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NON COMMUTATIVE L_p SPACES II

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Let M be a w^*-algebra (Von Neumann algebra), τ a semifinite, faithful, normal trace on M. There exists a w^*-dense (i.e., dense in the $\sigma(M, M_*)$-topology, where M_* is the predual of M) *-ideal J of M such that τ is a linear functional on J, and

$$x \mapsto \|x\|_p = \tau(|x|^p)^{1/p} \quad 1 \leqq p < \infty$$

(where $|x| = (x^*x)^{1/2}$) is a norm on J. The completion of J in this norm is $L_p(M, \tau)$ (see [**2**], [**8**], [**7**], and [**4**]).

If M is abelian, in which case there exists a measure space (X, μ) such that $M = L_\infty(X, \mu)$, then $L_p(X, \tau)$ is isometric, in a natural way, to $L_p(X, \mu)$. A natural question to ask is whether this situation persists if M is non-abelian. In a previous paper [**5**] it was shown that it is not possible to have a linear mapping

$$T : L_p(M, \tau) \rightarrow L_p(X, \mu), \quad p > 2$$

(where τ is a finite trace and (X, μ) a finite measure space) isometric on normal elements, unless M is abelian. In this note, this result is extended to the general case, thus showing that these non-commutative L_p spaces constitute a class of Banach spaces distinct from classical ones.

THEOREM 1. *Let M be a w^*-algebra, τ a semifinite faithful normal trace on M. Let (X, μ) be a measure space, $p > 2$, and*

$$T : L_p(M, \tau) \rightarrow L_p(X, \mu)$$

a linear mapping, isometric on normal elements. Then M is w^*-isomorphic to a w^*-subalgebra of $L_\infty(X, \mu)$, and hence is abelian.

For convenience, the proof will be broken in a series of lemmas, yielding some results on the way which will be needed later. The basic ideas of Lemmas 1 and 2 are contained in [**5**].

LEMMA 1. *Let $e, f \in M$ be projections such that $ef = fe = 0$ and $\tau(e)$, $\tau(f) < \infty$. Let X_e, $X_f \subseteq X$ be the supports of $T(e)$, $T(f) \in L_p(X, \mu)$ respectively. Then*

(i) *$X_e \cap X_f = \emptyset$ (modulo μ-null sets)*

(ii) *If $g = e + f$, $M_g = gMg$, a w^*-subalgebra of M, then $\forall\, x \in L_p(M_g, \tau)$,* supp *$Tx \subseteq X_g$ (modulo μ-null sets).*

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Proof. (i)

$$\|e+f\|_p^p + \|e-f\|_p^p = \tau(|e+f|^p) + \tau(|e-f|^p)$$
$$= \tau(e+f) + \tau(e+f) = 2\tau(e) + 2\tau(f) = 2\|e\|_p^p + 2\|f\|_p^p$$

since $ef = 0$. Therefore

$$\|Te + Tf\|_p^p + \|Te - Tf\|_p^p = 2\|Te\|_p^p + 2\|Tf\|_p^p$$

which implies (see e.g. [**6**]) that $(Te)(\omega)\cdot(Tf)(\omega) = 0$ for μ-almost all $\omega \in X$, i.e., that $X_e \cap X_f = \emptyset$, modulo μ-null sets.

(ii) We have

$$(Tg)(\omega) = (Te)(\omega) + (Tf)(\omega).$$

Therefore if $(Te)(\omega) \neq 0$, then $(Tf)(\omega) = 0$, so that $(Tg)(\omega) = (Te)(\omega) \neq 0$, i.e., $\omega \in \operatorname{supp} Tg = X_g$. Therefore if $e \in M$ is a projection smaller than g, then $\operatorname{supp} Te \subseteq X_g$. But each $x \in M_g$ may be approximated, in the operator norm, by a finite linear combination of such projections (by the spectral theorem), that is, by a $y \in M_g$ such that $\operatorname{supp} Ty \subseteq X_g$. Now

$$\|Tx - Ty\|_p \leqq 2\|x-y\|_p = 2\|g(x-y)g\|_p \leqq 2\|g\|_p^2\|x-y\|$$

(the first inequality resulting from the fact that T, being isometric on normal elements, has norm at most 2) so that Tx cannot be nonzero on a set of positive measure disjoint from X_g. Since each $x \in L_p(M_g, \tau)$ may be approximated, in the $\|\cdot\|_p$-norm, by elements of M_g, the lemma follows.

The next corollary will be needed later.

COROLLARY 1. *Let $E \subseteq M$ be a maximal family of pairwise orthogonal projections such that $\tau(e) < \infty$ for $e \in E$. Let X_e be as in Lemma 1. Then*

$$x \in L_p(M,\tau) \Rightarrow \operatorname{supp} Tx \subseteq \cup\{X_e : e \in E\}.$$

Proof. By semifiniteness, $\sum\{e : e \in E\} = 1$. Replacing each finite subfamily of E by its sum, one may construct a family $F \subseteq M$ of projections of finite trace, increasing to the identity of M. Let $x \in L_p(M,\tau)$ and $\epsilon > 0$ be given. Choose $y \in J$ such that $\|x-y\|_p < \epsilon/4$.

Now, since $p > 2$, for each $f \in F$

$$\|y - fyf\|_p^p = \tau(|y-fyf|^p) \leqq \||y-fyf|^{p-2}\|\tau(|y-fyf|^2)$$
$$\leqq (2\|y\|)^{p-2}\cdot\|y-fyf\|_2^2 \leqq (2\|y\|)^{p-2}(\|y-yf\|_2 + \|yf-fyf\|_2)^2$$
$$\leqq (2\|y\|)^{p-2}(\|y-yf\|_2 + \|y-fy\|_2\|f\|)^2$$

and

$$\|y-yf\|_2^2 = \tau((y^*-fy^*)(y-yf))$$
$$= \tau(y^*y - y^*yf - fy^*y + fy^*yf)$$
$$= \tau(y^*y - y^*yf) = \tau(y^*y - fy^*yf)$$

since

$$\tau(fy^*y) = \tau(f^2y^*y) = \tau(fy^*yf)$$

by centrality, and similarly

$$\|y - fy\|_2^2 = \tau(y^*y - y^*fy) = \tau(yy^* - yy^*f) = \tau(yy^* - fyy^*f).$$

Since F is an increasing family,

$$\sup \{fy^*yf : f \in F\} = y^*y \quad \text{and}$$
$$\sup \{fyy^*f : f \in F\} = yy^*.$$

Hence by normality of τ

$$\sup \{\tau(fy^*yf) : f \in F\} = \tau(y^*y)$$

and

$$\sup \{\tau(fyy^*f) : f \in F\} = \tau(yy^*).$$

Therefore one can choose $f \in F$ so that

$$\|y - fyf\|_p < \epsilon/4.$$

Then

$$\|Tx - T(fyf)\|_p \leqq 2\|x - fyf\|_p \leqq 2\|x - y\|_p + 2\|y - fyf\|_p < \epsilon.$$

Now $fyf \in M_f \subseteq L_p(M_f, \tau)$, hence $T(fyf)$ is supported in X_f by Lemma 1. Thus

$$\text{supp } T(fyf) \subseteq \cup \{X_e : e \in E\}$$

and hence also

$$\text{supp } Tx \subseteq \cup \{X_e : e \in E\}$$

LEMMA 2. *Let $e \in M$ be such that $\tau(e) < \infty$. Then there exists a w^*-continuous isometric *-homomorphism*

$$T_e : M_e \rightarrow L_\infty(X_e, \mu) \subseteq L_\infty(X, \mu)$$

defined by

$$T_e(x) = \frac{Tx}{Te} \quad (x \in M_e).$$

In particular, M_e is abelian.

Proof. Consider the measure μ_e defined on μ-measurable subsets A of X by

$$\mu_e(A) = \tau(e)^{-1}\int_A |(Te)(\omega)|^p d\mu(\omega).$$

μ_e is supported in X_e, and equivalent to the restriction of μ to X_e (since $|(Te)(\omega)|^p > 0$ μ-almost everywhere on X_e) so that $L_\infty(X_e, \mu_e) = L_\infty(X_e, \mu)$, a w^*-subalgebra of $L_\infty(X, \mu)$. Moreover, μ_e is a probability measure, since

$$\mu_e(X_e) = \tau(e)^{-1}\|Te\|_p{}^p = \tau(e)^{-1}\|e\|_p{}^p = 1.$$

Also scale the trace on M_e by defining a new trace t by

$$t(x) = \tau(e)^{-1}\tau(x), \quad x \in M_e$$

so that $t(e) = 1$.

Clearly $L_p(M_e, t)$, the completion of M_e in the norm $|||\cdot|||_p$ defined by

$$|||x|||_p{}^p = t(|x|^p) = ||x||_p{}^p\tau(e)^{-1},$$

coincides as a topological vector space with $L_p(M_e, \tau)$, and hence is a closed subspace of $L_p(M, \tau)$.

For $x \in M_e$, Tx/Te is a well-defined μ-measurable function supported in X_e, by Lemma 1, and hence μ_e-measurable. If $x \in M_e$ is normal,

$$\int_{X_e} \left|\frac{Tx}{Te}\right|^p d\mu_e = \tau(e)^{-1}\int |Tx|^p d\mu = \tau(e)^{-1}\|Tx\|_p{}^p$$
$$= \tau(e)^{-1}\|x\|_p{}^p = |||x|||_p{}^p.$$

This shows that the mapping $x \mapsto Tx/Te$ extends to a linear mapping

$$\bar{T}_e : L_p(M_e, t) \rightarrow L_p(X_e, \mu_e)$$

which is isometric on normal elements, and such that $\bar{T}_e(e) = 1$ (= the characteristic function of X_e). By Theorem 3 of [**5**], the restriction T_e of $\bar{T}_e$ to M_e is a w^*-continuous isometric $*$-isomorphism of M_e onto its range, a w^*-subalgebra of $L_\infty(X_e, \mu_e) = L_\infty(X_e, \mu)$, and hence of $L_\infty(X, \mu)$. In particular, M_e must be abelian.

COROLLARY 2. *M is abelian.*

Proof. As in the proof of Corollary 1, consider a family $F \subseteq M$ of projections of finite trace increasing to the identity of M. Then for each $x \in M$, the net $\{fxf : f \in M\}$ tends, in the w^*-topology, to x. Since multiplication is jointly w^*-continuous on norm-bounded subsets of M, we have, for $x, y \in M$,

$$xy = w^* - \lim\,(fxf \cdot fyf) = w^* - \lim\,(fyf \cdot fxf) = yx$$

because $fxf, fyf \in M_f$, an abelian algebra by Lemma 2.

LEMMA 3. *Let $E \subseteq M$ be as in Corollary 1. Then the mapping*

$$S : M \rightarrow L_\infty(X, \mu)$$
$$x \mapsto \sum \{T_e(xe) : e \in E\}$$

is an isometric w^*-isomorphism of M onto its range, a w^*-subalgebra of $L_\infty(X, \mu)$.

Proof. By semifiniteness, $\sum e = 1$. Therefore $x = \sum xe$ (the sum converging in the w^*-topology). But $xe = exe \in M_e$, since M is abelian. It is clear that

$$\|x\| = \sup \{\|xe\| : e \in E\}.$$

For each $e \in E$, we have a w^*-continuous isometric $*$-homomorphism

$$T_e : M_e \to L_\infty(X, \mu)$$

and $T_e(M_e) \subseteq L_\infty(X_e, \mu)$ (Lemma 2). Further, if $e, f \in E$, $e \neq f$, then $X_e \cap X_f = \emptyset$ (Lemma 1) and hence

$$T_e(M_e) \cap T_f(M_f) = 0.$$

Since, $\forall\, x \in M$,

$$\|\sum T_e(xe)\|_\infty = \sup \|T_e(xe)\|_\infty = \sup \|xe\| = \|x\|$$

the mapping S in the statement of the lemma is well-defined, $*$-linear, isometric, and multiplicative, since, for $x, y \in M$,

$$\begin{aligned} S(x)S(y) &= \sum \{T_e(xe) : e \in E\}. \sum \{T_f(yf) : f \in E\} \\ &= \sum \{T_e(xe) \cdot T_e(ye) : e \in E\} \end{aligned}$$

(for $T_e(xe) \cdot T_f(yf) = 0$ almost everywhere if $e \neq f$)

$$= \sum \{T_e(xye) : e \in E\} = S(xy).$$

Finally, $S(M) = \sum \oplus \{T_e(M_e) : e \in E\}$ is a w^*-subalgebra of $L_\infty(X, \mu)$, being the direct sum of the w^*-subalgebras $T_e(M_e)$. It thus follows automatically that S is a w^*-isomorphism of M onto $S(M)$ ([**3**], I.4, Corollary 1 of Theorem 2).

This concludes the proof of Theorem 1. The last restriction to be removed is the requirement that p be greater than 2. If $p = 2$, $L_p(M, \tau)$ is a Hilbert space, and one can go no further, unless something further is known about the isometry T, such as positivity preservation [**1**] (which, of course, holds for a $*$-homomorphism such as S). For $1 \leqq p < 2$, duality may be used:

THEOREM 2. *Let M be a w^*-algebra, τ a faithful normal semifinite trace on M, (X, μ) a measure space. For $1 \leqq p \leqq \infty$, $p \neq 2$, let*

$$T : L_p(M, \tau) \to L_p(X, \mu)$$

be an onto linear isometry. Then M is w^*-isomorphic to $L_\infty(X, \mu)$ and hence is abelian.

Proof. (i) Suppose $1 \leqq p < 2$. It is well known [**2**] that the dual of $L_p(M, \tau)$ is $L_q(M, \tau)$ where $1/q + 1/p = 1$, so that $q > 2$. (Here $L_\infty(M, \tau) = M$.) The dual map

$$T^* : L_q(X, \mu) \rightarrow L_q(M, \tau)$$

defined by

$$\tau((T^*x)y) = \int_X x(\omega) \cdot (Ty)(\omega) d\mu(\omega),$$

$(x \in L_q(X, \mu), y \in L_p(M, \tau))$ is an onto isometry, and therefore so is its inverse

$$(T^*)^{-1} : L_q(M, \tau) \rightarrow L_q(X, \mu) \quad q > 2.$$

Thus the problem is reduced to the case $p > 2$.

(ii) We use the same notation as in Theorem 1. By Corollary 1,

$$\text{supp } Tx \subseteq \cup \{X_e : e \in E\} \ \forall \ x \in L_p(M, \tau).$$

Therefore, since T is onto, we must have $\cup \{X_e : e \in E\} = X$, and the union is disjoint, by Lemma 1. Moreover, supp $Tx \subseteq X_e$ if and only if $x \in L_p(M_e, \tau)$. For, if supp $Tx \subseteq X_e$, then writing

$$Tx = \sum \{T(xf) : f \in E\}$$

one concludes that $T(xf) = 0$ if $f \neq e$, since supp $T(xf) \subseteq X_f$ and $X_f \cap X_e = \emptyset$ by Lemma 1. Thus $xf = 0$ if $f \neq e$, and hence $x = xe \in L_p(M_e, \tau)$. The converse is Lemma 1. Therefore T, restricted to $L_p(M_e, \tau)$ is an isometry onto $L_p(X_e, \mu)$. Hence it induces an onto isometry

$$\bar{T}_e : L_p(M_e, t) \rightarrow L_p(X_e, \mu_e)$$

(see the proof of Lemma 2). Both $\bar{T}_e$ and its inverse, when restricted to M_e and $L_\infty(X_e, \mu_e)$ respectively, preserve the operator (supremum) norm ([**5**], Theorem 3), and since these restrictions are inverses of each other, it follows that M_e and $L_\infty(X_e, \mu_e)$ are isomorphic.

In view of Lemma 3, the proof is complete if one observes that M is the direct sum of $\{M_e : e \in E\}$, and $L_\infty(X, \mu) = L_\infty(\cup X_e, \mu)$ is the direct sum of $\{L_\infty(X_e, \mu) : e \in E\}$ where each $L_\infty(X_e, \mu) = L_\infty(X_e, \mu_e)$ since μ_e and μ are equivalent on X_e.

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