then the Strong Axiom follows from the remarks preceeding the statement of this lemma. (We note here that a normal base $\boldsymbol{Z}$ is an *EF-proximity base* for $\delta = \delta(\boldsymbol{Z})$ in the sense of Njåstad [13]).

Let (X, τ) be a T_1-space and let $\boldsymbol{Z}$ be a separating base on X. Let Σ_X be the family of all bunches in (X, δ) where $\delta = \delta(\boldsymbol{Z})$ and let Σ_X be assigned the absorption or A-topology. Let $W(\boldsymbol{Z})$ be the family of all $\boldsymbol{Z}$-ultrafilters with the Wallman topology. Define a map b: $W(Z) \to \Sigma_X$ by

$$b(\boldsymbol{F}) = \{E \subset X : E^- \in \boldsymbol{F}\},$$

the bunch generated by $\boldsymbol{F}$. If $\boldsymbol{F}_1$, $\boldsymbol{F}_2$ are two $\boldsymbol{Z}$-ultrafilters then clearly $b(\boldsymbol{F}_1) \neq b(\boldsymbol{F}_2)$ i.e. b is one-to-one. Moreover, $E^- \in \boldsymbol{F}$ for every $\boldsymbol{F} \in \boldsymbol{P} \subset W(\boldsymbol{Z})$ iff E absorbs $b(\boldsymbol{P}) \subset \Sigma_X$. Hence b is a homeomorphism from $W(\boldsymbol{Z})$ into Σ_X, and we have the following result:

THEOREM 2.2. *Let (X,τ) be a T_1-space and $\boldsymbol{Z}$ a separating base on X. Then the Wallman compactification $W(\boldsymbol{Z})$ is homeomorphic to the space of bunches generated by all $\boldsymbol{Z}$-ultrafilters in X with the A-topology.*

LEMMA 2.3. *Let $\boldsymbol{Z}$ be a separating base on a T_1-space X and let $\delta = \delta(\boldsymbol{Z})$ be the corresponding LO-proximity on X. If $\boldsymbol{F}$ is a $\boldsymbol{Z}$-ultrafilter in X then*

$$\sigma(\boldsymbol{F}) = \{A \subset X : A\, \delta\, F \text{ for every } F \in \boldsymbol{F}\}$$

is a cluster, called the cluster generated by $\boldsymbol{F}$.

PROOF. We need only prove: $A, B \in \sigma(\boldsymbol{F})$ implies $A\, \delta\, B$, for the other axioms follow easily. If $A \not\delta B$ then there exist $L_A, L_B \in \boldsymbol{Z}$ such that $A \subset L_A$, $B \subset L_B$ and $L_A \cap L_B = \varnothing$. But $L_A\, \delta\, F$ for every $F \in \boldsymbol{F}$ implies $L_A \cap F \neq \varnothing$ for every $F \in \boldsymbol{F}$ and this in turn implies that $L_A \in \boldsymbol{F}$. Similarly $L_B \in \boldsymbol{F}$ and so $L_A \cap L_B \neq \varnothing$, a contradiction.

LEMMA 2.4. *Let $\boldsymbol{Z}$ be a normal base on a Tychonoff space X and let $\delta = \delta(\boldsymbol{Z})$. If $\boldsymbol{F}$ is a prime $\boldsymbol{Z}$-filter then*

$$\sigma(\boldsymbol{F}) = \{A \subset X : A\, \delta\, F \text{ for every } F \in \boldsymbol{F}\}$$

is a cluster in X. Conversely, given a cluster σ in X, there exists a unique $\boldsymbol{Z}$-ultrafilter $\sigma \cap \boldsymbol{Z}$ in X which generates σ.

PROOF. For the first part we need only prove that if $A, B \in \sigma(\boldsymbol{F})$, then $A\, \delta\, B$. If, on the contrary, $A \not\delta B$ then there exist L_A, L_B in $\boldsymbol{Z}$ such that $A \subset L_A$, $B \subset L_B$ and $L_A \cap L_B = \varnothing$. Since $\boldsymbol{Z}$ is normal, there exist L'_A, L'_B in $\boldsymbol{Z}$ such that $L_A \subset X - L'_A$, $L_B \subset X - L'_B$ and $L'_A \cup L'_B = X$. Clearly L'_A, L'_B do not belong to $\boldsymbol{F}$ which implies that $X = L'_A \cup L'_B \notin \boldsymbol{F}$ (since $\boldsymbol{F}$ is prime), a contradiction. To prove the converse, we note that $\sigma \cap \boldsymbol{Z}$ satisfies the conditions of Lemma 2.10 of Gagrat and Naimpally [9] and so there exists a prime $\boldsymbol{Z}$-filter $\boldsymbol{F} \subset \sigma$. Clearly $\boldsymbol{F} \subset \sigma \subset \sigma(\boldsymbol{F})$ and hence $\sigma = \sigma(\boldsymbol{F})$. Let $\boldsymbol{G}$ be the unique $\boldsymbol{Z}$-ultrafilter containing $\boldsymbol{F}$. Then $\sigma(\boldsymbol{F}) = \sigma(\boldsymbol{G}) = \sigma$. Finally, since $L_1, L_2 \in \sigma \cap \boldsymbol{Z}$ iff $L_1 \cap L_2 \neq \varnothing$, $\boldsymbol{G} = \sigma \cap \boldsymbol{Z}$ and the uniqueness follows.

Let $\boldsymbol{Z}$ be a normal base on a Tychonoff space X and let $\delta = \delta(\boldsymbol{Z})$ be the induced EF-proximity as defined in Lemma 2.1. By Theorem 3.10 of Gagrat and Naimpally [9], the map $\theta\colon b(W(\boldsymbol{F})) \subset \Sigma_X \to X$ (the Smirnov compactification of X i.e. the family of all clusters in X with the A-topology) given by $\theta(b(\boldsymbol{F})) = \sigma(\boldsymbol{F})$ is continuous. From Lemma 2.4 it follows that θ is one-to-one and onto $\boldsymbol{X}$. Hence we have the main result of this section.

THEOREM 2.5. *Let X be a Tychonoff space with a normal base $\boldsymbol{Z}$ and let $\delta = \delta(\boldsymbol{Z})$. Then the Wallman compactification $W(\boldsymbol{Z})$ is homeomorphic to the δ-Smirnov compactification $\boldsymbol{X}$ of X.*

We note that the Wallman topology on $W(\boldsymbol{Z})$ is the same as the A-topology on $W(\boldsymbol{Z})$.

We now list some results concerning a normal base $\boldsymbol{Z}$ on a Tychonoff space X. Most of the proofs, being routine, are omitted. (cf. Gillman and Jerison [10]).

LEMMA 2.6. (*a*) $X \in \boldsymbol{Z}$.

(b) *Given a neighbourhood U of $x \in X$, there exists a neighbourhood $L \in \boldsymbol{Z}$ of x such that $L \subset U$.*

(c) *If $p \in X$ is a cluster point of a $\boldsymbol{Z}$-filter $\boldsymbol{F}$ on X, then there exists a $\boldsymbol{Z}$-ultrafilter $\boldsymbol{U}$ containing $\boldsymbol{F}$ which converges to p.*

(d) *If $\boldsymbol{F}$ is a prime $\boldsymbol{Z}$-filter on X, then the following are equivalent*:

(i) *$\boldsymbol{F}$ converges to p,*

(ii) *p is a cluster point of $\boldsymbol{F}$,*

(iii) $\{p\} = \cap \{F: F \in \boldsymbol{F}\}$.

Hence every prime $\boldsymbol{Z}$-filter has at most one cluster point.

(e) *For $p \in X$ define $A_p = \{L \in \boldsymbol{Z}: p \in L\}$. Then*:

(i) *p is a cluster point of a $\boldsymbol{Z}$-filter $\boldsymbol{F}$ iff $\boldsymbol{F} \subset A_p$,*

(ii) *A_p is the unique $\boldsymbol{Z}$-ultrafilter converging to p,*

(iii) *Distinct $\boldsymbol{Z}$-ultrafilters cannot have a common cluster point.*

LEMMA 2.7. *Let $\boldsymbol{Z}$ be a normal base on a Tychonoff space X and let $X \subset T \subset W(\boldsymbol{Z})$. Then*

(i) $\mathrm{Cl}_T(L) \cap X = L$ *for each* $L \in \boldsymbol{Z}$

(ii) $\mathrm{Cl}_T(L_1 \cap L_2) = \mathrm{Cl}_T(L_1) \cap \mathrm{Cl}_T(L_2)$ *for all* $L_1, L_2 \in \boldsymbol{Z}$,

(iii) *Every point of T is the limit of a unique $\boldsymbol{Z}$-ultrafilter on X.*

LEMMA 2.8. *Let X be a dense subspace of a Tychonoff space Y and let $\boldsymbol{Z}$ be a normal base on X. If*

(i) *$\{Cl_Y(L): L \in \boldsymbol{Z}\}$ is a separating family of closed sets in Y such that for $y_1, y_2 \in Y$, $y_1 \neq y_2$ there exist L_1, L_2 in $\boldsymbol{Z}$ satisfying $y_i \in Cl_Y(L_i)$, $i = 1,2$ and $L_1 \cap L_2 = \emptyset$ and*

$$\text{(ii)} \bigcap_{i=1}^{n} Cl_Y(L_i) = Cl_Y\left(\bigcap_{i=1}^{n} L_i\right) L_i \in \boldsymbol{Z},\ 1 \leqq i \leqq n,$$

then Y is homeomorphic to a subspace of $W(\boldsymbol{Z})$.

PROOF. For each $y \in Y$, define $\boldsymbol{U}_y = \{L \in \boldsymbol{Z}:\ y \in Cl_Y(L)\}$. Then, from (i) and (ii) it easily follows that $\boldsymbol{U}_y$ is a $\boldsymbol{Z}$-ultrafilter on Y. Define $f\colon Y \to W(\boldsymbol{Z})$ by $f(y) = \boldsymbol{U}_y$, for each $y \in Y$. Then f is clearly one-to-one. To show that f is continuous, we must show that if $f(y) \notin f(A)^-$ then $y \notin A^-$, $y \in Y$, $A \subset Y$. If $f(y) \notin f(A)^-$, then there exists an $L_1 \in \boldsymbol{Z}$ such that L_1 absorbs $f(A)$ but does not belong to $f(y)$. This shows that $A^- \subset Cl_Y(L_1)$ and $y \notin Cl_Y(L_1)$ i.e. $y \notin A^-$. Finally we prove that f is closed. Suppose $\boldsymbol{U}_y \in f(X) \cap f(A)^-$. Then every L

which absorbs $f(A)$ also belongs to $\boldsymbol{U}_y$ i.e. $y \in Cl_Y(L)$ for each $L \in \boldsymbol{Z}$ such that $A \subset Cl_Y(L)$. If $y \notin A^-$, then from (i) there exists an $L_2 \in \boldsymbol{Z}$ such that $y \in Y - Cl_Y(L_2) \subset Y - A^-$. Hence $A \subset Cl_Y(L_2)$ and $y \notin Cl_Y(L_2)$, a contradiction. Hence f is closed, showing that Y is homeomorphic to $f(X) \subset W(\boldsymbol{Z})$.

From now onwards to the end of this section, we assume that $\boldsymbol{Z}$ is a countably productive (c.p.) normal base on a Tychonoff space X. The following result corresponds to Theorem 8.6, page 117 of Gillman and Jerison [10].

LEMMA 2.9. *Let $\boldsymbol{Z}$ be a c.p. normal base on X and let $X \subset T \subset W(\boldsymbol{Z})$. Then the following are equivalent*:

(i) $\bigcap_{n=1}^{\infty} L_n = \varnothing$ *implies* $\bigcap_{n=1}^{\infty} Cl_T(L_n) = \varnothing$, $L_n \in \boldsymbol{Z}$.

(ii) $\bigcap_{n=1}^{\infty} Cl_T(L_n) = Cl_T\left(\bigcap_{n=1}^{\infty} L_n\right)$, $L_n \in \boldsymbol{Z}$.

(iii) *Every point of T is the limit of a unique real $\boldsymbol{Z}$-ultrafilter on X (i.e. a $\boldsymbol{Z}$-ultrafilter with c.i.p.).*

(iv) $X \subset T \subset \eta(\boldsymbol{Z})$.

PROOF. Obviously (iii) is equivalent to (iv) and (ii) implies (i). We now prove that (i) implies (iii). By Lemma 2.7, every point t of T is the limit of a unique $\boldsymbol{Z}$-ultrafilter $\boldsymbol{F}$. $\boldsymbol{F}$ is real since if $L_n \in \boldsymbol{F}$ and $\bigcap_{n=1}^{\infty} L_n = \varnothing$, then $\bigcap_{n=1}^{\infty} Cl_T(L_n) = \varnothing$. But $t \in \bigcap_{n=1}^{\infty} Cl_T(L_n)$, a contradiction. Finally (iv) implies (ii) follows from Theorem 1 of Alò and Shapiro [4].

3. Hausdorff Wallman Compactifications

Let $\boldsymbol{Z}$ be a normal base on a Tychonoff space X and let $\delta = \delta(\boldsymbol{Z})$ be the induced EF-proximity on X. We now use Theorem 2.5, in conjunction with the well-known results in EF-proximity spaces, to obtain some of the recent results in Wallman compactifications.

Let $\boldsymbol{Z}$, $\boldsymbol{Z}'$ be two normal bases on X. Then $\boldsymbol{Z}$ is said to *separate* $\boldsymbol{Z}'$ iff $L_1', L_2' \in \boldsymbol{Z}'$, $L_1' \cap L_2' = \varnothing$ implies that there are $L_1, L_2 \in \boldsymbol{Z}$ such that $L_1' \subset L_1$, $L_2' \subset L_2$ and $L_1 \cap L_2 = \varnothing$. Clearly $\boldsymbol{Z}$ separates $\boldsymbol{Z}'$ iff $\delta(\boldsymbol{Z}') < \delta(\boldsymbol{Z})$. In the theory of EF-proximity spaces it is known that there is a one-to-one order isomorphism between EF-proximities on X and the corresponding Smirnov compactifications. Theorem 2.5 together with this result, gives the following:

THEOREM 3.1. (Steiner and Steiner [16]). *If $\boldsymbol{Z}$, $\boldsymbol{Z}'$, are two normal bases on X then $W(\boldsymbol{Z}') \leqq W(\boldsymbol{Z})$ if and only if $\boldsymbol{Z}$ separates $\boldsymbol{Z}'$.*

COROLLARY 3.2 (Steiner [15]) *$W(\boldsymbol{Z}) = W(\boldsymbol{Z}')$, if and only if $\boldsymbol{Z}$, $\boldsymbol{Z}'$ mutually separate each other.*

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We now recall some of the known results (see Alò and Shapiro [2], Banaŝchewski [6]).

LEMMA 3.3. *Let* $\mathbf{Z}$ *be a normal base on* X. *Then*

(a) $Cl_{W(\mathbf{Z})}(L_1 \cap L_2) = Cl_{W(\mathbf{Z})}(L_1) \cap Cl_{W(\mathbf{Z})}(L_2)$ *for all* $L_1, L_2 \in \mathbf{Z}$.

(b) $\{Cl_{W(\mathbf{Z})}(L): L \in \mathbf{Z}\}$ *is a base for closed subsets of* $W(\mathbf{Z})$.

(c) For each $y_1, y_2 \in W(\mathbf{Z})$, $y_1 \neq y_2$, *there exist* L_1, L_2 *in* $\mathbf{Z}$ *such that* $y_i \in Cl_{W(\mathbf{Z})}(L_i)$ $i = 1, 2$, *and* $L_1 \cap L_2 = \varnothing$.

(d) $Cl_{W(\mathbf{Z})}(L_1 \cup L_2) = Cl_{W(\mathbf{Z})}(L_1) \cup Cl_{W(\mathbf{Z})}(L_2)$ *for all* L_1, L_2 *in* $\mathbf{Z}$.

(e) *For each* $p \in W(\mathbf{Z})$ *and for each neighbourhood* V *of* p, *there exists an* L *in* $\mathbf{Z}$ *such that* $p \in Cl_{W(\mathbf{Z})}(L) \subset V$.

We now find necessary and sufficient conditions for a Hausdorff compactification Y of a space X to be Wallman.

THEOREM 3.4. *A necessary and sufficient condition for a Hausdorff compactification* Y *of* X *to be Wallman is that* X *has a normal base* $\mathbf{Z}$ *such that*

(i) $Cl_Y(L_1 \cap L_2) = Cl_Y(L_1) \cap Cl_Y(L_2)$ *for all* L_1, L_2 *in* $\mathbf{Z}$, *and*

(ii) *for* $y_1, y_2 \in Y$, $y_1 \neq y_2$, *there exist* L_1, L_2 *in* $\mathbf{Z}$ *such that* $y_i \in Cl_Y(L_i)$ $i = 1, 2$ *and* $L_1 \cap L_2 = \varnothing$.

PROOF. Since the necessity is obvious from Lemma 3.3, we need prove only the sufficiency. The Hausdorff compactification Y of X induces an EF-proximity δ_0 on X given by: $A \, \delta_0 \, B$ iff $Cl_Y(A) \cap Cl_Y(B) \neq \varnothing$. Condition (i) implies that $\delta_0 > \delta(\mathbf{Z})$ and hence there is a continuous function f from Y onto $W(\mathbf{Z})$ which is the unique extension of the identity map. The proof is complete if we show that f is one-to-one. This follows from (ii), since $y_1, y_2 \in Y$, $y_1 \neq y_2$ implies there exist L_1, L_2 in $\mathbf{Z}$ such that $L_1 \in f(y_1)$ but not $f(y_2)$ and $L_2 \in f(y_2)$ but not $f(y_1)$.

COROLLARY 3.5. (Banaŝchewski). *Theorem* 3.4 *is true if* (ii) *is replaced by*: $\{Cl_Y(L): L \in \mathbf{Z}\}$ *is a base for closed subsets of* X.

COROLLARY 3.6. (Alò and Shapiro). *Theorem* 3.4 *is true if* (ii) *is replaced by*: *for each* $p \in Y$ *and each neighbourhood* V *of* p, *there exists an* L *in* $\mathbf{Z}$ *such that* $p \in Cl_Y(L) \subset V$.

COROLLARY 3.7. (Njåstad). *A Hausdorff compactification* Y *of* X *is Wallman if and only if the corresponding EF-proximity has a productive base of closed sets.*

A family $\hat{\mathbf{Z}}$ of closed sets in Y has the *trace property* w.r.t. X iff

$$\bigcap_{i=1}^{n} \{L_i: L_i \in \hat{\mathbf{Z}}\} \neq \varnothing \text{ implies that } \left(\bigcap_{i=1}^{n} L_i \cap X\right) \neq \varnothing.$$

If Y is homeomorphic to $W(\mathbf{Z})$, then $\hat{\mathbf{Z}} = \{Cl_Y(L): L \in \mathbf{Z}\}$ has the trace property w.r.t. X and is a normal base in Y. Hence

COROLLARY 3.8. (Steiner) *A Hausdorff compactification Y of X is Wallman if and only if Y has a normal base $\hat{\mathbf{Z}}$ with the trace property w.r.t. X.*

4. Wallman Realcompactifications

Although the space $W(\mathbf{Z})$ corresponding to a normal base $\mathbf{Z}$ on X is always compact, in general, the space $\eta(\mathbf{Z})$ corresponding to a c.p. normal base $\mathbf{Z}$ need not be realcompact (see Steiner and Steiner [14]). Alò and Shapiro [4] have however shown that $\eta(\mathbf{Z})$ is $\mathbf{Z}^*$-realcompact ($\mathbf{Z}^* = \{Cl_{\eta(\mathbf{Z})}(L): L \in \mathbf{Z}\}$) i.e. every real $\mathbf{Z}^*$-ultrafilter on $\eta(\mathbf{Z})$ converges. They have also shown that in the special case when $\mathbf{Z} = Z(X)$, $\eta(\mathbf{Z})$ is the Hewitt realcompactification of X. This raises the problem of finding necessary and sufficient conditions on a c.p. normal base $\mathbf{Z}$ on X for $\eta(\mathbf{Z})$ to be realcompact. In this section we solve this problem and show that our result is an improvement of a result (sufficient conditions for $\eta(\mathbf{Z})$ to be realcompact) of Alò and Shapiro [5], Steiner and Steiner [14].

In this connection we shall find the notion of "Q-closure" due to Mrowka [12] useful. The *Q-closure* of a nonempty subset A of X is the set of all $p \in X$ such that every G_δ set containing p intersects A. It is known that the Q-closure of a subset is always realcompact. Alò and Shapiro [4] have shown that for a c.p. normal base $\mathbf{Z}$ on X, if X^Q denotes the Q-closure of X in $W(\mathbf{Z})$, then

LEMMA 4.1. $X \subset \eta(\mathbf{Z}) \subset X^Q \subset W(\mathbf{Z})$.
(We are supposing $X \subset \eta(\mathbf{Z})$ via the homeomorphism).

LEMMA 4.2. *$\eta(\mathbf{Z})$ is realcompact if and only if $\eta(\mathbf{Z}) = X^Q$.*
We now prove the main result of this section.

THEOREM 4.3. *$\eta(\mathbf{Z})$ is realcompact if and only if* (R) $\bigcap_{n=1}^{\infty} Cl_{X^Q}(L_n) = Cl_{X^Q}(\bigcap_{n=z1}^{\infty} L_n)$, $L_n \in \mathbf{Z}$.

PROOF. If $\mathbf{Z}$ satisfies (R), then from Lemma 2.9, $X^Q \subset \eta(\mathbf{Z})$. Hence Lemmas 4.1, 4.2 together show that $\eta(\mathbf{Z})$ is realcompact. Conversely, if $\eta(\mathbf{Z})$ is realcompact then $\eta(\mathbf{Z}) = X^Q$ (4.2) and hence by Lemma 2.9 the condition (R) holds.

We now show that the above theorem implies a result of Steiner and Steiner [14]; a similar result has been obtained by Alò and Shapiro [5]. A sequence of sets $\{L_n\}$ in $\mathbf{Z}$ is a *nest* iff there is a sequence $\{L'_n\}$ in $\mathbf{Z}$ such that

$$X - L'_{n+1} \subset L_{n+1} \subset X - L'_n \subset L_n, \qquad n \in N.$$

$\mathbf{Z}$ is *nest generated* iff for each $L \in \mathbf{Z}$, there is a nest $\{L_n\}$ such that $L = \bigcap_{n=1}^{\infty} L_n$. We note here that the concept of a "strong delta normal base" $\mathbf{Z}$ introduced by

Alò and Shapiro in [5] is the same as that of a "nest generated c.p. normal base" $\boldsymbol{Z}$. In [5] and [14] it is shown that if $\boldsymbol{Z}$ is a nest generated c.p. normal base on X, then $\eta(\boldsymbol{Z})$ is realcompact. Theorem 4.3, together with the following lemma implies this result.

LEMMA 4.4. *If $\boldsymbol{Z}$ is a nest generated c.p. normal base on X, then*

$$\bigcap_{n=1}^{\infty} Cl_{X^Q}(L_n) = Cl_{X^Q} \bigcap_{n=1}^{\infty} L_n$$

for all $L_n \in \boldsymbol{Z}$.

PROOF. In view of Lemma 2.9, it is sufficient to prove that

$$\bigcap_{n=1}^{\infty} L_n = \varnothing \text{ implies } \bigcap_{n=1}^{\infty} Cl_X Q(L_n) = \varnothing.$$

If, on the contrary, $p \in \bigcap_{n=1}^{\infty} Cl_{X^Q}(L_n)$, then from Theorem 2.2 of Steiner and Steiner [14], each $L_n = Z_n \cap X$ and $Cl_{W(\boldsymbol{Z})}(L_n) \subset Z_n$ for some $Z_n \in Z(W(\boldsymbol{Z}))$. Then $Z = \bigcap_{n=1}^{\infty} Z_n$ is a G_δ set in $W(\boldsymbol{Z})$ and belongs to $Z(W(\boldsymbol{Z}))$. (Gillman and Jerison [10]). For each $n \in N$,

$$p \in \bigcap_{n=1}^{\infty} Cl_{X^Q}(L_n) \subset \bigcap_{n=1}^{\infty} Z_n = Z.$$

Since $p \in X^Q$, $Z \cap X \neq \varnothing$ which implies that $\bigcap_{n=1}^{\infty} L_n \neq \varnothing$.

Corollary 3.2 provides a motivation for the next result.

THEOREM 4.5. *Suppose $\boldsymbol{Z}$ and $\boldsymbol{Z}'$ are two c.p. normal bases on a Tychonoff space X. Then $\eta(\boldsymbol{Z})$ is homeomorphic to $\eta(\boldsymbol{Z}')$ if and only if*

$$\bigcap_{n=1}^{\infty} Cl_T(F_n) = Cl_T\left(\bigcap_{n=1}^{\infty} F_n\right) \textit{for}$$

(i) $T = \eta(\boldsymbol{Z})$, $F_n \in \boldsymbol{Z}'$, *and*

(ii) $T = \eta(\boldsymbol{Z}')$, $F_n \in \boldsymbol{Z}$.

PROOF. Necessity is trivial and we need prove only sufficiency. We first note that by Theorem 3.2 the given conditions imply that $W(\boldsymbol{Z})$ is homeomorphic to $W(\boldsymbol{Z}')$ and so $\eta(\boldsymbol{Z})$ (resp. $\eta(\boldsymbol{Z}')$) is homeomorphic to a subspace of $W(\boldsymbol{Z}')$ (resp. $W(\boldsymbol{Z})$). Hence by Theorem 2.9, for each $p \in \eta(\boldsymbol{Z})$, there exists a $q \in \eta(\boldsymbol{Z}')$ which converges to p. But this means that p also converges to q as the following argument shows. If p does not converge to q, then q is not a cluster point of p. Then there exists an $L_1 \in p$ such that $q \notin Cl_{\eta(\boldsymbol{Z})}(L_1)$ and hence there exists a $G_1 \in \boldsymbol{Z}'$ such that $L_1 \subset G_1$ and $G_1 \notin q$ i.e. $G_1 \supset L_1$ and $G_1 \cap G_2 = \varnothing$ for some $G_2 \in q$. But $L_1 \in p$ implies

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$$p \in Cl_{\eta(\mathbf{Z})}(L_1) \subset Cl_{\eta(\mathbf{Z})}(G_1)$$

and also $p \in Cl_{\eta(\mathbf{Z})}(G_2)$ since q converges to p and $G_2 \in q$. Hence

$$Cl_{\eta(\mathbf{Z})}(G_1) \cap Cl_{\eta(\mathbf{Z})}(G_2) = Cl_{\eta(\mathbf{Z})}(G_1 \cap G_2) \neq \emptyset$$

a contradiction. So we define $f\colon \eta(\mathbf{Z}) \to \eta(\mathbf{Z}')$ by $f(p) = q$ for each $p \in \eta(\mathbf{Z})$ iff q converges to p. Then f is well defined, one-to-one and onto $\eta(\mathbf{Z}')$. In view of symmetry, the proof is complete if we show that f is continuous. Suppose $p \in \eta(\mathbf{Z})$, $A \subset \eta(\mathbf{Z})$, $f(p) = q \notin Cl_{\eta(\mathbf{Z}')} f(A)$. Then there is a $G \in \mathbf{Z}'$ such that G absorbs $f(A)$ but $G \notin q$. Then $A \subset Cl_{\eta(\mathbf{Z})}(G)$ and $p \notin Cl_{\eta(\mathbf{Z})}(G)$ and hence $p \notin Cl_{\eta(\mathbf{Z})}(A)$, showing that f is continuous.

Corollary 4.6. *$\eta(\mathbf{Z})$ is homeomorphic to vX, the Hewitt realcompactification of X if and only if*

$$Cl_T\left(\bigcap_{n=1}^{\infty} F_n\right) = \bigcap_{n=1}^{\infty} Cl_T(F_n)$$

for (i) $T = \eta(\mathbf{Z})$, $F_n \in Z(X)$
(ii) $T = vX$, $F_n \in \mathbf{Z}$.

We next consider the problem: if Y is a $\mathbf{Z}^*$-realcompactification of X, then what are the necessary and sufficient conditions for Y to be homeomorphic to $\eta(\mathbf{Z})$? Motivation for the solution of this problem is provided by the results (proved in Section 3) of Banasĉhewski (Corollary 3.5), Njåstad (Corollary 3.7) and Steiner (Corollary 3.8) concerning Wallman compactifications.

Theorem 4.7. (*cf.* 3.5). *For a c.p. normal base $\mathbf{Z}$ on X let Y be a $\mathbf{Z}^*$-realcompactification of X. Then Y is homeomorphic to $\eta(\mathbf{Z})$ if and only if*

(i) $Cl_Y(\bigcap_{n=1}^{\infty} L_n) = \bigcap_{n=1}^{\infty} Cl_Y(L_n)$, *for all $L_n \in \mathbf{Z}$ and*

(ii) $\{Cl_Y(L)\colon L \in \mathbf{Z}\}$ *is a separating family of closed sets in Y such that for y_1, y_2 in Y, $y_1 \neq y_2$, there exist L_i in $\mathbf{Z}$ satisfying $y_i \in Cl_Y(L_i)$, $i = 1,2$ and $L_1 \cap L_2 = \emptyset$.*

Proof. Necessity follows easily from Lemma 2.9. To prove the sufficiency we first note that by Lemma 2.8, Y is homeomorphic to a subspace of $W(\mathbf{Z})$. Hence from Lemma 2.9, each $y \in Y$ is a limit of a unique real $\mathbf{Z}$-ultrafilter $\mathbf{F}_y \in \eta(\mathbf{Z})$. Also for each $\mathbf{F} \in \eta(\mathbf{Z})$, $\{Cl_Y(F)\colon F \in \mathbf{F}\}$ is a real $\mathbf{Z}^*$-ultrafilter in Y in view of(i), and as Y is $\mathbf{Z}^*$-realcompact, it must converge to some point in Y. Hence the map $f\colon Y \to \eta(Z)$ defined by $f(y) = \mathbf{F}_y$, where $\mathbf{F}_y$ is the real $\mathbf{Z}$-ultrafilter converging to y, is well defined, one-to-one and onto. Continuity of f is proved as in the proof of Theorem 4.5. Finally, we prove that f is closed. If $y \in Y$, $f(y) = \mathbf{F}_y \in Cl_{\eta(\mathbf{Z})} f(A)$, $A \subset Y$ then every $L \in \mathbf{Z}$ which absorbs $f(A)$ belongs to $\mathbf{F}_y$ i.e. $y \in Cl_Y(L)$ for each $L \in \mathbf{Z}$ such that $A \subset Cl_Y(L)$ and hence from (ii), $y \in Cl_Y(A)$ i.e. f is closed.

COROLLARY 4.8 (cf. 3.7). *A $\boldsymbol{Z}^*$-realcompactification Y of X is homeomorphic to $\eta(\boldsymbol{Z})$ if and only if*

(i) $Cl_Y(\bigcap_{n=1}^{\infty} L_n) = \bigcap_{n=1}^{\infty} Cl_Y(L_n)$, $L_n \in \boldsymbol{Z}$, *and*

(ii) *there exists a c.p. normal base $\boldsymbol{A}$ for closed sets in Y such that $\boldsymbol{Z} = \boldsymbol{A} \cap X$.*

Let Y be called a *Wallman realcompactification* of X iff Y is homeomorphic to $\eta(\boldsymbol{Z})$ for some c.p. normal base $\boldsymbol{Z}$ on X. We now prove an analogue of Steiner's result.

THEOREM 4.9. (c.f. 3.8). *Y is a Wallman realcompactification of X if and only if*

(i) *Y possesses a c.p. normal base $\boldsymbol{A}$ with the trace property w.r.t. X and*

(ii) *Y is $(\boldsymbol{A} \cap X)^*$-realcompact. If Y has such an $\boldsymbol{A}$, then Y is homeomorphic to $\eta(\boldsymbol{A} \cap X)$.*

PROOF. If Y is homeomorphic to $\eta(\boldsymbol{Z})$ then setting $\boldsymbol{A} = \{Cl_Y(L): L \in \boldsymbol{Z}\}$ the necessity follows from Lemma 2.9. Conversely, if such an $\boldsymbol{A}$ exists, we set $\boldsymbol{Z} = \boldsymbol{A} \cap X$. Obviously $\boldsymbol{Z}$ is a separating base. We prove that $\boldsymbol{Z}$ is normal. For $A_1, A_2 \in \boldsymbol{A}$, $(A_1 \cap X) \cap (A_2 \cap X) = \varnothing$ implies $A_1 \cap A_2 = \varnothing$. Since $\boldsymbol{A}$ is normal there are B_1, B_2 in $\boldsymbol{A}$ such that $A_i \subset Y - B_i$, $i = 1,2$, $B_1 \cup B_2 = Y$. Hence $A_i \cap X \subset X - (B_i \cap X)$, $i = 1,2$ and $(B_1 \cap X) \cup (B_2 \cap X) = X$. Finally as $\boldsymbol{A}$ is closed under countable intersections so is $\boldsymbol{Z}$ and since $Cl_Y(A \cap X) = A$ for all A in $\boldsymbol{A}$, conditions of Theorem 4.7 are satisfied. Thus Y is homeomorphic to $\eta(\boldsymbol{Z})$.

The following is an analogue of Steiner's Theorem 4 in [15] and follows easily from the above theorem. See also Alò and Shapiro [3], [5].

THEOREM 4.10. *If Y possesses a c.p. normal base $\boldsymbol{A}$ of regular closed sets, then Y is a Wallman realcompactification of each of its dense subspace, X for which Y is $(\boldsymbol{A} \cap X)^*$-realcompact.*

In conclusion, we would like to mention that we have not been able to determine whether an analogue of Corollary 3.6 holds true for Wallman realcompactifications.

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