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Closure of singular foliations: the proof of Molino's conjecture

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Abstract

In this paper we prove the conjecture of Molino that for every singular Riemannian foliation $(M,\mathcal{F})$, the partition $\overline{\mathcal{F}}$ given by the closures of the leaves of $\mathcal{F}$ is again a singular Riemannian foliation.

1. Introduction

Given a Riemannian manifold M, a singular Riemannian foliation $\mathcal{F}$ on M is, roughly speaking, a partition of M into smooth connected and locally equidistant submanifolds of possibly varying dimension (the *leaves* of $\mathcal{F}$), which is spanned by a family of smooth vector fields. The precise definition, given in § 2, was suggested by Molino, by combining the concepts of *transnormal system* of Bolton [Bol73] and of *singular foliation* by Sussmann [Sus73].

A typical example of a singular Riemannian foliation is the decomposition of a Riemannian manifold M into the orbits of an isometric group action G on M. Such a foliation is called *homogeneous*. Another example of a foliation is given by the partition of a Euclidean vector bundle $E \to L$, endowed with a metric connection, into the *holonomy tubes* around the zero section (cf. Example 2.7). Such a foliation, which we call a *holonomy foliation*, will be a sort-of prototype in the structural results that will appear later on. Holonomy foliations are in general not homogeneous (the zero section L is always a leaf but in general not a homogeneous manifold); however, they are *locally homogeneous*, in the sense that the infinitesimal foliation at every point of E is homogeneous (cf. §§ 2.3 and 2.4). This construction is related to other important types of foliations, like polar foliations [Toe06] or Wilking's dual foliation to the Sharafutdinov projection [Wil07]; see Remark 2.8.

In general, the leaves of a singular Riemannian foliation might not be closed, even in the simple cases defined above. In the homogeneous case, consider for example the foliation on the flat torus T^2 by parallel lines, of irrational slope. These are non-closed orbits of an isometric $\mathbb{R}$-action on T^2.

Given a (regular) Riemannian foliation $(M,\mathcal{F})$ with non-closed leaves, Molino proved that replacing the leaves of $\mathcal{F}$ with their closure yields a new singular Riemannian foliation $\overline{\mathcal{F}}$ (cf. [Mol88, ch. 5]). Moreover, he conjectured that the same result should hold true if one starts with a singular Riemannian foliation and this has become known, in recent decades, as *Molino's conjecture*.

Molino proved that the closure $\overline{\mathcal{F}}$ of a singular Riemannian foliation $(M,\mathcal{F})$ is a transnormal system [Mol88], thus leaving to prove that it is a singular foliation as well. Moreover, in [Mol94]

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he suggested a strategy to prove the conjecture for the case of *orbit-like foliations*, i.e. foliations which, roughly speaking, are locally diffeomorphic to the orbits of some proper isometric group action around each point (cf. § 2.4). A formal alternative proof in this case can be found in [AR17]. Molino's conjecture was also proved for *polar foliations* and then *infinitesimally polar foliations* in [Ale06] and [AL11], respectively.

These partial results do not cover every possible foliation. Since the 1980s there are examples of non-orbit-like foliations and, in recent years, there was shown the existence of a remarkably large class of 'infinitesimal' foliations that are neither homogeneous nor polar, the so-called Clifford foliations [Rad14] (these infinitesimal foliations have been shown, however, to have an algebraic nature; cf. [LR16]). Therefore, it is important to give a complete answer to the conjecture, to fully understand the semi-local dynamic of singular Riemannian foliations.

The goal of this paper is to prove the full Molino's conjecture.

Theorem (Molino's conjecture). *Let $(M,\mathcal{F})$ be a singular Riemannian foliation on a complete manifold M and let $\overline{\mathcal{F}}=\{\overline{L}\mid L\in\mathcal{F}\}$ be the partition of M into the closures of the leaves of $\mathcal{F}$. Then $(M,\overline{\mathcal{F}})$ is a singular Riemannian foliation.*

This result is in fact a direct consequence of the following.

Main theorem. *Let $(M,\mathcal{F})$ be a singular Riemannian foliation, let L be a (possibly not closed) leaf, and let U be an ϵ-neighbourhood around the closure of L. Then for ϵ small enough, there are a metric g^{ℓ} on U and a singular foliation $\widehat{\mathcal{F}}^{\ell}$ such that:*

(1) *$(U,\mathrm{g}^{\ell},\widehat{\mathcal{F}}^{\ell})$ is an orbit-like singular Riemannian foliation;*
(2) *the foliation $\widehat{\mathcal{F}}^{\ell}$ coincides with $\mathcal{F}$ on $\overline{L}$;*
(3) *the closure of $\widehat{\mathcal{F}}^{\ell}$ is contained in the closure of $\mathcal{F}$.*

In short, the foliation $\widehat{\mathcal{F}}^{\ell}$ is obtained by first constructing the *linearized foliation* $\mathcal{F}^{\ell}$ of $\mathcal{F}$ in U, which is a subfoliation of $\mathcal{F}$ spanned by the first-order approximations, around L, of the vector fields tangent to $\mathcal{F}$ (see § 2.5 for a precise definition). The foliation $\widehat{\mathcal{F}}^{\ell}$ is then obtained from $\mathcal{F}^{\ell}$ by taking the 'local closure' of the leaves of $\mathcal{F}^{\ell}$. The foliations $\mathcal{F},\mathcal{F}^{\ell},\widehat{\mathcal{F}}^{\ell}$, together with their closures, are then related by the following inclusions:

$$\begin{array}{ccccc}\mathcal{F} & \supseteq & \mathcal{F}^{\ell} & \subseteq & \widehat{\mathcal{F}}^{\ell}\\ \cap\!\!\!\!\!\!\;\vert & & \cap\!\!\!\!\!\!\;\vert & & \cap\!\!\!\!\!\!\;\vert\\ \overline{\mathcal{F}} & \supseteq & \overline{\mathcal{F}^{\ell}} & = & \overline{\widehat{\mathcal{F}}^{\ell}}.\end{array}$$

Example 1.1. Consider a Euclidean vector bundle E over a complete Riemannian manifold L, with a metric connection ∇^E and a connection metric g^E (cf. Example 2.7). Let H_p denote the holonomy group of (E,∇^E) at p, acting by isometries on the Euclidean fiber E_p, and let $(E_p,\mathcal{F}^0_p)$ be a singular Riemannian foliation preserved by the H_p-action. Finally, let K_p be the maximal connected group of isometries of E_p that fixes each leaf of $\mathcal{F}^0_p$ as a set.

Letting $\mathcal{F}$ be the partition of E into the holonomy translates of the leaves of $\mathcal{F}^0_p$ (i.e. for every leaf $\mathcal{L}\in\mathcal{F}^0_p$, $L_{\mathcal{L}}$ denotes the set of points in E that can be reached via ∇^E-parallel translation from a point in $\mathcal{L}$), then $\mathcal{F}$ is a singular Riemannian foliation. In this case, the linearized foliation $\mathcal{F}^{\ell}$ is the foliation by the holonomy translates of the K_p-orbits in E_p, and the local closure of $\widehat{\mathcal{F}}^{\ell}$ is the foliation by the holonomy translates of the $\overline{K}_p$-orbits in E_p, where $\overline{K}_p$ denotes the closure of K_p in $\mathsf{O}(E_p)$.

This can be restated in the language of groupoids: defining H as the holonomy groupoid of the connection ∇^E, then $\mathcal{F} = \{H(\mathcal{L}) \mid \mathcal{L} \in \mathcal{F}_p^0\}$, $\mathcal{F}^\ell$ is given by the orbits of HK_p, and its local closure $\hat{\mathcal{F}}^\ell$ is given by the orbits of $H\overline{K}_p$.

This paper is organized as follows: after a section of preliminaries (§ 2) we show how Molino's conjecture follows from the main theorem (§ 3). In § 4 we fix the setup in which we work for the rest of the paper. In § 5 we define three distributions of the tangent bundle TU. We first use these to obtain information on the local structure of $\mathcal{F}^\ell$ and define the local closure $\widehat{\mathcal{F}}^\ell$ (§ 6) and then to define the metric g^ℓ used in the main theorem (§ 7). In this final section we also prove the main theorem.

2. Preliminaries

Given a Riemannian manifold (M, g), a partition $\mathcal{F}$ of M into complete connected submanifolds (the *leaves* of $\mathcal{F}$) is called a *transnormal system* if geodesics starting perpendicular to a leaf stay perpendicular to all leaves, and a *singular foliation* if every vector tangent to a leaf can be locally extended to a vector field everywhere tangent to the leaves.

A singular Riemannian foliation will be denoted by the triple $(M, \mathrm{g}, \mathcal{F})$. However, if the Riemannian metric of M is understood, we will drop it and simply write $(M, \mathcal{F})$.

The following notation will be used throughout the rest of the paper. Given a point $p \in M$, the leaf of $\mathcal{F}$ through p will be denoted by L_p. A small relatively compact open subset $P \subset L$ is called a *plaque*. The tangent and normal spaces to L_p at p are denoted by T_pL_p and ν_pL_p, respectively. Given some $\epsilon > 0$, $\nu_p^\epsilon L_p$ denotes the set of vectors $x \in \nu_pL_p$ with norm $< \epsilon$. If ϵ is small enough that the normal exponential map $\exp : \nu_p^\epsilon L_p \to M$ is a diffeomorphism onto the image, such image is called a *slice* of L_p at p and it is denoted by S_p. The *slice foliation* $\mathcal{F}|_{S_p}$ denotes the partition of S_p into the connected components of the intersections $L \cap S_p$, where $L \in \mathcal{F}$.

2.1 Vector fields of a singular Riemannian foliation

We review here the main notations about vector fields of a singular Riemannian foliation.

A vector field V is called *vertical* if it is tangent to the leaves at each point. The set of smooth vertical vector fields is a Lie algebra, which is denoted by $\mathfrak{X}(M, \mathcal{F})$.

A vector field X is called *foliated* if its flow takes leaves to leaves or, equivalently, if $[X, V] \in \mathfrak{X}(M, \mathcal{F})$ for every $V \in \mathfrak{X}(M, \mathcal{F})$. Any vertical vector field is foliated, but there are other foliated vector fields. A vector field is called *basic* if it is both foliated and everywhere normal to the leaves.

2.2 Homothetic transformation lemma

One of the most fundamental results in the theory of singular Riemannian foliations is the homothetic transformation lemma. A deeper discussion of this lemma, with proof and applications, can be found in Molino [Mol88, ch. 6], in particular Lemma 6.1 and Proposition 6.7.

Let $(M, \mathcal{F})$ be a singular foliation, let L be a leaf of $\mathcal{F}$, and let $P \subset L$ be a plaque. Let $\epsilon > 0$ be such that the normal exponential map $\exp : \nu^\epsilon P \to M$ is a diffeomorphism onto its image $B_\epsilon(P)$. For any two radii $r_1, r_2 = \lambda r_1$ in $(0, \epsilon)$, it makes sense to define the *homothetic transformation*

$$h_\lambda : B_{r_1}(P) \to B_{r_2}(P), \quad h_\lambda(\exp v) = \exp \lambda v.$$

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The leaves of $\mathcal{F}$ intersect $B_{r_i}(P)$, $i=1,2$, in plaques that foliate $B_{r_i}(P)$. We call $\mathcal{F}|_{B_{r_i}(P)}$ the foliation of $B_{r_i}(P)$ into the path components of such intersections. One then has the following result.

Theorem 2.1 (Homothetic transformation lemma). *The homothetic transformation h_λ takes the leaves of $(B_{r_1}, \mathcal{F}|_{B_{r_1}(P)})$ onto the leaves of $(B_{r_2}, \mathcal{F}|_{B_{r_2}(P)})$.*

This result still holds, more generally, if we replace the plaque P by an open subset B of some submanifold $N \subset M$ which is a union of leaves of the same dimension. In this case, we consider some $\epsilon > 0$ such that $\exp : \nu^\epsilon B \to M$ is a diffeomorphism onto the image $B_\epsilon(B)$, and define the homothetic transformation *around* B, $h_\lambda : B_r(B) \to B_{\lambda r}(B)$, as before. In this case, an analogous version of the homothetic transformation lemma applies.

2.3 Infinitesimal foliation

Let $(M, \mathcal{F})$ be a singular Riemannian foliation, $p \in M$ a point, and S_p a slice at p.

Definition 2.2 (Infinitesimal foliation at p). The *infinitesimal foliation of $\mathcal{F}$ at p*, denoted by $(\nu_p L_p, \mathcal{F}_p)$, is defined as the partition of $\nu_p L_p$ whose leaf at $v \in \nu_p L_p$ is given by

$$L_v = \{ w \in \nu_p L_p \mid \exp_p tw \in L_{\exp_p tv}\ \forall t > 0 \text{ small enough} \},$$

where $L_{\exp_p tv}$ denotes the leaf of $(S_p, \mathcal{F}|_{S_p})$ through $\exp_p tv$.

The leaf L_v is well defined because, by the homothetic transformation lemma, if $\exp_p t_0 w$ belongs to the same leaf of $\exp_p t_0 v$ for some small t_0, then $\exp_p tw$ belongs to the same leaf of $\exp_p tv$ for every $t \in (0, t_0)$. In the following proposition we collect the important facts about infinitesimal foliations that we will need.

Theorem 2.3. *Given a singular Riemannian foliation $(M, \mathcal{F})$ and a point $p \in M$ with infinitesimal foliation $(\nu_p L_p, \mathcal{F}_p)$, then:*

(1) *the foliation $(\nu_p L_p, \mathcal{F}_p)$ is a singular Riemannian foliation with respect to the flat metric g_p at p;*
(2) *the normal exponential map $\exp_p : \nu_p^\epsilon L_p \to M$ sends the leaves of $\mathcal{F}_p$ to the leaves of $(S_p, \mathcal{F}|_{S_p})$;*
(3) *$(\nu_p L_p, \mathcal{F}_p)$ is invariant under rescalings $r_\lambda : \nu_p L_p \to \nu_p L_p$, $r_\lambda(v) = \lambda v$.*

Proof. (1) [Mol88, Proposition 6.5].

(2) Follows from the definitions of infinitesimal foliation and of slice foliation.

(3) Via the exponential map $\exp : \nu^\epsilon L_p \to S_p$, this corresponds to the homothetic transformation lemma on S_p. □

The following fact will become very useful.

Proposition 2.4. *Given singular Riemannian foliations $(M, \mathcal{F})$, $(M', \mathcal{F}')$ and a foliated diffeomorphism $\phi : U \to U'$, between open sets U, U' of M, M' respectively, sending a point $p \in U$ to $p' \in U'$, the differential of ϕ induces a linear, foliated isomorphism $\phi_* : (\nu_p L_p, \mathcal{F}_p) \to (\nu_{p'} L_{p'}, \mathcal{F}'_{p'})$.*

Proof. By substituting $(M', \mathrm{g}', \mathcal{F}')$ with $(M, \phi^* g', \mathcal{F})$, the problem can be reduced to the case where $M = M'$, $\phi = \mathrm{id}$, $p = p'$, and $\mathcal{F} = \mathcal{F}'$ is a singular Riemannian foliation with respect to

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two metrics, g and $\tilde{\mathrm{g}}$. In the following, we will denote with a 'tilde' ($\tilde{}$) every geometric object related to the metric $\tilde{\mathrm{g}}$, and without the tilde any geometric object related to g.

Let S_p (respectively $\widetilde{S}_p$) denote a slice at p with respect to g (respectively $\tilde{\mathrm{g}}$). Consider the set $\{X_1, \dots, X_k\} \subset \mathfrak{X}(M, \mathcal{F})$, $k = \dim L_p$, of vector fields such that $\{X_1(p), \dots, X_k(p)\}$ is a basis of T_pL_p. Denote by Φ_i^t the flow of X_i and define $\Phi_{(t_1,\dots,t_k)} = \Phi_k^{t_k} \circ \cdots \circ \Phi_1^{t_1}$.

Around p, both S_p and $\widetilde{S}_p$ are transverse to $\mathrm{span}(X_1, \dots, X_k)$ and, up to possibly replacing S_p and $\widetilde{S}_p$ with smaller open subsets, we can assume that for every $q \in S_p$ there exists a unique $\tilde{q} \in \widetilde{S}_p$ of the form $\tilde{q} = \Phi_{(t_1,\dots,t_k)}(q)$. This gives rise to a map $H : S_p \to \widetilde{S}_p$, $H(q) = \tilde{q}$ which is differentiable and, since q and $\tilde{q}$ belong to the same leaf of $\mathcal{F}$, sends the leaves of $\mathcal{F}|_{S_p}$ to the leaves of $\mathcal{F}|_{\widetilde{S}_p}$. In other words, there is a foliated diffeomorphism $H : (S_p, \mathcal{F}|_{S_p}) \to (\widetilde{S}_p, \mathcal{F}|_{\widetilde{S}_p})$.

Consider the composition ψ of foliated diffeomorphisms

$$(\nu_p^\epsilon L_p, \mathcal{F}_p) \xrightarrow{\exp_p} (S_p, \mathcal{F}|_{S_p}) \xrightarrow{H} (\widetilde{S}_p, \mathcal{F}|_{\widetilde{S}_p}) \xrightarrow{\widetilde{\exp}_p^{-1}} (\tilde{\nu}_p^\epsilon L_p, \widetilde{\mathcal{F}}_p).$$

For any $\lambda \in (0, 1)$, one can define a new foliated diffeomorphism

$$\psi_\lambda : (\nu_p^{\epsilon/\lambda} L_p, \mathcal{F}_p) \to (\tilde{\nu}_p^{\epsilon/\lambda} L_p, \widetilde{\mathcal{F}}_p), \quad \psi_\lambda(v) = \frac{1}{\lambda}\psi(\lambda v).$$

As $\lambda \to 0$, the maps ψ_λ converge to the differential $d_0\psi$ of ψ at 0. This is an invertible linear map (in particular a diffeomorphism) and, as a limit of foliated maps, it is itself foliated. Therefore, the map

$$\phi_* := d_0\psi : (\nu_p L, \mathcal{F}_p) \longrightarrow (\tilde{\nu}_p L, \widetilde{\mathcal{F}}_p)$$

satisfies the statement of the proposition. □

Remark 2.5. Given a singular Riemannian foliation $(M, \mathcal{F})$ and a submanifold $N \subset M$ which is a union of leaves of the same dimension, the infinitesimal foliation at a point $p \in M$ splits as a product $(\nu_p(L_p, N) \times \nu_p N, \{\text{pts.}\} \times \mathcal{F}_p|_{\nu_p N})$, where $\nu_p(L_p, N) = \nu_p L_p \cap T_p N$. In this case, the foliation $(\nu_p N, \mathcal{F}_p|_{\nu_p N})$ is the 'essential part' of the infinitesimal foliation $(\nu_p L_p, \mathcal{F}_p)$. By abuse of notation, we will call the foliation $(\nu_p N, \mathcal{F}_p|_{\nu_p N})$ the *infinitesimal foliation at* p as well and denote it by $\mathcal{F}_p$.

Given a singular Riemannian foliation $(M, \mathcal{F})$ and a point $p \in M$, the infinitesimal foliation $(\nu_p L_p, \mathcal{F}_p)$ at p contains the origin as a leaf of $\mathcal{F}_p$. Based on this fact, we make the following definition.

Definition 2.6 (Infinitesimal foliation). An *infinitesimal foliation* is a singular Riemannian foliation $(V, \mathcal{F})$ on a Euclidean vector space, with the origin $\{0\}$ being a zero-dimensional leaf.

2.4 Homogeneous and orbit-like foliations

A singular Riemannian foliation $(M, \mathcal{F})$ is called *homogeneous* (sometimes *Riemannian homogeneous*) if there exists a connected Lie group G acting by isometries on M, whose orbits are precisely the leaves of $\mathcal{F}$. Furthermore, a singular Riemannian foliation $(M, \mathcal{F})$ is called *orbit-like* if at every point $p \in M$, the infinitesimal foliation $(\nu_p L_p, \mathcal{F}_p)$ is closed and homogeneous.

Example 2.7 (Holonomy foliations). An example of an orbit-like foliation, which will be useful to keep in mind later on, can be constructed as follows. Consider a Riemannian manifold L and a Euclidean vector bundle E over L, that is, a vector bundle over L with an inner product $\langle\, , \rangle_p$

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on each fiber E_p, $p \in L$. Let ∇^E be a metric connection on E, i.e. a connection on E such that, for every vector field X on L and sections ξ, η of E, one has

$$X\langle \xi, \eta\rangle = \langle \nabla^E_X \xi, \eta\rangle + \langle \xi, \nabla^E_X \eta\rangle.$$

Given (E, ∇^E), there is an induced Riemannian metric g^E on E, called the *connection metric*. Moreover, ∇^E induces a *parallel transport* on E: given $\xi_0 \in E_p$ and a curve $\gamma : [0,1] \to L$ with $\gamma(0) = p$, there exists a unique lift $\xi : [0,1] \to E$ of γ, with $\xi(0) = \xi_0$ such that $\nabla^E_{\gamma'(t)}\xi(t) = 0$ for every $t \in [0,1]$. On E one can now define a foliation $\mathcal{F}^E$, by declaring two vectors $\xi_1, \xi_2 \in E$ in the same leaf if they can be connected to one another via a composition of parallel transports. The leaves of $\mathcal{F}^E$ are usually referred to as the *holonomy tubes* around the zero section $L \subset E$ and they define a singular Riemannian foliation on (E, g^E). Moreover, the infinitesimal foliation at any point of E is homogeneous: in fact, for any point p along the zero section L, one can first construct the holonomy group H_p of the connection ∇^E, which acts by isometries on the fiber E_p in such a way that the orbits of the identity component H_p^0 are precisely the leaves of the infinitesimal foliation of $\mathcal{F}^E$ at p. Similarly, the infinitesimal foliation at a point $\xi \in E_p$ is given by the orbits in $\nu_\xi L_\xi$ of the identity component of the stabilizer $H_\xi \subset H_p$ of ξ. The foliation $\mathcal{F}^E$ coincides with its own linearization with respect to the zero section (see definition in § 2.5). Moreover, if the leaves of $\mathcal{F}^E$ are closed, then $(E, \mathrm{g}^E, \mathcal{F}^E)$ is an orbit-like foliation.

Remark 2.8. When $L \subset M$ is a submanifold of somewhat special geometry, the holonomy foliation on the normal bundle E of L, endowed with the Levi-Civita connection, induces via the normal exponential map a foliation on the whole of M. For example, if L has *parallel focal structure*, then the induced foliation on M is a *polar foliation* [Toe06]. If M is a complete, non-compact manifold with sectional curvature $\geqslant 0$ and L is a *soul* of M [CG72], then the induced foliation on M is Wilking's *dual foliation* to the Sharafutdinov projection [Wil07].

Although in principle the property of being orbit-like might depend on the metric, the following proposition shows in fact that being orbit-like is invariant under foliated diffeomorphisms.

Proposition 2.9. *The following hold.*

1. *Given a foliated linear isomorphism $\varphi : (V, \mathcal{F}) \to (V', \mathcal{F}')$ between infinitesimal foliations, $(V, \mathcal{F})$ is homogeneous if and only if $(V', \mathcal{F}')$ is homogeneous.*
2. *Given a foliated diffeomorphism $\phi : (M, \mathcal{F}) \to (M', \mathcal{F}')$ between singular Riemannian foliations, $(M, \mathcal{F})$ is orbit-like if and only if $(M', \mathcal{F}')$ is orbit-like.*

Proof. (1) By the symmetric roles of V and V', it is enough to show that if $(V, \mathcal{F})$ is homogeneous, so is $(V', \mathcal{F}')$. Suppose that $(V, \mathcal{F})$ is homogeneous and therefore the foliation $\mathcal{F}$ is spanned by Killing fields. Recall that a vector field X on a Euclidean space (V, g) is Killing if and only if it is of the form $X(v) = Av$, where A is a skew-symmetric endomorphism of V, in the sense that $\mathrm{g}(Av, v) = 0$ for every $v \in V$. Letting $\{X_1, \ldots, X_k\}$ denote a set of Killing fields on V spanning the foliation $\mathcal{F}$, the set $\{Y_1, \ldots, Y_k\}$ with $Y_i(v') = \varphi_*(X_i(\varphi^{-1}(v')))$ spans the foliation $\mathcal{F}'$ as well. Since φ is a linear map and the vector fields X_i are linear, it follows that Y_i can be written as $Y_i(v') = B_i v'$ for some endomorphism $B_i : V' \to V'$, $i = 1, \ldots, k$. Since $(V', \mathrm{g}', \mathcal{F}')$ is a singular Riemannian foliation, the leaf $L_{v'}$ through v' lies in a distance sphere from the origin and in particular $\mathrm{g}'(T_{v'}L_{v'}, v') = 0$. Since $Y_i(v')$ is tangent to $L_{v'}$, it follows that

$$0 = \mathrm{g}'(Y_i(v'), v') = \mathrm{g}(B_i v', v').$$

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In other words, B_i is skew symmetric and thus Y_i is a Killing field as well. Therefore, the foliation $(V', \mathcal{F}')$ is spanned by Killing vector fields and hence it is homogeneous as well.

(2) Up to exchanging the roles of M and M', it is enough to show that if $(M, \mathcal{F})$ is orbit-like, so is $(M', \mathcal{F}')$. Fixing a point $p \in M$, Proposition 2.4 states that the foliated diffeomorphism ϕ induces a foliated linear isomorphism $\phi_* : (\nu_p L_p, \mathcal{F}_p) \longrightarrow (\nu_{p'} L_{p'}, \mathcal{F}_{p'})$, where $p' = \phi(p)$. Since $(M, \mathcal{F})$ is orbit-like, it follows that $(\nu_p L_p, \mathcal{F}_p)$ is closed and homogeneous. From the first point above it follows that $(\nu_{p'} L_{p'}, \mathcal{F}_{p'})$ is homogeneous as well and by the continuity of ϕ_* one has that $(\nu_{p'} L_{p'}, \mathcal{F}_{p'})$ is closed. Since p' was chosen arbitrarily, it follows that $(M', \mathcal{F}')$ is orbit-like. □

2.5 Linearization and linearized foliation

Let $(M, \mathcal{F})$ be a singular Riemannian foliation, $B \subset M$ a submanifold saturated by leaves, and $U \subset M$ an ϵ-tubular neighbourhood of B with metric projection $\mathsf{p} : U \to B$. Given a vector field V in U tangent to the leaves of $\mathcal{F}$, it is possible to produce a new vector field V^ℓ, called the *linearization of V with respect to B*, as follows:

$$V^\ell = \lim_{\lambda \to 0} (h_\lambda^{-1})_* (V|_{h_\lambda(U)}),$$

where $h_\lambda : U \to U$ denotes the homothetic transformation around B. From [MR15, Proposition 5], the linearization V^ℓ is a smooth vector field invariant under the homothetic transformation h_λ and it coincides with V along B. On U, consider the module $\mathfrak{X}(U, \mathcal{F})^\ell$ given by the linearization, with respect to B, of the vector fields in $\mathfrak{X}(U, \mathcal{F})$:

$$\mathfrak{X}(U, \mathcal{F})^\ell = \{V^\ell \mid V \in \mathfrak{X}(U, \mathcal{F})\}.$$

Let D be the pseudogroup of local diffeomorphisms of U, generated by the flows of linearized vector fields, and let $(U, \mathcal{F}^\ell)$ be the partition of U into the orbits of diffeomorphisms in D. By Sussmann [Sus73, Theorem 4.1], such orbits are (possibly non-complete) smooth submanifolds of M. Moreover, as noted by Molino [Mol88, Lemma 6.3], this foliation is spanned, at each point, by the vector fields in $\mathfrak{X}^\ell(U, \mathcal{F})$.

We call $(U, \mathcal{F}^\ell)$ the *linearized foliation of $\mathcal{F}$ with respect to B*. We will show, later, that the leaves of the linearized foliation are actually complete and have a particularly nice local structure (cf. § 6).

Given a point $p \in B$, define $U_p = \mathsf{p}^{-1}(p) \subset U$ and let $\mathcal{F}_p$ (respectively $(\mathcal{F}^\ell)_p$) denote the partition of U_p into the connected components of $L \cap U_p$, as L ranges through the leaves of $\mathcal{F}$ (respectively $\mathcal{F}^\ell$). If U_p is given the flat metric g_p of $\nu_p B$ via the exponential map $\exp_p : \nu_p^\epsilon B \to U_p$, then $\mathcal{F}_p$ corresponds to the infinitesimal foliation at p (cf. Remark 2.5), which justifies the notation of $\mathcal{F}_p$ for this foliation. Furthermore, as noted in [Mol88, § 6.4], $(\mathcal{F}^\ell)_p$ is given by the linearization of $(U_p, \mathrm{g}_p, \mathcal{F}_p)$ with respect to the origin. In other words, $(\mathcal{F}^\ell)_p = (\mathcal{F}_p)^\ell$ and it makes sense to denote this foliation simply by $\mathcal{F}^\ell_p$. Moreover, letting $\mathsf{O}(\mathcal{F}_p)$ denote the Lie group of (linear) isometries of (U_p, g_p) sending every leaf to itself, one has the following result.

Proposition 2.10. *The foliation $(U_p, \mathcal{F}^\ell_p)$ is homogeneous, given by the orbits of the identity component H_p of $\mathsf{O}(\mathcal{F}_p)$.*

Proof. We identify here U_p with a neighbourhood of the origin in $\nu_p B$ via the exponential map and we think of $\mathcal{F}^\ell_p$ as the linearization of $\mathcal{F}_p$.

Given a vector field $V \in \mathfrak{X}(U_p, \mathcal{F}_p)$, its linearization V^ℓ is linear, in the sense that $V^\ell_p = A \cdot p$ for some $A \in \mathrm{End}(U_p)$. Since $\mathcal{F}_p$ is a singular Riemannian foliation, the leaves are tangent to the distance spheres around the origin and therefore perpendicular to the radial directions from the origin: $\langle V_p^\ell, p\rangle = 0$. In other words, $V_p^\ell = A\cdot p$ with A skew symmetric, which implies that the flow of V^ℓ is an isometry of U_p. Moreover, since V^ℓ is everywhere tangent to the leaves of $\mathcal{F}_p^\ell$, the flow of V^ℓ is a one-parameter group in H_p, moving every leaf of $(\mathcal{F}_p)^\ell$ to itself. In particular, the orbits of H_p are contained in the leaves of $(\mathcal{F}_p)^\ell$.

However, by definition of H_p, the tangent space of an H_p-orbit through a point $q \in U_p$ is given by

$$T_q(H_p\cdot q) = \{W_q \mid W \text{ Killing vector field tangent to the leaves of } \mathcal{F}_p\}$$

and such vector fields coincide precisely with the vector fields in $\mathfrak{X}(U_p, \mathcal{F}_p)^\ell$. Therefore, $H_p\cdot q$ is the integral manifold of $\mathfrak{X}(U_p, \mathcal{F}_p)^\ell$ through q. □

3. Molino's conjecture, assuming the main theorem

Before proving the main theorem, we show how Molino's conjecture follows from it as a corollary.

Proof of Molino's conjecture. Let $(M, \mathcal{F})$ be a singular Riemannian foliation and let $\overline{\mathcal{F}}$ denote the closure of $\mathcal{F}$. Molino himself proved that $\overline{\mathcal{F}}$ is a partition into complete smooth closed submanifolds and that $\overline{\mathcal{F}}$ is a transnormal system. Therefore, in order to prove the conjecture, it is enough to show that for any leaf $L \in \mathcal{F}$ with closure $\overline{L}$ and any vector $v \in \nu(L, \overline{L}) := \nu L \cap T\overline{L}$, there exists a smooth extension of v to a vector field V everywhere tangent to the leaves of $\overline{\mathcal{F}}$.

Let U be a tubular neighbourhood of $\overline{L}$ and let $(U, \widehat{\mathcal{F}}^\ell)$ be the foliation satisfying the main theorem. Since $\widehat{\mathcal{F}}^\ell$ coincides with $\mathcal{F}$ along $\overline{L}$, it follows that L is a leaf of $\widehat{\mathcal{F}}^\ell$ as well. Since $\widehat{\mathcal{F}}^\ell$ is an orbit-like foliation, by [AR17, Theorem 1.6], given $v \in \nu(L, \overline{L})$ there is a vector field V extending v which is tangent to the closure of $\widehat{\mathcal{F}}^\ell$. Since this closure is contained in $\overline{\mathcal{F}}$, it follows that V is also tangent to $\overline{\mathcal{F}}$ and this ends the proof of the conjecture. □

4. The setup

Fix a leaf L and a distance tube $U = B_\epsilon(L)$ around $\overline{L}$. Using the normal exponential map $\exp : \nu\overline{L} \to M$, U can be identified with the ϵ-tube $\nu^\epsilon\overline{L}$ around the zero section. By the homothetic transformation lemma, the pull-back foliation $\exp^{-1}\mathcal{F}$ on $\nu^\epsilon\overline{L}$ is invariant under the rescalings $r_\lambda : \nu^\epsilon\overline{L} \to \nu^\epsilon\overline{L}$, $r_\lambda(p, v) = (p, \lambda v)$ for any $\lambda \in (0, 1)$.

For this reason, in the following sections we will be considering the (slightly more general) setup:

- U is the ϵ-tube around the zero section of some Euclidean vector bundle $E \to B$ (in our case $B = \overline{L}$), with projection $\mathsf{p} : U \to B$;
- g is a Riemannian metric on U with the same radial function as the Euclidean metric on each fiber of E;
- $(U, \mathrm{g}, \mathcal{F})$ is a singular Riemannian foliation on U, invariant under rescalings r_λ. In particular, the zero section B is saturated by leaves and the projection p sends leaves onto leaves;
- the restriction $\mathcal{F}_B = \mathcal{F}|_B$ is a regular Riemannian foliation;
- for every leaf $L \subseteq B$ and any point $p \in L$, the normal exponential map $\nu_p^\epsilon L \to U$ is an embedding.

5. Three distributions

Let $(U, \mathrm{g}, \mathcal{F})$, $\mathsf{p} : U \to B$ be as in § 4. In order to prove the main theorem, it is first needed to produce a nicer metric on U and for this we first need to split the tangent space of U into

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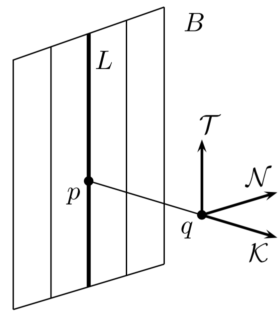

Figure 1. The distributions at q.

three components. The first, $\mathcal{K} = \ker \mathsf{p}_*$, is the distribution tangent to the fibers of p. For the remaining two notice that, since the foliation $(B, \mathcal{F}_B)$ is regular, the tangent bundle TB splits into a tangent and a normal part to the foliation: $TB = T\mathcal{F}|_B \oplus \nu\mathcal{F}|_B$. The last two distributions will be constructed as (appropriately chosen) extensions $\mathcal{T}$ and $\mathcal{N}$ of $T\mathcal{F}|_B$ and $\nu\mathcal{F}|_B$, respectively, to the whole of U; see Figure 1.

5.1 The distribution $\mathcal{T}$

From [Ale10], there exists a distribution $\widehat{\mathcal{T}}$ of rank $\dim \mathcal{F}|_B$, which extends $T\mathcal{F}|_B$ and is everywhere tangent to the leaves of $\mathcal{F}$.

The distribution $\mathcal{T}$ is simply defined as the *linearization of* $\widehat{\mathcal{T}}$ *with respect to* B, as follows: consider a family of vector fields $\{V_\alpha\}_\alpha$ spanning $\widehat{\mathcal{T}}$. Since $\widehat{\mathcal{T}}|_B$ is tangent to B, the vector fields V_α lie tangent to B as well and therefore it makes sense to consider their linearization V_α^ℓ with respect to B. By the properties of the linearization, these linearized vector fields still span a smooth distribution of the same rank as $\widehat{\mathcal{T}}$, which we call $\mathcal{T}$.

5.2 The distribution $\mathcal{N}$

At each point $q \in U$ with $\mathsf{p}(q) = p$, the slice $S_p = \exp_p(\nu_p^\epsilon L_p)$ contains q as well as the whole p-fiber U_p through p. In particular, $\mathcal{K}_q$ lies tangent to S_p. Moreover, S_p comes equipped with a flat metric g_p, inherited from the metric on $\nu_p L$ via the diffeomorphism $\exp_p : \nu^\epsilon L_p \to S_p$.

Define $\widehat{\mathcal{N}}_q$ as the subspace of $T_q S_p$ which is g_p-orthogonal to $\mathcal{K}_q$. Finally, define $\mathcal{N}$ as the *linearization* of $\widehat{\mathcal{N}}$, as defined in the previous section.

The distributions $\widehat{\mathcal{N}}$ and $\mathcal{N}$ satisfy the following property.

Proposition 5.1. *For every smooth $\mathcal{F}_B$-basic vector field X_0 along a plaque P in B, there exists a smooth extension X to an open subset of U such that:*

(i) *X is foliated and tangent to $\widehat{\mathcal{N}}$;*
(ii) *the linearization X^ℓ of X with respect to B is tangent to $\mathcal{N}$ and it is foliated with respect to both $\mathcal{F}$ and $\mathcal{F}^\ell$.*

Proof. (1) Fix a leaf L in B, a plaque $P \subset L$, and a parametrization

$$\varphi : (-1, 1)^k \to P \subset L,$$

where $k = \dim \mathcal{F}_B$. We first show that there exists a small neighbourhood of P in U, on which any $\mathcal{F}_B$-basic vector field X_0 along P can be extended to a foliated vector field X', *whose restriction to $\mathsf{p}^{-1}(P)$ is tangent to $\widehat{\mathcal{N}}$.*

Let $\partial_{y_1}, \ldots, \partial_{y_k}$ be coordinate vector fields on P and let $Y_1, \ldots, Y_k$ denote vector fields, linearized *with respect to* L, that extend $\partial_{y_1}, \ldots, \partial_{y_k}$ to a neighbourhood of P in U. There is a foliated diffeomorphism

$$\begin{aligned} F : (P \times S_p, P \times \mathcal{F}_{S_p}) &\longrightarrow (U, \mathcal{F}) \\ (\varphi(y_1, \ldots, y_k), q) &\longmapsto \Phi_k^{y_k} \circ \cdots \circ \Phi_1^{y_1}(q), \end{aligned}$$

where $\Phi_i^{y_i}$ is the flow of Y_i, after time y_i.

Furthermore, the foliation $(P \times S_p, P \times \mathcal{F}_{S_p})$ locally splits as

$$(P \times \nu_p(L, B) \times \nu_p B, P \times \{\text{pts.}\} \times \mathcal{F}|_{\nu_p B}),$$

where $\nu(L, B) = \nu L \cap TB$. Moreover, if S_p is endowed with the Euclidean metric g_p on $\nu_p L$, the splitting $S_p = \nu_p(L, B) \times \nu_p B$ is in fact Riemannian.

The map F satisfies the following.

- The set $P \times \{0\} \times \nu_p^\epsilon B$ is sent to $\mathsf{p}^{-1}(P) = \nu^\epsilon B|_P$.
- The set $P \times \nu_p^\epsilon(L, B) \times \{0\}$ is sent to a neighbourhood of P in B.
- Since F is defined via linearized vector fields, each fiber $\{p'\} \times S_p \subset P \times S_p$ is sent, via F, to the slice $S_{p'}$, isometrically with respect to the flat metrics on S_p and $S_{p'}$ (cf. [MR15]).

From the last point, it follows that the distribution of $P \times \nu_p(L, B) \times \nu_p B$ tangent to the second factor is sent, along $\nu^\epsilon B|_P$, precisely to the distribution $\widehat{\mathcal{N}}$.

Any $\mathcal{F}_B$-basic vector field X_0 along P corresponds, via F, to a vector field along $P \times \{0\} \times \{0\}$ of the form $(0, x_0, 0)$, where $x_0 \in \nu_p(L, B)$ is a fixed vector. One can clearly extend such a vector field to the foliated vector field $X' = F_*(0, x_0, 0)$. Since F is a foliated map, the vector field X' is a foliated vector field, whose restriction to B is tangent to B by the second point above. Moreover, by the discussion above the restriction of X' to $\mathsf{p}^{-1}(P)$ is tangent to $\widehat{\mathcal{N}}$.

This proves the first claim, made at the beginning of the proof. In particular, since the plaque P was chosen arbitrarily, this shows that the distribution $\widehat{\mathcal{N}}$ is *foliated*: that is, given a vector y tangent to $\widehat{\mathcal{N}}$ at a point q, there exists a foliated extension Y along a plaque containing q which is everywhere tangent to $\widehat{\mathcal{N}}$. It is easy to see that $\mathcal{K}$ and $\mathcal{T}$ are foliated as well. In particular, given the foliated vector field X', the (unique) decomposition

$$X' = X'_{\mathcal{K}} + X'_{\mathcal{T}} + X'_{\widehat{\mathcal{N}}}, \quad X'_{\mathcal{K}} \in \mathcal{K}, X'_{\mathcal{T}} \in \mathcal{T}, X'_{\widehat{\mathcal{N}}} \in \widehat{\mathcal{N}}$$

produces three vector fields $X'_{\mathcal{K}}$, $X'_{\mathcal{T}}$, $X'_{\widehat{\mathcal{N}}}$ which are foliated. In particular, the vector field $X = (X')_{\widehat{\mathcal{N}}}$ is foliated, everywhere tangent to $\widehat{\mathcal{N}}$, and it extends $X_0 = (X_0)_{\widehat{\mathcal{N}}}$ to an open set of U, as we needed to show.

(2) Since X is tangent to $\widehat{\mathcal{N}}$, its linearization X^ℓ is tangent to the linearization of $\widehat{\mathcal{N}}$, which is $\mathcal{N}$. Moreover, since X is foliated and $r_\lambda : U \to U$ is a foliated map, $X^\ell = \lim_{\lambda \to 0} (r_\lambda^{-1})_* X \circ r_\lambda$ is foliated as well. Finally, since X is foliated, for every vector field V tangent to $\mathcal{F}$ one has that $[X, V]$ is also tangent to $\mathcal{F}$. Since r_λ is a diffeomorphism, one computes

$$\begin{aligned} [X^\ell, V^\ell] &= \lim_{\lambda \to 0} [(r_\lambda^{-1})_* X \circ r_\lambda, (r_\lambda^{-1})_* V \circ r_\lambda] \\ &= \lim_{\lambda \to 0} (r_\lambda^{-1})_* [X, V] \circ r_\lambda \\ &= [X, V]^\ell. \end{aligned}$$

Since the linearization V^ℓ are precisely the vector fields generating $\mathcal{F}^\ell$, it follows from the equation above that $[X^\ell, V^\ell]$ is tangent to $\mathcal{F}^\ell$ whenever V^ℓ is and therefore X^ℓ is foliated with respect to $\mathcal{F}^\ell$. $\square$

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6. Structure of $\mathcal{F}^\ell$ and the local closure $\widehat{\mathcal{F}}^\ell$

Using the extensions X^ℓ defined in Proposition 5.1, one can prove the following.

Proposition 6.1. *Around any point $p \in B$ there is a neighbourhood W of p in B such that $(\mathsf{p}^{-1}(W), \mathcal{F}^\ell|_{\mathsf{p}^{-1}(W)})$ is foliated diffeomorphic to a product*

$$(\mathbb{D}^k \times \mathbb{D}^{m-k} \times U_p, \mathbb{D}^k \times \{\text{pts.}\} \times \mathcal{F}^\ell_p),$$

where $k = \dim \mathcal{F}|_B$ and $m = \dim B$.

Proof. Let W be a coordinate neighbourhood of B around p, with a foliated diffeomorphism $\varphi : (W, \mathcal{F}|_W) \to (\mathbb{D}^k \times \mathbb{D}^{m-k}, \mathbb{D}^k \times \{\text{pts.}\})$. Let $\partial/\partial y_1, \dots, \partial/\partial y_k$ denote a basis of vector fields in W tangent to the leaves of $\mathcal{F}|_W$ and let $V_1, \dots, V_k$ denote vector fields on $\mathsf{p}^{-1}(W)$, linearized with respect to B, extending $\partial/\partial y_i$, $i = 1, \dots, k$, and spanning the foliation $\mathcal{T}$. Similarly, let $\partial/\partial x_1, \dots, \partial/\partial x_{m-k}$ denote a basis of basic vector fields in W normal to the leaves and let $X^\ell_1, \dots, X^\ell_{m-k}$ denote linearized vector fields in $\pi^{-1}(W)$ defined as in Proposition 5.1, extending the vectors $\partial/\partial x_i$, $i = 1, \dots, m-k$. Finally, define Φ^t_i and Ψ^t_i as the flows of V_i and X^ℓ_i, respectively, after time t, and let

$$\begin{aligned} G : \mathbb{D}^k \times \mathbb{D}^{m-k} \times U_p &\longrightarrow \mathsf{p}^{-1}(W) \\ ((t_1, \dots, t_k), (s_1, \dots, s_{m-k}), q) &\longmapsto \Phi^{t_k}_k \circ \dots \circ \Phi^{t_1}_1 \circ \Psi^{s_{m-k}}_{m-k} \circ \dots \circ \Psi^{s_1}_1(q). \end{aligned}$$

Since the V_i and X^ℓ_i are linearized, they take fibers of $\mathbb{D}^k \times \mathbb{D}^{m-k} \times U_p \to \mathbb{D}^k \times \mathbb{D}^{m-k}$ to fibers of $\mathsf{p} : \mathsf{p}^{-1}(W) \to W$. Since the flows Ψ_i send the leaves of $\mathcal{F}^\ell$ to leaves and the flows Φ_i take the leaves of $\mathcal{F}^\ell$ to themselves, the leaves of $\mathbb{D}^k \times (\mathbb{D}^{m-k}, \{\text{pts.}\}) \times (U_p, \mathcal{F}^\ell_p)$ are sent into the leaves of $\mathcal{F}^\ell$. Since the differential dG is invertible at $(0, 0, p) \in \mathbb{D}^k \times \mathbb{D}^{m-k} \times U_p$, it is a diffeomorphism around $G(0, 0, p) = p$ and, by dimensional reasons, the leaves of $(\mathbb{D}^k \times \mathbb{D}^{m-k} \times U_p, \mathbb{D}^k \times \{\text{pts.}\} \times \mathcal{F}^\ell_p)$ are mapped diffeomorphically onto the leaves of $(\mathsf{p}^{-1}(W), \mathcal{F}^\ell|_{\mathsf{p}^{-1}(W)})$. ◻

The local closure of $\mathcal{F}^\ell$

Even though $(U_p, \mathcal{F}^\ell_p)$ is homogeneous for every $p \in B$, it might be the case that its leaves are not closed, which happens when the group $H_p \subseteq \mathsf{O}(U_p)$ defined in Proposition 2.10 is not closed. To obviate this problem we define a new foliation $\widehat{\mathcal{F}}^\ell$, called the *local closure* of $\mathcal{F}^\ell$, such that $\mathcal{F}^\ell \subset \widehat{\mathcal{F}}^\ell \subset \overline{\mathcal{F}^\ell}$ and whose restriction $\widehat{\mathcal{F}}^\ell_p$ to each p-fiber U_p is homogeneous and closed.

Recall that $\mathcal{F}^\ell$ is defined by the orbits of the pseudogroup D of local diffeomorphisms, generated by the flows of linearized vector fields. For each $q \in U_p$, consider the closure $\overline{H}_p$ of H_p in $\mathsf{O}(U_p)$ and define the $\widehat{\mathcal{F}}^\ell$-leaf $\widehat{L}_q$ through q to be the D-orbit of $\overline{H}_p \cdot q$:

$$\widehat{L}_q = \{q' = \Phi(h \cdot q) \mid \Phi \in \mathsf{D}, h \in \overline{H}_p\}.$$

Let $\sim$ denote the relation $q \sim q'$ if and only if $q' = \Phi(h \cdot q)$ for some $\Phi \in \mathsf{D}$ and $h \in \overline{H}_p$. In this way, the leaf of $\widehat{\mathcal{F}}^\ell$ through q can be rewritten as $\{q' \in U \mid q' \sim q\}$. As for the other foliations, for every $p \in B$ we define $(U_p, \widehat{\mathcal{F}}^\ell_p)$ to be the partition of U_p into the connected components of the intersections of U_p with the leaves in $\widehat{\mathcal{F}}^\ell$.

Proposition 6.2. *The following hold:*

(1) *$\widehat{\mathcal{F}}^\ell$ is a well-defined partition of U;*

(2) *for every $p \in B$, the leaves of $\widehat{\mathcal{F}}^\ell_p$ are the orbits of $\overline{H}_p$ on U_p.*

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Proof. (1) One must prove that the relation $\sim$ defined above is an equivalence relation. For this, notice that, since any $\Phi \in \mathsf{D}$ defines a foliated isometry between $(U_p, \mathcal{F}^\ell_p)$ and $(U_{\Phi(p)}, \mathcal{F}^\ell_{\Phi(p)})$ for any $p \in B$, in particular it defines a foliated isometry between the respective closures $(U_p, \overline{H}_p)$ and $(U_{\Phi(p)}, \overline{H}_{\Phi(p)})$. In particular, for any $h \in \overline{H}_p$ and $\Phi \in \mathsf{D}$, one has $h' = \Phi \circ h \circ \Phi^{-1} \in \overline{H}_{\Phi(p)}$.

– Reflexivity of $\sim$: if $q' \sim q$, then $q' = \Phi(h(q))$ for some $h \in \overline{H}_p$ and $\Phi \in \mathsf{D}$. Then $q' = h'(\Phi(q))$, where $h' = \Phi \circ h \circ \Phi^{-1} \in \overline{H}_{\Phi(p)}$ and therefore $q = \Phi^{-1}((h')^{-1}q')$, which means that $q \sim q'$.

– Transitivity of $\sim$: if $q' \sim q$ and $q'' \sim q'$, then $q' = \Phi(h(q))$ and $q'' = \Psi(g(q'))$ for some $h \in \overline{H}_p$, $g \in \overline{H}_{\Phi(p)}$, and $\Phi, \Psi \in \mathsf{D}$. Then $q'' = (\Psi \circ \Phi)((g' \circ h)(q))$, where $g' = \Phi^{-1} \circ g \circ \Phi \in \overline{H}_p$, and therefore $q'' \sim q$.

(2) Let L' denote a leaf of $\widehat{\mathcal{F}}^\ell$. From (1), the intersection of L' with U_p is a union of orbits of $\overline{H}_p$. On the other hand, we claim that the intersection $L' \cap U_p$ consists of countably many orbits of $\overline{H}_p$, so that each connected component of such intersection must consist of a single $\overline{H}_p$-orbit. From the definition of $\widehat{\mathcal{F}}^\ell$, it is enough to prove that the subgroup $\mathsf{D}_p \subset \mathsf{D}$ of diffeomorphisms fixing p moves every $\overline{H}_p$-orbit in $L' \cap U_p$ to at most countably many orbits. For this, consider a piecewise-smooth loop $\gamma : [0, 1] \to L_p$ with $\gamma(0) = \gamma(1) = p$. Using linearized vector fields with γ as integral curve, one can construct a continuous path $\Phi_t : [0, 1] \to \mathsf{D}$ of diffeomorphisms such that $\Phi_0 = \mathrm{id}_U$ and $\Phi_t(p) = \gamma(t)$, as described in [MR15, Corollary 7]. Fixing some $\overline{H}_p$-orbit $\mathcal{O}$ in $L' \cap U_p$, its image $\Phi_1(\mathcal{O})$ is again some $\overline{H}_p$-orbit, which depends only on the class $[\gamma] \in \pi_1(L_p, p)$ and not on the actual path γ, nor on the specific choice of Φ_t. This gives a map

$$\partial : \pi_1(L_p, p) \to \{\overline{H}_p\text{-orbits in } L' \cap U_p\}.$$

This map admits a section, namely: for every orbit $\mathcal{O}'$ in $L' \cap U_p$, take a path γ in L' from a point in a (fixed) orbit $\mathcal{O}$ to a point in $\mathcal{O}'$. Under the projection $\mathsf{p} : U \to B$, the composition $\mathsf{p} \circ \gamma$ is a loop in L_p. The section of ∂ sends $\mathcal{O}'$ to $[\mathsf{p} \circ \gamma] \in \pi_1(L_p, p)$. In particular, the map ∂ is surjective and therefore the set of $\overline{H}_p$-orbits in $L' \cap U_p$ has at most the cardinality of $\pi_1(L_p, p)$, which is at most countable since L_p is a manifold. $\square$

As a corollary of Propositions 6.2 and 6.1, one gets the following result.

COROLLARY 6.3. *Let $(U, \mathcal{F})$ be a singular Riemannian foliation as in § 4, let $\mathcal{F}^\ell$ be its linearized foliation, and $\widehat{\mathcal{F}}^\ell$ the local closure. Then $\widehat{\mathcal{F}}^\ell$ is a singular foliation with complete leaves. Moreover, around each point $p \in B$ there is a neighbourhood W of p in B such that $(\mathsf{p}^{-1}(W), \widehat{\mathcal{F}}^\ell|_{\mathsf{p}^{-1}(W)})$ is foliated diffeomorphic to a product*

$$(\mathbb{D}^k \times \mathbb{D}^{m-k} \times U_p, \mathbb{D}^k \times \{\text{pts.}\} \times \{\textit{orbits of } \overline{H}_p \subseteq \mathsf{O}(U_p)\}),$$

which can be given the structure of a singular Riemannian foliation.

Once it is shown that $\widehat{\mathcal{F}}^\ell$ is also a transnormal system with respect to some metric, then by the corollary above it is globally a singular Riemannian foliation.

7. A new metric

Let $\mathcal{T}, \mathcal{N}, \mathcal{K}$ be the distributions as in the previous section. Clearly, one has $TU = \mathcal{T} \oplus \mathcal{N} \oplus \mathcal{K}$.

Now define the new metric g^ℓ on U, as the metric defined by the following properties:

- $\mathcal{T} \oplus \mathcal{N}$ and $\mathcal{K}$ are orthogonal with respect to g^ℓ;
- $\mathrm{g}^\ell|_{\mathcal{T}\oplus\mathcal{N}} = \mathsf{p}^*\mathrm{g}_B$, where g_B denotes the restriction of the original metric on B. In particular, $\mathcal{T}$ and $\mathcal{N}$ are also orthogonal to one another;

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- for any $q \in U_p$, recall that $\mathcal{K}_q = T_q U_p$ and define $\mathrm{g}^\ell|_{\mathcal{K}_q} = \mathrm{g}_p$ as the flat metric on U_p induced from $\exp_p : \nu_p B \to U_p$.

These conditions characterize the metric g^ℓ uniquely. The most useful property of this metric is the following.

Proposition 7.1. *The triples $(U, \mathrm{g}^\ell, \mathcal{F}^\ell)$ and $(U, \mathrm{g}^\ell, \widehat{\mathcal{F}}^\ell)$ are singular Riemannian foliations.*

Proof. The arguments for $\mathcal{F}^\ell$ and $\widehat{\mathcal{F}}^\ell$ are identical; therefore, we will only check the proposition for $(U, \mathrm{g}^\ell, \widehat{\mathcal{F}}^\ell)$ (which is the only case we need for the main theorem anyway).

Moreover, the statement is local in nature; therefore, it is enough to prove the statement on certain open sets covering the whole of U. For any point $p \in B$, let W denote a neighbourhood of p in B and $\mathsf{p}^{-1}(W)$ a neighbourhood of p in U. We need to check that $(\mathsf{p}^{-1}(W), \mathrm{g}^\ell, \widehat{\mathcal{F}}^\ell)$ is a singular Riemannian foliation. To prove this, we apply [Ale10, Proposition 2.14], which states that it is enough to check two conditions:

(1) $(\mathsf{p}^{-1}(W), g', \widehat{\mathcal{F}}^\ell)$ is a singular Riemannian foliation with respect to *some* Riemannian metric g';
(2) for every stratum $\Sigma \subset \mathsf{p}^{-1}(W)$ (i.e. union of leaves of the same dimension), the restriction of $\widehat{\mathcal{F}}^\ell$ to Σ is a (regular) Riemannian foliation.

The first condition is satisfied by Corollary 6.3. The second condition is equivalent to checking that, for every leaf $\widehat{L}_q$ of $\widehat{\mathcal{F}}^\ell|_{\mathsf{p}^{-1}(W)}$ and every basic vector field X along $\widehat{L}_q$ tangent to the stratum through $\widehat{L}_q$, the norm $\|X\|_{\mathrm{g}^\ell}$ is constant along $\widehat{L}_q$.

By definition of the metric g^ℓ, the space $\nu \widehat{L}_q$ is given by $\mathcal{N}|_{\widehat{L}_q} \oplus (\nu \widehat{L}_q \cap \mathcal{K})$. From Proposition 5.1, along $\widehat{L}_q$ the space $\mathcal{N}$ is spanned by linearized vector fields X_i^ℓ, which are then g^ℓ-basic (i.e. foliated *and* g^ℓ-orthogonal to the leaves). In particular, any basic vector field $\overline{X}$ along $\widehat{L}_q$ splits as a sum $\overline{X} = \overline{X}_1 + \overline{X}_2$, where $\overline{X}_1$ is tangent to $\mathcal{N}$, $\overline{X}_2$ is tangent to $\mathcal{N}' := \nu \widehat{L}_q \cap \mathcal{K}$, and $\mathrm{g}^\ell(\overline{X}_1, \overline{X}_2) = 0$. Therefore, it is enough to check independently that for every basic vector field $\overline{X}$ along $\widehat{L}_q$, tangent to either $\mathcal{N}$ or $\mathcal{N}'$, the norm of $\overline{X}$ is constant along $\widehat{L}_q$.

If $\overline{X}$ is tangent to $\mathcal{N}$, then by the construction in Proposition 5.1 it projects to some basic vector field X along $\widehat{L}_p \subset B$. Since $(B, \mathrm{g}_B, \mathcal{F}|_B)$ is a Riemannian foliation, the norm $\|X\|_{\mathrm{g}_B}$ is constant along $\widehat{L}_p$. By the construction of the metric g^ℓ, one has $\|\overline{X}\|_{\mathrm{g}^\ell} = \|X\|_{\mathrm{g}_B}$ and, therefore, the norm of $\overline{X}$ is constant along $\widehat{L}_q$.

If $\overline{X}$ is tangent to $\mathcal{N}'$, then it is tangent to any fiber $U_{p'}$, $p' \in \widehat{L}_p$. The restriction $\overline{X}|_{U_{p'}}$ is a basic vector field of $(U_{p'}, \widehat{\mathcal{F}}^\ell_{p'})$ along $\widehat{L}_q \cap U_{p'}$ and therefore the norm $\|\overline{X}|_{U_{p'}}\|_{\mathrm{g}_{p'}}$ is locally constant along $\widehat{L}_q \cap U_{p'}$. By the construction of g^ℓ, it follows that $\|\overline{X}|_{U_{p'}}\|_{\mathrm{g}^\ell}$ is also locally constant along each $\widehat{L}_q \cap U_{p'}$. However, given two points $p', p'' \in \widehat{L}_p$ and a vertical, foliated vector field V^ℓ whose flow Φ moves p' to p'', one also has that Φ moves $U_{p'}$ isometrically to $U_{p''}$ and $\overline{X}|_{U_{p'}}$ to $\overline{X}|_{U_{p''}}$. In particular, $\|\overline{X}|_{U_{p'}}\|_{\mathrm{g}^\ell} = \|\overline{X}|_{U_{p'}}\|_{\mathrm{g}_{p'}}$ does not really depend on the point $p' \in \widehat{L}_p$ and it is actually constant along the whole leaf $\widehat{L}_q$. $\square$

With this in place, one can finally prove the main theorem.

Proof of the main theorem. Let U be an ϵ-tubular neighbourhood around the closure $\overline{L}$ of a leaf $L \in \mathcal{F}$. Letting $B = \overline{L}$, we are under the assumptions of § 4. In particular, it is possible to define the linearized foliation $\mathcal{F}^\ell$ on U, its local closure $\widehat{\mathcal{F}}^\ell$, and the metric g^ℓ as in Proposition 7.1.

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It is clear by construction that $\widehat{\mathcal{F}}^\ell|_{\overline{L}} = \mathcal{F}|_L$ and that the closure of $\widehat{\mathcal{F}}^\ell$ is contained in the closure of $\mathcal{F}$. Moreover, by Corollary 6.3, the foliation $(U, \mathrm{g}^\ell, \widehat{\mathcal{F}}^\ell)$ is, locally around each point, foliated diffeomorphic to the orbit-like foliation $(\mathbb{D}^k \times \mathbb{D}^{m-k} \times U_p, \mathbb{D}^k \times \{\text{pts.}\} \times \widehat{\mathcal{F}}^\ell_p)$. By Proposition 2.9, the foliation $(U, \mathrm{g}^\ell, \widehat{\mathcal{F}}^\ell)$ is orbit-like as well and this concludes the proof. □

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