# TYPE RADICALS

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**1. The lower radical of a module type.** For a ring $R$ with unit, the module type $t(R)$ was defined in [**6**] as follows: $t(0)=0$; $t(R)=d$ if every free $R$-module has invariant rank; $t(R)=(c,k)$ for integers $c,k\geqq 1$ if every free $R$-module of rank $<c$ has invariant rank, while a free module of rank $h\geqq c$ has rank $h+nk$ for any integer $n\geqq 0$. The module types form a lattice under the ordering $0<(c,k)<d$ and $(c',k')\leqq(c,k)$ if and only if $c'\leqq c$ and $k'\mid k$. Two of the basic theorems on types are:

A. [**6**; Theorem 2, p. 115] *If $R\to R'$ is a unit-preserving homomorphism, then $t(R')\leqq t(R)$.*

B. [**6**; Theorem 3, p. 116]

$$t\left(\sum_1^n \oplus R_i\right)=\bigcup_1^n t(R_i).$$

We recall that in the definition of the Kurosh lower radical (as modified by Sulinski, Anderson and Divinsky in [**10**]) for a class $\mathcal{M}$ of rings, the class $\mathcal{M}_1$ is the homomorphic closure of $\mathcal{M}$, and for any ordinal $\beta>1$ the class $\mathcal{M}_\beta$ consists of rings $R$ such that every non-zero homomorphic image has a non-zero ideal in $\mathcal{M}_\alpha$ for some $\alpha<\beta$. Then the lower radical class defined by $\mathcal{M}$ is $\mathcal{L}(\mathcal{M})=\bigcup\mathcal{M}_\beta$, taken over all ordinals $\beta$. We note that, in fact [**10**; Theorem 1, p. 420], $\mathcal{L}(\mathcal{M})=\mathcal{M}_\omega$, where $\omega$ is the smallest infinite ordinal. If we define $\mathcal{T}_\alpha=\{R\mid t(R)\leqq\alpha\}$, then, by Theorem A, $(\mathcal{T}_\alpha)_1=\mathcal{T}_\alpha$. We further note that $\mathcal{T}_0=\{0\}$ is already (trivially) a radical class. We shall require the following lemma.

LEMMA 1. *If an ideal $I$ of a ring $R$ has a unit, then $R=I\oplus I^*$, where $I^*$ is the annihilator of $I$ in $R$.*

*Proof.* The Pierce decomposition relative to the unit of $I$ clearly yields $R=I+I^*$, and since $I$ has a unit this sum must be direct.

For use in the following theorem, and at several other points, we record the following:

CONSTRUCTION 1. Let $\{\alpha_i\}_{i\in\mathcal{I}}$ be an arbitrary set of module types. By [**1**; Theorem 9, p. 130] there exists for each $i$ a ring $R_i$ with $t(R_i)=\alpha_i$. Let $R$ be the direct sum $\sum_{i\in\mathcal{I}}\oplus R_i$. Note that if $\mathcal{I}$ is infinite then $R$ does not have module type (since it has no unit), but nevertheless every non-zero image has a non-zero ideal with type. This follows from the fact that if $I$ is a proper ideal of $R$ then at least one $R_i\nsubseteq I$. Thus $R/I$ has a non-zero ideal $R+I/I\cong R_i/R_i\cap I$, and, by Theorem $A$, $t(R_i/R_i\cap I)\leqq\alpha_i$.

THEOREM 1. $\mathcal{L}(\mathcal{T}_\alpha)=(\mathcal{T}_\alpha)_2$, *and for $\alpha>0$, $\mathcal{L}(\mathcal{T}_\alpha)\neq(\mathcal{T}_\alpha)_1$.*

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*Proof.* For the first statement it is sufficient to show that $(\mathcal{T}_\alpha)_3 = (\mathcal{T}_\alpha)_2$. Thus suppose there exists $R \in (\mathcal{T}_\alpha)_3$, $R \notin (\mathcal{T}_\alpha)_2$. Then, by definition, $R$ has an image $\bar{R}$ with a non-zero ideal $I \in (\mathcal{T}_\alpha)_2$ but no ideal in $(\mathcal{T}_\alpha)_1$. Since $I \in (\mathcal{T}_\alpha)_2$ it has a non-zero ideal $J \in (\mathcal{T}_\alpha)_1 = \mathcal{T}_\alpha$. Thus, by Lemma 1, $I = J \oplus J^*$. Let $x \in J$ and $y \in \bar{R}$; then $xy \in I$ so that $xy = z + z^*$ for some $z \in J$ and $z^* \in J^*$. But if $e$ is the unit of $J$, this yields $xy = exy = ez = z \in J$. Similarly $yx \in J$, so $J$ is an ideal of $\bar{R}$, contradicting the condition that $\bar{R}$ should have no ideals from $\mathcal{T}_\alpha$.

To establish the second statement of the theorem, let $R$ be the ring of Construction 1 with $\mathcal{J}$ infinite and all $\alpha_i = \alpha$. Then $R \notin \mathcal{T}_\alpha$ but every non-zero image has a non-zero ideal $I$ with $t(I) \leqq \alpha$. Thus $R \in (\mathcal{T}_\alpha)_2$.

**2. The lower maxit radicals.** In [9] the module type was used to construct for a general ring an invariant (called the " maxit " of the ring) which coincides with the module type for rings with unit. In the present paper we shall sharpen the definition of [9] in the following way: we shall extend the module type lattice to a *lattice of maxits* by permitting $c$ and $k$ in $(c, k)$ to take on values $\omega$ and 0 respectively, in addition to all positive integers. The order in this extended lattice (and hence the lattice operations) is defined as for module types, noting that $c \leqq \omega$ for all $c$ and $k \mid 0$ for all $k$. We now define the *maxit* $m(R)$ of a ring $R$ as follows:

(i) Whenever $R$ has module type, $m(R) = t(R)$.

(ii) For all other rings, let $\mathcal{W}$ be the set consisting of all modular ideals of $R$ together with $R$ itself; then $m(R) = \bigcup_{I \in \mathcal{W}} m(R/I)$.

Remark 1. From this definition it follows that a ring $R$ is a Brown–McCoy radical ring if and only if $m(R) = 0$. This is clear since $R$ is a Brown–McCoy radical if and only if it has no modular ideals [8, p. 134]. But then $m(R) = m(R/R) = t(0) = 0$.

Remark 2. On the other hand, since in the extended lattice such maxits as $(\omega, k)$ are now available, it is clear that $m(R) = d$ if and only if there exists some $I \in \mathcal{W}$ such that $t(R/I) = d$.

Remark 3. From the definition of the maxit it is also clear that, if $\bar{R}$ is a homomorphic image of $R$, then $m(\bar{R}) \leqq m(R)$.

Remark 4. It is easily seen that the method of proof of [9; Theorem 3.2, p. 131] can be applied to yield $m\left(\sum_1^n \oplus R_i\right) = \bigcup_1^n m(R_i)$. This result will be extended (see Theorem 2) to infinite direct sums or complete direct sums (to be written $\sum_c \oplus R_i$).

For a given ring $R$, define

$$G(a) = \{xa - x + ay - y + \sum (x_i a y_i - x_i y_i)\} \quad \text{for all} \quad x, y, x_i, y_i \in R. \tag{1}$$

Lemma 2. $m(R) = \bigcup_{a \in R} t(R/G(a))$.

*Proof.* If $I$ is a modular ideal of $R$, then there exists an identity $a$ of $R$ modulo $I$. Thus $G(a) \subseteq I$ and from the natural homomorphism $R/G(a) \to R/I$ and Theorem A, $t(R/I) \leqq t(R/G(a))$. Thus $m(R) = \bigcup_{I \in \mathcal{W}} t(R/I) \leqq \bigcup_{a \in R} t(R/G(a))$.

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But also $G(a)\in\mathscr{W}$; so we have the reverse inequality.

Let $\{a_i\}$ $(i = 1, 2, \ldots, n)$ be a set of members of $R$, and write

$$b = \sum_{k=1}^{n} (-1)^{k+1} \sum_{i_1<\ldots<i_k} a_{i_1} \ldots a_{i_k}. \tag{2}$$

LEMMA 3. $G(b) \subseteq \bigcap_{1}^{n} G(a_i)$.

*Proof.* For a given $i$, let $f$ be any polynomial in $\{a_i\}$ over the integers not containing $a_i$ as a term. It is clear that $f(a_1 \ldots a_i \ldots a_n) - f(a_1 \ldots 1 \ldots a_n) \in G(a_i)$, where " 1 " simply indicates the deletion of any $a_i$ from any term. But it is easy to see from (2) that if 1 is substituted for $a_i$ in $b - a_i$ the result is zero. Hence $b - a_i \in G(a_i)$. Thus $b$ is a unit modulo $G(a_i)$ for each $i$, and so is a unit modulo $\bigcap G(a_i)$.

PROPOSITION 1. (*a*) *If* $m(R) = (c, k)$ *for integers* $c, k \geqq 1$, *then there exists an ideal* $I$ *of* $R$ *such that* $t(R/I) = (c, k)$.

(*b*) *If* $m(R) = (\omega, k)$, *then for some infinite ascending chain of integers* $c_1 < c_2 < \ldots$ *there exists a chain* $I_1 \supset I_2 \supset \ldots$ *of ideals such that* $t(R/I_i) = (c_i, k)$.

(*c*) *Similarly, if* $m(R) = (c, 0)$, *there is an infinite ascending chain* $\{k_i\}$ *of proper divisors and* $t(R/I_i) = (c, k_i)$.

(*d*) *Similarly, if* $m(R) = (\omega, 0)$, *there are ascending chains* $\{c_i\}$ *and* $\{k_i\}$ *and* $t(R/I_i) = (c_i, k_i)$.

*Proof.* (*a*) Let $m(R) = (c, k)$ for $c, k \geqq 1$. Choose one $a_i$ (if such exists) for which $t(R/G(a_i)) = \alpha_i$ for each $\alpha_i \leqq (c, k)$. The set $\{a_i\}$ so chosen is finite, say $i = 1, \ldots, n$. Define $b$ by (2); then it follows from Lemma 3 and Theorem A that $t(R/G(b)) \geqq \bigcup_{1}^{n} t(R/G(a_i))$. But, by Lemma 2, the right side of this inequality is $m(R)$, and since by definition $t(R/G(b)) \leqq m(R)$, equality follows.

(*b*) If $m(R) = (\omega, k)$, then we again have only a finite set $k_i | k$ for which a choice $a_i$ exists with $t(R/G(a_i)) = (c_i, k_i)$ for some $c_i$. Now $k$ is clearly the least common multiple of the $\{k_i\}$. Thus if $b_1$ is the $b$ defined by (2) for this set of $\{a_i\}$, then, by the same argument as in (*a*) above, $t(R/G(b_1)) \geqq (c, k)$ for some $c$. But $t(R/G(b_1)) < (\omega, k)$, so we must have $t(R/G(b_1)) = (c_1, k)$ for some $c_1$. Since $m(R) = (\omega, k)$, there must, by Lemma 2, exist some $b'$ such that $t(R/G(b')) = (c', k')$ for which $c' > c_1$ (of course with some $k' | k$). Defining, as in (2), $b_2 = b_1 + b' - b_1 b'$, we have $G(b_2) \subseteq G(b_1) \cap G(b')$ so that $t(R/G(b_2)) \geqq (c_1, k) \cup (c', k') = (c', k)$. Thus $t(R/G(b_2)) = (c_2, k)$ for some $c_2 > c_1$. The process clearly continues. It is also clear that similar constructions exist for cases (*c*) and (*d*).

Let $I$ be an ideal of $R$ and $a \in I$. Write $G_I(a)$ for the ideal of $I$ defined by (1), restricting $x, y, x_i, y_i$ to be elements of $I$.

LEMMA 4. $G_I(a) = G(a) \cap I$.

*Proof.* Clearly $G_I(a) \subseteq G(a) \cap I$; so suppose that $z \in G(a) \cap I$. Then $z - aza = (z - za) + (za - aza) \in G_I(a)$. But from $z \in G(a)$ and $a \in I$ it follows that $aza \in G_I(a)$. Thus $z \in G_I(a)$, and so $G(a) \cap I \subseteq G_I(a)$.

Let $\{R_j\}_{j\in J}$ be a set of rings. If either $R=\sum\oplus R_j$ or $R=\sum_c \oplus R_j$, then from the natural homomorphism $R\to R_j$ it follows, by Remark 3, that $\bigcup m(R_j)\leqq m(R)$. Let $a_j\in R_j$ be the projection of $a\in R$, and write $G_j=G_{R_j}(a_j)$.

THEOREM 2. *(a) If* $R=\sum\oplus R_j$, *then* $m(R)=\bigcup_{j\in J} m(R_j)$.

*(b) If* $R=\sum_c\oplus R_j$ *and there exists* $a\in R$ *such that* $\bigcup t(R_j/G_j)=(\omega,k)$, $(c,0)$, *or* $(\omega,0)$, *then* $m(R)=d$. *Otherwise* $m(R)=\bigcup_{j\in J} m(R_j)$.

*Proof.* (*a*) Let $a\in\sum\oplus R_j$; then $a=\sum_1^n a_j$. Thus $G(a)=\left(\sum_1^n\oplus G_j\right)\oplus\left(\sum_{j>n}\oplus R_j\right)$ and it is easily seen that $R/G(a)\cong\sum_1^n\oplus R_j/G_j$. From Theorem B, it follows that $t(R/G(a))=\bigcup_1^n t(R_j/G_j)\leqq\bigcup_{j\in J} m(R_j)$. Since $a$ was arbitrary, by Lemma 2, we have $m(R)\leqq\bigcup_{j\in J} m(R_j)$ and hence equality.

(*b*) Let $a\in\sum_c\oplus R_j$; then again it is easy to see that $G(a)=\sum_c\oplus G_j$ and $R/G(a)\cong\sum_c\oplus R_j/G_j$. From the projection $R/G(a)\to R_j/G_j$ we have $t(R_j/G_j)\leqq t(R/G(a))$. Thus, if $\bigcup t(R_j/G_j)$ is not a module type, then $t(R/G(a))=d$ and so $m(R)=d$.

Now if $t(R_j/G_j)=0$ for all $j\in J$, then all $R_j=G_j$ and so $R=G(a)$. If this is true for all $a\in R$, then, by Lemma 2, $0=m(R)=\bigcup m(R_j)$.

Thus suppose for a given $a\in R$ we have $\bigcup t(R_j/G_j)=(c,k)$, so that all $t(R_j/G_j)\leqq(c,k)$. Then if $G_j\neq R_j$, by the matrix criterion for rings of given module type (see [1] or [6]), there exist $c+k$ by $c$ and $c$ by $c+k$ matrices $A_j$, $B_j$ with elements in $R_j$ such that

$$A_jB_j-a_jI_{c+k}\equiv 0 \pmod{G_j} \quad\text{and}\quad B_jA_j-a_jI_c\equiv 0 \pmod{G_j},$$

where we write $a_jI_c$ for the diagonal matrix with $a_j$ on the diagonal (that is, a unit matrix over $R_j$ modulo $G_j$).

Let $A=\sum_c\oplus A_j$ and $B=\sum_c\oplus B_j$ (note: insert a zero matrix whenever $G_j=R_j$). Then clearly $AB-aI_{c+k}$ and $BA-aI_c\equiv 0 \pmod{G(a)}$; so, by the matrix criterion for module type, $t(R/G(a))\leqq(c,k)$. Thus $m(R)=\bigcup_a t(R/G(a))\leqq\bigcup_a\bigcup_{j\in J} t(R_j/G_j)$. By reversing the order of unions in the last expression, this yields $m(R)\leqq\bigcup m(R_j)$ and hence equality.

We recall that a class of rings is called *hereditary* if it includes all ideals of all of its members.

THEOREM 3. *The class* $\mathcal{M}_\alpha=\{R\mid m(R)\leqq\alpha\}$ *is a hereditary class.*

*Proof.* Let $R\in\mathcal{M}_\alpha$ and $a\in I$, where $I$ is an ideal of $R$. Since $m(R)\leqq\alpha$, it follows that $t(R/G(a))\leqq\alpha$. The Pierce decomposition $x=ax-(ax-x)$ shows that $R=I+G(a)$. Thus $R/G(a)\cong I/I\cap G(a)$ and by Lemma 4 this equals $I/G_I(a)$. From Lemma 2 it follows that $m(I)\leqq\alpha$.

COROLLARY 1. *If* $m(R)<d$, *then, for an ideal* $I$ *of* $R$, *we have the following results.*

*(a) If* $I$ *is Noetherian, it is included in the Brown–McCoy radical of* $R$.

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(*b*) *If I is commutative, it is included in the Jacobson radical of R.*

(*c*) *If I is Artinian, it is included in the nil radical of R.*

*Proof.* Note first that, if $I$ satisfies one of the conditions of (*a*), (*b*) or (*c*), then so does every homomorphic image of $I$. Now, by Theorem 3, $m(I) < d$ and, if $m(I) \neq 0$, then $I$ has a non-zero homomorphic image $\bar{I}$ with $t(\bar{I}) \leqq m(I)$. But by [**4**; p. 32] if $\bar{I}$ is Noetherian, and hence by [**4**; Theorem 29, p. 71] if $\bar{I}$ is Artinian, or by [**2**; p. 563] if $\bar{I}$ is commutative, then $t(\bar{I}) = d$, contradicting $t(\bar{I}) \leqq m(I)$. Thus $m(I) = 0$ and so $I$ is a Brown–McCoy radical ring. Hence if $I$ is commutative it is a Jacobson radical ring, or if it is Artinian it is nil.

THEOREM 4. $\mathscr{L}(\mathscr{M}_\alpha) = (\mathscr{M}_\alpha)_2$ *and is a hereditary radical class.*

*Proof.* By Remark 3, $\mathscr{M}_\alpha = (\mathscr{M}_\alpha)_1$ and, by Theorem 3, it is a hereditary class. Also any nilpotent ring $R$ is Brown–McCoy radical, so $m(R) = 0 \leqq \alpha$. Thus it follows from [**10**; Theorem 2, p. 420] that $\mathscr{L}(\mathscr{M}_\alpha) = (\mathscr{M}_\alpha)_2$. Now, by [**3**; Theorem 1.4, p. 29], the lower radical of any hereditary class is hereditary, and so $\mathscr{L}(\mathscr{M}_\alpha)$ is a hereditary class.

Although $\mathscr{M}_\alpha$ may not in general be a radical class, the ideals $I$ of $R$ satisfying $I \in \mathscr{M}_\alpha$ or $R/I \in \mathscr{M}_\alpha$ have certain maximal properties. We conclude this section by recording these properties.

PROPOSITION 2. (*a*) *Every ring R has an ideal I maximal relative to* $I \in \mathscr{M}_\alpha$.

(*b*) *If R has an ideal I with* $m(I) = \alpha$, *then it has an ideal maximal relative to this property.*

*Proof.* To show that Zorn's Lemma applies, let $\{I_i\}$ be a chain of ideals with $m(I) \leqq \alpha$ (or $= \alpha$ in case (*b*)). Let $I = \bigcup I_i$, so that, by Theorem 3, $m(I) \geqq \bigcup m(I_i)$. Suppose $m(I) \nleqq \alpha$ (or $\neq \alpha$ in case (*b*)). Then there exists some $a \in I$ such that $t(I/G_I(a)) = \beta \nleqq \alpha$ (or $> \alpha$ in case (*b*)). Now $a \in I$ and so $a \in I_i$ for some $I_i$ in the chain. Writing $G_i = G_{I_i}(a)$, we obtain from Lemma 4 $G_i = G_I(a) \cap I_i$, and, as in the proof of Theorem 3, $I/G_I(a) \cong I_i/G_i$. Thus $t(I_i/G_i) = \beta$, violating the definition of $I_i$. Hence $m(I) \leqq \alpha$ (or $= \alpha$ in case (*b*)) and so Zorn's Lemma applies.

PROPOSITION 3. *If R contains an ideal H such that* $t(R/H) = \alpha$, *then it contains an ideal maximal relative to this property.*

*Proof.* If $\alpha = 0$, then $H = R$ satisfies the proposition. Thus suppose $\alpha > 0$. To show that Zorn's Lemma applies, let $H = \bigcup H_i$ for a chain $\{H_i\}$ for which $t(R/H_i) = \alpha$. By Theorem A, $t(R/H) \leqq t(R/H_i) = \alpha$, and suppose $t(R/H) = \beta < \alpha$. Thus $\beta \neq d$ and if $\beta = 0$ then $H = R$. But for any $H_i$, if $a_i$ is the identity of $R$ modulo $H_i$, this means that $a_i$ belongs to some $H_j$ in the chain. Now $a_i \notin H_i$; so $H_j \nsubseteq H_i$ and hence $H_i \subseteq H_j$. But this is also impossible, for then $G(a_i) \subseteq H_j$ and thus, since $a_i \in H_j$, $R \subseteq H_j$.

We therefore suppose that $\beta = (c, k)$ and let $a$ be an identity of $R$ modulo $H$. As was remarked in the proof of Theorem 2, the matrix criterion then requires the existence of $c+k$ by $c$ and $c$ by $c+k$ matrices $S$ and $T$ such that

$$U = ST - aI_{c+k} \equiv 0 \pmod{H} \quad \text{and} \quad V = TS - aI_c \equiv 0 \pmod{H}.$$

Now there is only a finite number of elements in the matrices $U$ and $V$, and since all are contained in $H$, they must in fact all be contained in some $H_i$ of the chain. If $a_i$ is the identity of $R$ modulo $H_i$, then $aa_i - a \in H_i$. On the other hand, $a$ is an identity of $R$ modulo $H$, so that $aa_i - a_i \in H$ and so is in $H_j$, for some $H_j$ in the chain. One of the two must be larger, say $H_j \supseteq H_i$. Then $a$ is a unit of $R$ modulo $H_j$, and since $U, V \equiv 0 \pmod{H_j}$, we have $t(R/H_j) \leqq \beta$, contradicting the definition of $H_j$. We conclude that $t(R/H) = \alpha$, and so Zorn's Lemma applies.

COROLLARY 2. *For $\alpha$ a module type, if $m(R/H) = \alpha$ for some ideal $H$ of $R$, then $R$ contains an ideal maximal relative to this property.*

*Proof.* If $\alpha = 0$, let $H = \bigcup H_i$ for a chain of ideals with $m(R/H_i) = 0$. By Remark 3, $m(R/H) \leqq m(R/H_i) = 0$. Thus $m(R/H) = 0$, and so Zorn's Lemma applies. Thus suppose that $m(R/H) = \alpha > 0$. By Proposition 1 (or Remark 2 if $\alpha = d$), there exists an ideal $I \supseteq H$ such that $t(R/I) = \alpha$. By Proposition 3, we may assume that $I$ is maximal relative to this property. Suppose there could exist $H_1 \supset I$ such that $m(R/H_1) = \alpha$. By the same argument, there would then exist $I_1 \supseteq H_1$ such that $t(R/I_1) = \alpha$, violating the maximality of $I$.

**3. Simple type radicals.** Let $\mathscr{W}$ be the set consisting of all maximal modular ideals of $R$, together with $R$ itself. Define $\overline{m}(R) = \bigcup_{I \in \mathscr{W}} m(R/I)$, and write $\overline{\mathscr{M}}_\alpha = \{R \mid \overline{m}(R) \leqq \alpha\}$. Then, by definition, $\mathscr{M}_\alpha \subseteq \overline{\mathscr{M}}_\alpha$.

LEMMA 5. *$\overline{m}(R/I) \leqq \overline{m}(R)$ for all ideals $I$ of $R$.*

*Proof.* If $H/I$ is maximal in $R/I$, then $H$ is maximal in $R$. Since $(R/I)/(H/I) \cong R/H$, it follows that $\overline{m}(R/I) = \bigcup t(R/H)$, the union being taken over all $H \supseteq I$; hence result.

LEMMA 6. *If $I$ is an ideal of $R$, then $\overline{m}(I) \leqq \overline{m}(R)$.*

*Proof.* If $I$ has no maximal modular ideals, then $\overline{m}(I) = 0$ and the Lemma is satisfied trivially. Now it is well-known that $H_1$ is a maximal modular ideal of $I$ if and only if $H_1 = I \cap H$ for some maximal modular ideal $H$ of $R$. Thus $I \nsubseteq H$, so that $R = I + H$, and hence $I/H_1 \cong R/H$. It follows that $\overline{m}(I) = \bigcup m(I/H_1) \leqq \overline{m}(R)$.

Let $\mathscr{S}$ be the class of all simple rings, partitioned into $\mathscr{S}_\alpha = \{R \in \mathscr{S} \mid m(R) = \overline{m}(R) \leqq \alpha\}$ and $\mathscr{S}'_\alpha = \mathscr{S} - \mathscr{S}_\alpha$. One way to use this partitioning would be to consider $\mathscr{L}(\mathscr{S}_\alpha)$. Since $\mathscr{S}_\alpha \subseteq \mathscr{M}_\alpha$ we have $\mathscr{L}(\mathscr{S}_\alpha) \subseteq \mathscr{L}(\mathscr{M}_\alpha)$. However, let $\mathscr{N}$ be the class of all zero rings (rings with trivial multiplication: $xy = 0$ for all $x$, $y$ in the ring) without minimal ideals. Note that $\mathscr{N}$ is non-empty since we can impose a trivial multiplication on any additive abelian group without minimal subgroups (such as the additive group of the integers). Now it is known [**3**; Theorem 1.15, p. 39] that, if a class $\mathscr{U}$ is homomorphically closed and the class $\mathscr{V}$ is hereditary, then $\mathscr{U} \cap \mathscr{V} = 0$ implies that $\mathscr{L}(\mathscr{U}) \cap \mathscr{V} = 0$. Clearly the class $\mathscr{N}$ is hereditary and $(\mathscr{S}_\alpha)_1 \cap \mathscr{N} = 0$, so that $\mathscr{L}(\mathscr{S}_\alpha) \cap \mathscr{N} = 0$. On the other hand the zero rings are Brown–McCoy radical; so $\mathscr{N} \subseteq \mathscr{M}_\alpha$. Thus $\mathscr{L}(\mathscr{S}_\alpha)$ is properly contained in $\mathscr{L}(\mathscr{M}_\alpha)$. We might ask the relation of $\mathscr{L}(\mathscr{T}_\alpha)$ to $\mathscr{L}(\mathscr{S}_\alpha)$. However, these classes are not comparable: if $R$ is a simple radical ring, then $R \in \mathscr{S}_\alpha$ but $R \notin \mathscr{L}(\mathscr{T}_\alpha)$; on the other hand, the ring of integers $Z \in \mathscr{T}_d$ but has no minimal ideals and so is not in $\mathscr{L}(\mathscr{S}_d)$.

The upper radical defined by this partition [5; p. 22] can, however, be used to characterize $\bar{\mathcal{M}}_\alpha$. Write $\mathcal{U}(\mathcal{S}'_\alpha)$ for the upper radical class defined by $\mathcal{S}'_\alpha$. (Recall [5; p. 17] that $\mathcal{U}(\mathcal{S}'_\alpha) = \{R \mid \text{every non-zero homomorphic image } R/I \notin \mathcal{S}'_\alpha\}$.)

THEOREM 5. *$\bar{\mathcal{M}}_\alpha = \mathcal{U}(\mathcal{S}'_\alpha)$ and is thus a hereditary radical class.*

*Proof.* Let $R \in \bar{\mathcal{M}}_\alpha$, so that $\bar{m}(R) = \beta \leqq \alpha$. Then, by definition, for any simple image $R/H$ we have $m(R/H) \leqq \beta$, and so $R$ has no images in $\mathcal{S}'_\alpha$. Thus $\bar{\mathcal{M}}_\alpha \subseteq \mathcal{U}(\mathcal{S}'_\alpha)$. On the other hand, if $R \in \mathcal{U}(\mathcal{S}'_\alpha)$ but $R \notin \bar{\mathcal{M}}_\alpha$, then we would have $\bar{m}(R) = \beta \nleqq \alpha$. There would then exist some maximal modular $H$ such that $\bar{m}(R/H) = t(R/H) \nleqq \alpha$. Thus $R/H \in \mathcal{S}'_\alpha$, contradicting the definition of $\mathcal{U}(\mathcal{S}'_\alpha)$. Hence $\bar{\mathcal{M}}_\alpha = \mathcal{U}(\mathcal{S}'_\alpha)$ and so is a radical class, and by Lemma 6 it is hereditary.

We remark that, since $\mathcal{T}_\alpha \subseteq \mathcal{M}_\alpha \subseteq \bar{\mathcal{M}}_\alpha$ for all module types $\alpha$, it follows that $\mathcal{L}(\mathcal{T}_\alpha) \subseteq \mathcal{L}(\mathcal{M}_\alpha) \subseteq \bar{\mathcal{M}}_\alpha$. The first inequality is always proper since every $\mathcal{M}_\alpha$ contains all Brown–McCoy radical rings. However, it is an open question whether or not $\bar{\mathcal{M}}_\alpha$ differs from $\mathcal{L}(\mathcal{M}_\alpha)$ (or even from $\mathcal{M}_\alpha$). Of course, trivially, $\mathcal{M}_d = \mathcal{L}(\mathcal{M}_d) = \bar{\mathcal{M}}_d$ since $\mathcal{M}_d$ is the class of all rings.

We may also ask whether or not these radical classes differ for different $\alpha$. There are a number of open questions in this regard, but we may state the following:

1. *If $\alpha$ is a module type, then $\mathcal{L}(\mathcal{T}_\alpha)$ is non-zero, and if $\alpha \neq \beta$, then $\mathcal{L}(\mathcal{T}_\alpha) \neq \mathcal{L}(\mathcal{T}_\beta)$.*

*Proof.* We may assume that $\alpha \nleqq \beta$. If $\alpha = d$, then $\beta \neq d$ and any field is in $\mathcal{L}(\mathcal{T}_d)$ but not in $\mathcal{L}(\mathcal{T}_\beta)$. If $\alpha = (c, k)$, use the ring $V = V_{c,c+k}$ [1; Section 5, p. 221] (see also [6; Footnote 6, p. 130]) universal for rings of type $\alpha$; then $V \in \mathcal{T}_\alpha$. Suppose that $V \in \mathcal{L}(\mathcal{T}_\beta)$; then, by Theorem 1, there exists a non-zero ideal $I$ of $V$ such that $t(I) \leqq \beta$. But, aside from its unit, $V$ has no elements which are even local identities. Thus $V$ itself is the only ideal of $V$ which has type, and $t(V) \nleqq \beta$.

2. Since $\mathcal{M}_0$ is the class of all Brown–McCoy radical rings, $\mathcal{L}(\mathcal{M}_0) = \mathcal{M}_0$. Then, since $\mathcal{M}_\beta$, for any $\beta \neq 0$, contains rings with unit, $\mathcal{L}(\mathcal{M}_0) \neq \mathcal{L}(\mathcal{M}_\beta)$, $\beta \neq 0$.

On the other hand, $\mathcal{L}(\mathcal{M}_\beta)$ for $\beta \neq d$ contains no fields, and, since $\mathcal{M}_d = \mathcal{L}(\mathcal{M}_d)$ is the class of all rings, $\mathcal{L}(\mathcal{M}_d) \neq \mathcal{L}(\mathcal{M}_\beta)$, $\beta \neq d$.

3. *For given $\alpha \neq \beta$, if $\bar{\mathcal{M}}_\alpha \neq \bar{\mathcal{M}}_\beta$, then $\mathcal{L}(\mathcal{M}_\alpha) \neq \mathcal{L}(\mathcal{M}_\beta)$.*

*Proof.* We can assume that $\bar{\mathcal{M}}_\alpha \nsubseteq \bar{\mathcal{M}}_\beta$. Thus there must exist some $R$ with $\bar{m}(R) \leqq \alpha$ and $\bar{m}(R) \nleqq \beta$. It follows that $R$ has a simple image $\bar{R}$ with $t(\bar{R}) \leqq \alpha$ and $t(\bar{R}) \nleqq \beta$. Hence $\bar{R} \in \mathcal{M}_\alpha$, and, since it is simple, $\bar{R} \notin \mathcal{L}(\mathcal{M}_\beta)$.

4. *When $\alpha = (c, k)$ and $\beta = (c', k')$, with $k \neq k'$ (integers $\geqq 0$) and $c, c'$ arbitrary (integers $\geqq 1$, or one or both equal to $\omega$), then $\bar{\mathcal{M}}_\alpha \neq \bar{\mathcal{M}}_\beta$ and so $\mathcal{L}(\mathcal{M}_\alpha) \neq \mathcal{L}(\mathcal{M}_\beta)$.*

*Proof.* We may assume that $k \nmid k'$ (so $k' \neq 0$). If $k \neq 0$, there exists a simple ring $R$ [7; Theorem 2, p. 307] such that $t(R) = (1, k) \leqq \alpha$, $\nleqq \beta$. Thus $R \in \bar{\mathcal{M}}_\alpha$, $\notin \bar{\mathcal{M}}_\beta$. If $k = 0$, we may use Construction 1 with $R = \sum_1^\infty \oplus R_i$, where the $R_i$ are simple with $t(R_i) = (1, i)$. Then $\bar{m}(R) = (1, 0)$ and so again $R \in \bar{\mathcal{M}}_\alpha$, $\notin \bar{\mathcal{M}}_\beta$.

5. The only remaining case is $\alpha = (c,k)$, $\beta = (c',k)$, with $c \neq c'$, say $c' < c$. It is clear that if a simple ring exists for which $t(R) = (c,1)$, then an argument similar to those of the preceding paragraph would show that $\bar{\mathcal{M}}_\alpha \neq \bar{\mathcal{M}}_\beta$. However, the existence of such rings for $c > 1$ is an open question. A further question, in this case, is the following: even if $\bar{\mathcal{M}}_\alpha = \bar{\mathcal{M}}_\beta$, does it follow that $\mathcal{L}(\mathcal{M}_\alpha) = \mathcal{L}(\mathcal{M}_\beta)$?

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