INTERPOLATION BY POLYNOMIALS IN z AND z^{-1} IN THE ROOTS OF UNITY

A. SHARMA

1. Introduction. Given a function $f(z)$, continuous on C: $|z| = 1$ in the complex plane, there is a close analogy between approximation in the sense of least squares by polynomials on the unit circle and interpolation by polynomials in the nth roots of unity to the same function. For detailed discussion of the problem and its generalization for a suitable Jordan curve one can refer to Walsh **(3)** or to a recent paper by Curtiss **(2)**. More recently, Curtiss **(1)** has considered the problem of interpolation by polynomials in non-equally spaced points on the unit circle and has pointed out the limitations inherent in the problem. He has shown that if $f(z)$ is analytic in $|z| < 1$, continuous in $|z| \leqslant 1$, and sufficiently smooth in the neighbourhood of ξ (where $|\xi| = 1$ and ξ is not a root of unity), then the polynomial $L_n(f;z)$ which interpolates to $f(z)$ in the $(n-1)$th roots of unity and ξ has the property that

$$(1) \qquad \lim_{n\to\infty} L_n(f;z) = f(z) \qquad \text{for } |z| < 1.$$

If $f(z)$ is only R-integrable on C, then he has shown that without the hypothesis of analyticity for $|z| < 1$, but with the assumption that the divided difference $d(f|z, \xi)$ defined in (4) is R-integrable on C, (1) becomes

$$(2) \quad \lim_{n\to\infty} L_n(f;z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt - \frac{1}{2\pi i}\int_C d(f|t, \xi)dt, \qquad \text{for } |z| < 1.$$

Curtiss has has also shown that a similar situation prevails when more "mavericks" are included with the roots of unity. The object of this note is to study a similar situation for interpolatory polynomials $P_{m,n}(z, z^{-1})$ of degree m in z and n in z^{-1}. It will be shown that if we set

$$(3) \qquad P_{m,n}(z, z^{-1}) \equiv q_m^{(n)}(z) + r_n^{(m)}(z^{-1})$$

where $q_m^{(n)}(z)$ is the polynomial component and $r_n^{(m)}(z^{-1})$ is zero at infinity, then under the smoothness conditions required by Curtiss, $q_m^{(n)}(z)$ has the same property as $L_n(f;z)$ above, except that the extra term on the right in (2) does not appear. A similar result is true for $r_n^{(m)}(z^{-1})$ for $|z| > 1$.

The polynomial component $q_m^{(n)}(z)$ is not an interpolatory polynomial in general, but it has the property that it preserves polynomials of degree $\leqslant m$. We do not want to emphasize this property of $q_m^{(n)}(z)$ as a linear operator, but

Received January 12, 1965. This paper was written under U.S. Air Force Grant AF-, AFOSR-62-198 (AF49(638)-574) at Harvard University in 1961–62.

we shall illustrate its efficacy by stating an analogue of a result of Walsh (**3**, p. 153).

2. Interpolation in certain strong equidistributions. Consider a set of points S_{m+n+1}^{ν} formed of $(m+n-\nu+1)$th roots of unity and the ν points $S_\nu(S_\nu: \xi_1, \xi_2, \ldots, \xi_\nu)$ which are not roots of unity, ν being a fixed non-negative integer with $|\xi_i| = 1$; $i = 1, 2, \ldots, \nu$. These points, $m+n+1$ in number, are strongly equidistributed in the classical sense (**3**, p. 166). Let

$$d(f|z, \xi_1), d(f|z, \xi_1, \xi_2), \ldots$$

be the divided differences of order 1, 2, . . . given by

$$(4)\qquad \begin{aligned} d(f|z, \xi_1) &= \frac{f(z) - f(\xi_1)}{z - \xi_1}, \\ d(f|z, \xi_1, \xi_2) &= \frac{d(f|z, \xi_1) - d(f|\xi_1, \xi_2)}{z - \xi_2}. \end{aligned}$$

We shall prove the following theorem.

THEOREM 1. *If $P_{m,n}(z, z^{-1}; S_{m+n+1}^{\nu}) = P_{m,n}(z, z^{-1})$ is the polynomial of form* (3) *in z and z^{-1}, of degree m in z and of degree n in z^{-1} interpolating to $f(z)$ in the points S_{m+n+1}^{ν}, where $f(z)$ together with*

$$d(f|t, \xi_1, \xi_2, \ldots, \xi_k) \qquad (k = 1, 2, \ldots, \nu)$$

is R*-integrable on C, then*

$$(5)\qquad \begin{aligned} \lim_{m\to\infty} q_m^{(n)}(z) &= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt, && |z| < 1,\ n \geqslant \nu \text{ and fixed}, \\ \lim_{n\to\infty} r_n^{(m)}\left(\frac{1}{z}\right) &= \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt, && |z| > 1,\ m \geqslant \nu \text{ and fixed}. \end{aligned}$$

If $f(z)$ is analytic for $|z| < \rho > 1$, but has a singularity on $|z| = \rho$, then $q_m^{(n)}(z)$ converges maximally to $f(z)$ on C.

Moreover if $s_m(z; S_\nu)$ is the polynomial of degree m whose difference from $f(z)$ has a zero of order $(m - \nu)$ at the origin and vanishes in the point set S_ν, then

$$(6)\qquad \lim_{m\to\infty} [s_m(z; S_\nu) - q_m^{(n)}(z)] = 0$$

for $|z| < \rho^{2+\alpha}$, uniformly for $|z| \leqslant Z < \rho^{2+\alpha}$ where

$$\lim_{m,n\to\infty} (n/m) = \alpha.$$

Also

$$(7)\qquad \lim_{m,n\to\infty} r_n^{(m)}\left(\frac{1}{z}\right) = 0$$

for $|z| > (1/\rho)^{1/\alpha}$ and uniformly for $|z| \geqslant Z > (1/\rho)^{1/\alpha}$.

https://doi.org/10.4153/CJM-1967-002-6 Published online by Cambridge University Press

Remark. A sequence of polynomials $p_n(z)$ is said to converge to $f(z)$ on C *maximally* or with the greatest geometric degree of convergence, if

$$|f(z) - p_n(z)| < M/R^n, \qquad z \text{ on } C,$$

for every $R < \rho$, when M depends on R but not on n or z. Equivalently

$$\limsup_{n\to\infty} \mu^{1/n} = \frac{1}{\rho}, \qquad \mu_n = \max\{|f(z) - p_n(z)|, z \text{ on } C\}.$$

For the sake of simplicity, we shall prove the theorem for $\nu = 2$. Then $z^n P_{m,n}(z, z^{-1})$ is the polynomial of degree $m + n$ interpolating to $z^n f(z)$ in the points $S_{m+n+1}{}^2$. Hence

$$z^n P_{m,n}(z, z^{-1}) = \sum_{k=1}^{m+n-1} w^{nk} f(w^k) \frac{(z^{m+n-1} - 1)(z - \xi_1)(z - \xi_2) w^k}{(m + n - 1)(z - w^k)(w^k - \xi_1)(w^k - \xi_2)}$$
$$+ \xi_1{}^n f(\xi_1) \frac{(z^{m+n-1} - 1)(z - \xi_2)}{(\xi_1^{m+n-1} - 1)(\xi_1 - \xi_2)} + \xi_2{}^n f(\xi_2) \frac{(z^{m+n-1} - 1)(z - \xi_1)}{(\xi_2^{m+n-1} - 1)(\xi_2 - \xi_1)},$$

where w is a $(m + n - 1)$th root of unity with $w^{m+n-1} = 1$.

A simple calculation shows that for $n \geqslant \nu$,

$$q_m{}^{(n)}(z) = \sum_{k=1}^{m+n-1} \frac{w^{nk} f(w^k) w^k}{(m + n - 1)(w^k - \xi_1)(w^k - \xi_2)} \left[\frac{z^{m+1} - w^{(m+1)k}}{z - w^k} \right.$$
$$\left. - (\xi_1 + \xi_2) \cdot \frac{z^m - w^{mk}}{z - w^k} + \xi_1 \xi_2 \cdot \frac{z^{m-1} - w^{(m-1)k}}{z - w^k}\right]$$
$$+ \frac{(z^m - \xi_2 z^{m-1}) \xi_1{}^n f(\xi_1)}{(\xi_1{}^{m+n-1} - 1)(\xi_1 - \xi_2)} + \frac{\xi_2{}^n f(\xi_2)(z^m - \xi_1 z^{m-1})}{(\xi_2^{m+n-1} - 1)(\xi_2 - \xi_1)}.$$

Since

$$(z^{m+1} - w^{(m+1)k}) - (\xi_1 + \xi_2)(z^m - w^{mk}) + \xi_1 \xi_2 (z^{m-1} - w^{(m-1)k})$$
$$= z^{m-1}(z - \xi_1)(z - \xi_2) - w^{mk-k}(w^k - \xi_1)(w^k - \xi_2)$$

and since

$$\frac{x^n}{x^{m+n-1} - 1} = -\frac{1}{m + n - 1} \sum_{k=1}^{m+n-1} \frac{w^{nk} \cdot w^k}{w^k - x}, \qquad m \geqslant 2,$$

we can rewrite

$$q_m{}^{(n)}(z) = I_1 + I_2 + I_3 + I_4$$

where

$$I_1 = \sum_{k=1}^{m+n-1} \frac{w^{nk} \cdot f(w^k) w^k \cdot z^{m-1}(z - \xi_1)(z - \xi_2)}{(m + n - 1)(w^k - \xi_1)(w^k - \xi_2)(z - w^k)},$$

$$I_2 = -\sum_{k=1}^{m+n-1} \frac{w^{mk+nk} \cdot f(w^k)}{(m + n - 1)(z - w^k)},$$

$$I_3 = -\sum_{k=1}^{m+n-1} \frac{f(\xi_1) w^{nk} \cdot w^k (z^m - \xi_2 z^{m-1})}{(m + n - 1)(w^k - \xi_1)(\xi_1 - \xi_2)},$$

$$I_4 = -\sum_{k=1}^{m+n-1} \frac{f(\xi_2) w^{nk} \cdot w^k (z^m - \xi_1 z^{m-1})}{(m + n - 1)(w^k - \xi_2)(\xi_2 - \xi_1)}.$$

Using the definition of divided differences (4), we obtain on combining I_1, I_3, and I_4:

$$q_m^{(n)}(z) = I_2 - z^{m-1} \sum_{k=1}^{m+n-1} \frac{w^{nk+k} \cdot f(w^k)}{(m+n-1)(w^k - z)}$$
$$+ z^{m-1} \sum_{k=1}^{m+n-1} \frac{w^{nk} \cdot w^k}{m+n-1} d(f|w^k, \xi_1)$$
$$+ (z^m - \xi_1 z^{m-1}) \sum_{k=1}^{m+n-1} \frac{w^{nk} \cdot w^k}{m+n-1} d(f|w^k, \xi_1, \xi_2).$$

The last three sums can be looked upon, after multiplying and dividing by 2, as Riemann sums for the integrals

$$\frac{1}{2\pi i} \int_{|t|=1} t^n \frac{f(t)}{t-z} dt, \qquad \frac{1}{2\pi i} \int_{|t|=1} t^n d(f|t, \xi_1) dt$$

and

$$\frac{1}{2\pi i} \int_{|t|=1} t^n d(f|t, \xi_1, \xi_2) dt,$$

respectively and since $f(t)$, $d(f|t, \xi_1)$, and $d(f|t, \xi_1, \xi_2)$ are by hypothesis R-integrable, the result follows at once for fixed $n \geqslant \nu$, since $|z| < 1$. For $n \to \infty$, the result again follows by the Riemann–Lebesgue theorem.

The proof for the second part of (5) is similar. We just give the expression for $r_n^{(m)}(z^{-1})$ $(m \geqslant \nu)$:

$$r_n^{(m)}(z^{-1}) = \sum_{k=1}^{m+n-1} \frac{w^{nk} \cdot f(w^k) \cdot w^k \cdot \alpha(z, w^k)}{(m+n-1)(w^k - \xi_1)(w^k - \xi_2)(z - w^k)}$$
$$+ \frac{\xi_1^n f(\xi_1)(-z^{-n+1} + \xi_2 z^{-n})}{(\xi_1^{m+n-1} - 1)(\xi_1 - \xi_2)} + \frac{\xi_2^n f(\xi_2)(-z^{-n+1} + \xi_1 z^{-n})}{(\xi_2^{m+n-1} - 1)(\xi_2 - \xi_1)},$$

where

$$\alpha(z, w^k) = -z^{-n}(z - \xi_1)(z - \xi_2) + w^{mk-k}(\xi_1 - w^k)(\xi_2 - w^k).$$

After rearrangement, one can rewrite

$$r_n^{(m)}(z^{-1}) = -I_2 - z^{-n} \sum_{k=1}^{m+n-1} \frac{w^{-mk} f(w^k) \cdot w^k}{(m+n-1)(z - w^k)}$$
$$+ z^{-n} \sum_{k=1}^{m+n-1} \frac{w^{-mk} \cdot w^k}{m+n-1} d(f|w^k, \xi_1)$$
$$+ (z^{-n+1} - \xi_1 z^{-n}) \sum_{k=1}^{m+n-1} \frac{w^{-mk+k} \cdot w^k}{m+n-1} d(f|w^k, \xi_1, \xi_2).$$

For $|z| > 1$, the result now follows as in the first case.

If $f(z)$ is analytic in a circle $|z| < \rho$, but has a singularity on $|z| = \rho$, we can prove maximal convergence of $q_m^{(n)}(z)$ on C by observing that

$$f(z) - q_m^{(n)}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{z^{n-1} t^m (z - \xi_1)(z - \xi_2) - (t - \xi_1)(t - \xi_2)}{(t^{m+n-1} - 1)(t - \xi_1)(t - \xi_2)} \cdot \frac{f(t)}{(t - z)} dt$$

where Γ is the circle $|z| = R$, $1 < R < \rho$.

The details of the rest of the proof are omitted.

3. It is interesting to observe that (5) is independent of the extra term that occurs in (2). However both results require the existence of integrals involving divided differences of $f(t)$ at the points ξ_i. This is not surprising, as is seen by taking $f(z) = z^{-1}$, and $\nu = 1$ in the above theorem. The Lagrange polynomial of interpolation $L_m(f; z)$ is given by

$$L_m(f; z) = \frac{1}{z}\left[1 - \frac{(z^{m-1} - 1)(z - \xi)}{\xi}\right]$$

(**1**, p. 873) and so

$$\lim_{m\to\infty} L_m(f; z) = \frac{1}{\xi} \qquad \text{for } |z| < 1,$$

whereas $z^n P_{m,n}(z, z^{-1}) = z^{n-1}$ gives for $n \geqslant 1$, $q_m^{(n)}(z) \equiv 0$ for all m. However, if $n = 0$, $q_m^{(0)}(z) \equiv L_m(f; z) \to 1/\xi$ for $|z| < 1$ as $m \to \infty$.

The assumption $n \geqslant \nu$ is necessary in the theorem, since for $n < \nu$, the extra terms of the type which occur in (2) begin to appear in formula (5).

4. Analogue of a theorem of Walsh. We shall now give an analogue of a theorem of Walsh (**3**, p. 153) which appears to be stronger than the original. We formulate

THEOREM 2. *Let $f(z)$ be analytic for $|z| < \rho > 1$ but have a singularity on the circle $|z| = \rho$ (the case ρ infinite is not excluded). Let $P_{m,n}(z,z^{-1})$ be a polynomial of degree m in z and n in z^{-1} given by* (3), *which interpolates to $f(z)$* at the $(m + n - 1)$th *roots of unity; then $q_m^{(n)}(z)$ converges maximally to $f(z)$ on C: $|z| = 1$.*

Moreover, if $s_m(z)$ is the polynomial of degree m in z which coincides with $f(z)$ at the origin up to order $m + 1$, then

$$\lim_{m,n\to\infty} [s_m(z) - q_m^{(n)}(z)] = 0 \tag{8}$$

for $|z| < \rho^{2+\alpha}$ uniformly for $|z| \leqslant Z < \rho^{2+\alpha}$ where

$$\lim_{m,n\to\infty} \frac{n}{m} = \alpha. \tag{9}$$

Also

$$\lim_{m,n\to\infty} r_n^{(m)}(z^{-1}) = 0 \tag{10}$$

for $|z| > \rho^{-1/\alpha}$ uniformly for $|z| \geqslant Z > \rho^{-1/\alpha}$.

Further, if $P_{m,n'}(z, z^{-1})$ is a polynomial of degree m in z and of degree n' in z^{-1}, interpolating to $f(z)$ at the $(m + n' + 1)$th *roots of unity with*

$$P_{m,n'}(z, z^{-1}) = q_m^{(n')}(z) + r_{n'}^{(m)}(z^{-1}) \tag{11}$$

where $q_m^{(n')}(z)$ and $r_n^{(m)}(z^{-1})$ have obvious meanings as above, then

$$\lim_{n,n',m\to\infty} [q_m^{(n')}(z) - q_m^{(n)}(z)] = 0 \tag{12}$$

for $|z| < \rho^{\beta+2}$, *uniformly for* $|z| \leqslant Z < \rho^{\beta+2}$ *where*

$$\lim_{n',m\to\infty} (n'/m) = \beta \text{ with } \beta \leqslant \alpha.$$

We omit the proof.

Some special cases are of interest. If $n = km$, (8) holds for $|z| < \rho^{2+k}$ and (10) for $|z| > \rho^{-1/k}$. If $n = k + m$, (8) holds for $|z| < \rho^3$ and (10) for $|z| > \rho^{-1}$. For $n = k$ (a fixed positive integer) (8) holds for $|z| < \rho^2$, which is the case treated by Walsh **(3)** for $k = 0$.

Similarly for $m = n'k$, (8) holds for $|z| < \rho^{2+1/k}$ and (9) holds for $|z| > \rho^{-k}$; for $m = n' + k$, (8) holds for $|z| < \rho^3$ and (10) for $|z| > \rho^{-1}$.

Results about the degree of convergence of $q_m^{(n)}(z)$ to $f(z)$ and of $r_n^{(m)}(z^{-1})$ to zero can be also obtained. Indeed, we have

$$\text{(13)} \qquad \limsup_{m,n\to\infty} [\max\{|r_n^{(m)}(z^{-1})|, z \text{ on } C\}]^{1/n} \leqslant \left(\frac{1}{\rho}\right)^{1/\alpha}$$

where

$$\lim_{n,m\to\infty} \left(\frac{n}{m}\right) = \alpha,$$

and

$$\text{(14)} \qquad \limsup_{n,m\to\infty} [\max\{|f(z) - q_m^{(n)}(z)|, z \text{ on } C\}]^{1/m} = \frac{1}{\rho}.$$

Looking upon $q_m^{(n)}$ and $r_n^{(m)}$ as operators that map every continuous function $f(z)$ defined on the unit circle into the polynomials of degree m in z and of degree n in z^{-1} respectively by interpolation, as described earlier, we see that they are obviously linear operators. This observation enables us to formulate our result for functions that are analytic in an annulus. We thus have the following

COROLLARY. *Let* $f(z)$ *be analytic for* $1/\rho < |z| < \rho > 1$, *but not for* $1/\rho' < |z| < \rho'$, $\rho' > \rho$. *Let* $P_{m,n}(z, z^{-1})$ *be the polynomial of degree* m *in* z *and* n *in* z^{-1} *which interpolates to* $f(z)$ *at the* $(m + n + 1)$th *roots of unity and let* (3) *hold. If* $f = f_1 + f_2$ *where* f_1 *is analytic for* $|z| < \rho$ *and* f_2 *is analytic for* $|z| > 1/\rho$, *then* $q_m^{(n)}$ *converges maximally to* $f_1(z)$ *and* $r_n^{(m)}(z^{-1})$ *converges maximally to* $f_2(z)$.

Moreover, if $s_m(z)$ *is the polynomial of degree* m *in* z *which coincides with* $f_1(z)$ *at the origin up to order* $m + 1$, *then*

$$\text{(15)} \qquad \lim_{m,n\to\infty} [s_m(z) - q_m^{(n)}(z)] = 0$$

for $|z| < \rho^\alpha$, *uniformly for* $|z| \leqslant Z < \rho^\alpha$, *where* α *is given by* (9).

If $t_n(z^{-1})$ *is the polynomial of degree* n *in* z^{-1} *which coincides with* $f_2(z)$ *in the point at infinity, up to order* $n + 1$, *then*

$$\text{(16)} \qquad \lim_{m,n\to\infty} [t_n(z^{-1}) - r_n^{(m)}(z^{-1})] = 0$$

https://doi.org/10.4153/CJM-1967-002-6 Published online by Cambridge University Press

for $|z| > \rho^{-1/\alpha}$, *uniformly for* $|z| \geqslant Z > \rho^{-1/\alpha}$.

If $P_{m,n'}(z, z^{-1})$ *has the form* (11) *and if*

$$\lim_{n',n,m\to\infty} (n'/m) = \beta \leqslant \alpha,$$

then

$$\lim_{n',n,m\to\infty} [q_m^{(n')}(z) - q_m^{(n)}(z)] = 0 \tag{17}$$

for $|z| < \rho^{\beta}$, *uniformly for* $|z| \leqslant Z < \rho^{\beta}$; *a similar relation holds for* $r_n^{m}(z^{-1})$. *Further*

$$\limsup_{m,n\to\infty} [\max\{|f_1(z) - q_m^{(n)}(z)|, z \text{ on } C\}]^{1/m} = \begin{cases} 1/\rho, & \alpha \geqslant 1, \\ 1/\rho^{\alpha}, & \alpha \leqslant 1, \end{cases} \tag{18}$$

and

$$\limsup_{m,n\to\infty} [\max\{|f_2(z) - r_n^{(m)}(z)|, z \text{ on } C\}]^{1/n} = \begin{cases} (1/\rho)^{1/\alpha}, & \alpha \geqslant 1, \\ 1/\rho, & \alpha \leqslant 1. \end{cases} \tag{19}$$

It is enough to mention the following example. The formal proof of Theorem 2 and its corollary is exactly as in Walsh **(4)**.

Let $f(z) = f_1(z) + f_2(z)$ where

$$f_1(z) = \left(\rho - \frac{1}{\rho}\right)^{-1}(z - \rho)^{-1}, \qquad f_2(z) = (-1)\left(\rho - \frac{1}{\rho}\right)^{-1}\left(z - \frac{1}{\rho}\right)^{-1};$$

then

$$f_1(z) - q_m^{(n)}(z) = \frac{1}{\rho - 1/\rho}\,\frac{1}{z - \rho}\cdot\frac{\rho^n z^{m+1} - 1}{\rho^{m+n+1} - 1} + \frac{p^m z^m - 1}{\rho^{m+n+1} - 1}\cdot\frac{1}{z - 1/\rho}$$

and

$$f_2(z) - r_n^{(m)}(z^{-1}) = \frac{1}{\rho - 1/\rho}\,\frac{z^n - \rho^n}{(\rho^{n+m+1} - 1)z^n}\cdot\frac{1}{z - \rho} + \frac{z^n - \rho^{m+1}}{z^n(\rho^{m+n+1} - 1)}\cdot\frac{1}{z - 1/\rho}.$$

Also

$$s_m(z) - q_m^{(n)}(z) = \frac{1}{\rho - 1/\rho}\,\frac{z^{m+1} - \rho^{m+1}}{\rho^{m+1}(\rho^{m+n+1} - 1)}\cdot\frac{1}{z - \rho} + \frac{\rho^m z^{m-1}}{\rho^{m+n+1} - 1}\cdot\frac{1}{z - 1/\rho}$$

and

$$t_n(z^{-1}) - r_n^{(m)}(z^{-1}) = \frac{1}{\rho - 1/\rho}\,\frac{z^n - \rho^n}{\rho^{m+n+1} - 1}\cdot\frac{1}{(z - \rho)z^n} + \frac{p^n z^n - 1}{z^n \rho^n(\rho^{m+n+1} - 1)}\cdot\frac{1}{z - 1/\rho}.$$

5. If $f(z)$ is only given to be continuous (or R-integrable) on the unit circle, interpolation by polynomials $P_{m,n}(z, z^{-1})$ has the same character as when $m = n$ **(3**, p. 180**)**. To be more explicit, we state without proof the following

https://doi.org/10.4153/CJM-1967-002-6 Published online by Cambridge University Press

THEOREM 3. *Let* $f(z)$ *be* R-*integrable on* C: $|z| = 1$. *Let* $P_{m,n}(z, z^{-1})$, $q_m^{(n)}(z)$, $r_n^{(m)}(z^{-1})$ *have the same meaning as in Theorem* 2, *and let* $P_{m,n}(z, z^{-1})$ *interpolate to* $f(z)$ *at the* $(m + n - 1)$th *roots of unity. Then*

$$(20)\qquad \begin{cases} \lim\limits_{m\to\infty} q_m^{(n)}(z) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(t)}{t-z}\,dt, & |z| < 1, \\ \lim\limits_{n\to\infty} r_n^{(m)}(z^{-1}) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(t)}{t-z}\,dt, & |z| > 1, \end{cases}$$

the last integral being taken in the clockwise sense. The convergence is uniform for $|z| \leqslant r < 1$ *and* $|z| \geqslant 1/r > 1$ *respectively.*

If $f(z)$ *is analytic in the annulus* $1/\rho < |z| < \rho > 1$ (*in particular, for* $|z| < 1$ *or* $|z| > 1$) *the equations* (20) *are valid for* $|z| < \rho$ *and* $|z| > 1/\rho$ *respectively, with uniformity for* $|z| \leqslant R < \rho$ *and* $|z| \geqslant 1/R > 1/\rho$ *respectively. Also* $P_{m,n}(z, z^{-1}) \to f(z)$ *uniformly for* $1/R \leqslant |z| \leqslant R < \rho$.

The first part is Theorem 2 in **(5)**. If in (20) we set

$$f_1(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}\,dt = a_0 + a_1 z + \ldots, \qquad |z| < 1,$$

one might suspect, in analogy with Theorem 2, that the equation

$$\lim_{m\to\infty} [q_{m(n)}(z) - (a_0 + a_1 z + \ldots + a_m z^m)] = 0$$

could be established for certain values of z in modulus greater than R, at least if $f(z)$ has no singularity for $|z| \geqslant R$. That this is not so can be verified exactly as in Walsh **(3)**, by the example of the function

$$1/(z - \rho), \qquad 0 < \rho < 1.$$

The author is grateful to Professor Walsh and Professor Curtiss for many helpful suggestions and advice.

REFERENCES

1. J. H. Curtiss, *Polynomial interpolation in points equidistributed on the unit circle*, Pacific J. Math., *12* (1962), 863–877.

2. ——— *A stochastic treatment of some classical interpolation problems*, Proc. Fourth Berkeley Symposium Math. Statistics and Probability, Vol. II (Berkeley, 1961), pp. 79–93.

3. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain* (Providence, 1960), pp. 153–156, 179–182.

4. ——— *Interpolation and an analogue of the Laurent development*, Proc. Nat. Acad. Sci., *19* (1) (1933), 203–20.

5. J. L. Walsh and A. Sharma. *Least square approximation and interpolation in the roots of unity*, Pacific J. Math., *14* (1964), 727–730.

University of Alberta