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ISOPERIMETRIC INEQUALITIES ON SURFACES OF CONSTANT CURVATURE

HSU-TUNG KU, MEI-CHIN KU AND XIN-MIN ZHANG

ABSTRACT. In this paper we introduce the concepts of hyperbolic and elliptic areas and prove uncountably many new geometric isoperimetric inequalities on the surfaces of constant curvature.

1. **Introduction.** Isoperimetric problems for plane polygons and polyhedra in $\mathbf{R}^3$ date back to ancient times. Considerable contributions were made to this area in 19th century by Steiner, Lindelöf, Schwarz, Brunn, Minkowski, *etc*. Different methods and techniques that have evolved ever since cover many branches of mathematics such as combinatorics, calculus of variations, group theory, differential and integral geometry, geometric measure theory, *etc*. During the last two decades, advances in computer sciences, crystallography, geometric tomography and other practical sciences have shown that research in geometric extremum problems for polygons and polyhedra are more interesting and important than ever before. However, "very little is known about the isoperimetric problems for non-Euclidean polytopes. One reason may be that the measurement of volume in non-Euclidean space is rather complicated" [5, p. 213]. In this paper, we shall be concerned with isoperimetric problems for polygonal curves in non-Euclidean planes. It is known that the plane trigonometry laid the foundation for geometry of plane polygons, and the Heron's formula for a triangle and the Brahmagupta's formula for a cyclic quadrilateral are the two keystones. These facts inspired us to approach isoperimetric problems for non-Euclidean polygons by introducing "hyperbolic" and "elliptic" areas and lengths that are based on non-Euclidean Heron's formulas. We then are able to establish a unified Heron formula for a triangle, and a unified Brahmagupta formula for a cyclic quadrilateral on surfaces of constant curvature. These new definitions of length and area for polygons are compatible with existing ones, and can be used to give a unified isoperimetric inequality for polygonal curves on Euclidean plane, sphere, and hyperbolic plane. The main techniques used in this paper are discrete analytic inequalities which include many new interesting inequalities involving trigonometric functions and hyperbolic trigonometric functions. There is no doubt that these analytic inequalities are important in their own right, and applicable to many problems in other fields.

Let M be a complete, connected, simply connected surface with constant Gaussian

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curvature $K_M = 0, -1$, or 1 respectively, that is,

$$M = \begin{cases} \mathbf{R}^2, & \text{Euclidean plane,} & (K_M = 0) \\ \mathbf{H}^2(-1), & \text{hyperbolic surface,} & (K_M = -1) \\ \mathbf{S}^2(1), & \text{unit 2-sphere,} & (K_M = 1), \end{cases}$$

and let P_n be an n-sided polygon in M (*i.e.*, a simple closed curve consisting of n geodesic segments which is not smooth at n-vertices) with length $L(P_n)$ which encloses a domain of area $A(P_n)$. If $M = \mathbf{R}^2$, the classical isoperimetric inequality states that (*cf*. [6, 7, 8])

$$L^2(P_n) \geq 4d_n A(P_n), \quad d_n = n \tan \frac{\pi}{n}. \tag{1}$$

Equality holds if and only if P_n is regular. We shall prove that the isoperimetric inequalities similar to (1) also hold for P_n in the surfaces $\mathbf{H}^2(-1)$ and $\mathbf{S}^2(1)$.

Now let us fix some notations. We shall assume that P_n is cyclic, that is, the vertices $\{A_i\}_{1 \leq i \leq n}$ are on a circle of radius r (arranged in counterclockwise order and assuming $r < 1$ if $K_M = 1$) with center at O, and O is inside the domain bounded by P_n. Let B_i be the point lying on the geodesic joining A_i and A_{i+1} (setting $A_{n+1} = A_1$) so that OB_i is perpendicular to $A_i A_{i+1}$, $1 \leq i \leq n$. For $1 \leq i \leq n$, set

α_i = angle $\angle A_i O B_i$, β_i = angle $\angle O A_i B_i$,
a_i = length of $A_i B_i$, b_i = length of OB_i, and
F_i = area of the triangle $\triangle OA_iB_i$.

From the definition we have

$$L(P_n) = 2 \sum_{i=1}^{n} a_i, \tag{2}$$

$$A(P_n) = 2 \sum_{i=1}^{n} F_i. \tag{3}$$

In order to have unified statements for manifolds M with $K_M = 0, -1$, and 1, set

$$k = \begin{cases} p, & \text{if } M = \mathbf{R}^2, \\ h, & \text{if } M = \mathbf{H}^2(-1), \\ e, & \text{if } M = \mathbf{S}^2(1), \end{cases}$$

and

$$u_k(\omega) = \begin{cases} \omega, & \text{if } k = p, \\ uh(\omega), & \text{if } k = h, \\ u(\omega), & \text{if } k = e, \end{cases} \tag{4}$$

where $u = \sin, \cos, \tan$, *etc*. For instance,

$$\sin_k(\omega) = \begin{cases} \omega, & \text{if } k = p, \\ \sinh(\omega), & \text{if } k = h, \\ \sin(\omega), & \text{if } k = e. \end{cases}$$

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By using (4), the law of sines can be stated as follows. Let $\triangle ABC$ be a geodesic triangle in M with lengths of edges BC, CA and AB equal to a, b and c respectively. Then

$$\frac{\sin A}{\sin_k(a)} = \frac{\sin B}{\sin_k(b)} = \frac{\sin C}{\sin_k(c)}, \qquad k = p, h, \text{ and } e. \tag{5}$$

If we set $s = \frac{1}{2}(a+b+c)$ and F = area of $\triangle ABC$, then the well-known Heron formulas are given as follows (*cf.* [6, 10]):

$$F = \left\{s(s-a)(s-b)(s-c)\right\}^{1/2} \tag{6}$$

if $M = \mathbf{R}^2$, and if $M = \mathbf{H}^2(-1)$, then

$$2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}\sin\frac{F}{2} = \left\{\sinh s\sinh(s-a)\sinh(s-b)\sinh(s-c)\right\}^{1/2}. \tag{7}$$

For the triangle $\triangle OA_iA_{i+1}$ in $\mathbf{H}^2(-1)$, we have $a = 2a_i$, and $b = c = r$, hence (7) becomes

$$(1+\cosh r)\cosh a_i \sin F_i = \left\{\sinh s\sinh(s-a)\sinh(s-b)\sinh(s-c)\right\}^{1/2}. \tag{8}$$

Formulas similar to (7) and (8) also hold for $M = \mathbf{S}^2(1)$. It is well-known that the main distinction among the three geometries lies in the laws of sines (5) and cosines which together with (7) and (8) indicate that the study of trigonometry of such as $\sin_k(a)$, *etc.* for $k = p, h$ and e; and $(1+\cos_k r)\cos_k a_i$, $k = h, e$, *etc.*, are essential and important in these geometries. This motivates us to introduce the following concepts of hyperbolic and elliptic lengths and areas.

DEFINITION 1.1. (a) (Hyperbolic and Elliptic Lengths) Define

$$L_k(P_n) = 2\sum_{i=1}^{n}\sin_k(a_i), \qquad (k = p, h, e)$$

$$\tilde{L}_k(P_n) = 2\sum_{i=1}^{n}\sin_k(b_i), \qquad (k = p, h, e)$$

$$L_k(P_n; c) = 2\sum_{i=1}^{n}\left\{\cos_k(a_i) + \cos_k(b_i)\right\}, \qquad (k = h, e)$$

$$L_k(P_n; u, v) = 2\sum_{i=1}^{n}u_k(a_i)v_k(b_i), \qquad (k = h, e)$$

where $u, v = \sin, \cos, \tan$, *etc*.

(b) (Hyperbolic and Elliptic Areas) For $k = h,\ e$, define

$$A_k(P_n) = \left\{1+\cos_k(r)\right\}\sum_{i=1}^{n}\cos_k(a_i)\sin F_i,$$

$$\tilde{A}_k(P_n) = \left\{1+\cos_k(r)\right\}\sum_{i=1}^{n}\cos_k(b_i)\sin F_i, \quad \text{and}$$

$$\mathcal{A}_k(P_n) = A_k(P_n) + \tilde{A}_k(P_n).$$

Set $A_p(P_n) = \tilde{A}_p(P_n) = A(P_n)$ if $M = \mathbf{R}^2$.

(c) (Hyperbolic and Elliptic Constants)

$$d_n(P_n) = n\tan\frac{A(P_n)}{2n}, \qquad d_n^k(P_n) = n\tan(Q_n^k),$$
$$\delta_n(P_n) = n\cot\frac{A(P_n)}{2n}, \qquad \delta_n^k(P_n) = n\cot(Q_n^k),$$

where

$$Q_n^k = \begin{cases} \frac{(n-2)\pi - A(P_n)}{2n}, & \text{if } k = h \\ \frac{(n-2)\pi + A(P_n)}{2n}, & \text{if } k = e. \end{cases}$$

For $M = \mathbf{R}^2$, set $d_n^p(P_n) = d_n$.

Observe that $A(P_n) < 2\pi$, if $k = e$ because $r < 1$, and by Gauss-Bonnet theorem [3]

$$2\sum_{i=1}^n \beta_i = \begin{cases} (n-2)\pi - A(P_n), & \text{if } k = h \\ (n-2)\pi + A(P_n), & \text{if } k = e. \end{cases}$$

Hence we have

$$0 < A(P_n)/2n < \pi/n, \quad \text{and} \quad 0 < Q_n^k < \pi/2, \quad k = h, e. \tag{9}$$

In terms of hyperbolic and elliptic lengths and areas, we shall establish the following isoperimetric inequalities which generalize (1).

THEOREM 1.2.
(a) $L_k^2(P_n) \geq 4d_n A_k(P_n)$, $\quad (k = p, h, e)$
(b) $\tilde{L}_k^2(P_n) \geq 4d_n^k(P_n)\tilde{A}_k(P_n)$, $\quad (k = p, h, e)$
(c) $L_k^2(P_n; \sin, \sin) \geq 4d_n(P_n)\mathcal{A}_k(P_n)$, $\quad (k = h, e)$.
Equality holds in any of (a), (b) and (c) if and only if P_n is regular, that is, $a_1 = a_2 = \cdots = a_n$.

THEOREM 1.3. *Let $\delta_n = n\cot\frac{\pi}{n}$.*
(a) $L_k^2(P_n; \cos, \sin) \geq 4\delta_n A_k(P_n)$, $\quad (k = p, h, e)$
(b) $L_k^2(P_n; \sin, \cos) \geq 4\delta_n^k(P_n)\tilde{A}_k(P_n)$, $\quad (k = h, e)$
(c) $L_k^2(P_n; c) \geq 4\delta_n(P_n)\mathcal{A}_k(P_n)$, $\quad (k = h, e)$
Equality holds in any of (a), (b) and (c) if and only if P_n is regular.

In [14], Theorem 1.2(b) for $k = p$ was proved by Tang. Let us observe that the definition $A_k(P_n)$ for $k = h, e$ is quite natural and consistent with the definition of $A_p(P_n)$. For if P_n lies in $\mathbf{R}^2$, we have

$$A_p(P_n) = A(P_n) = \frac{1}{2}\sum_{i=1}^n (2a_i)b_i. \tag{10}$$

In section 3, we shall verify the following:

$$A_k(P_n) = \frac{1}{2}\sum_{i=1}^n \sin_k(2a_i)\sin_k(b_i), \quad (k = h, e). \tag{11}$$

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Hence, we can combine (10) and (11) so that (11) also holds for $k = p$. We can also have uniform statements for the Heron's formula and the formula of Brahmagupta (*cf.* Lemma 3.4). More precisely, we have

$$A_k(P_3) = \Big\{ s\big[s - 2s_k(a_1)\big]\big[s - 2s_k(a_2)\big]\big[s - 2s_k(a_3)\big]\Big\}^{\frac{1}{2}}, \tag{12}$$

where $k = p, h, e$, $s_k(a_i) = \sin_k(a_i)$, $i = 1, 2, 3$, and $s = s_k(a_1) + s_k(a_2) + s_k(a_3)$; and

$$A_k(P_4) = \Big\{ \big[s - 2s_k(a_1)\big]\big[s - 2s_k(a_2)\big]\big[s - 2s_k(a_3)\big]\big[s - 2s_k(a_4)\big]\Big\}^{\frac{1}{2}}, \tag{13}$$

where $s = \sum_{i=1}^{4} s_k(a_i)$, $k = p, h, e$. If $k = p$, (12) and (13) are simply the well-known Heron's formula (6) and Brahmagupta formula. These are simply some of our results which deal uniformly for three different geometries. In this paper we shall prove many new isoperimetric inequalities including those which generalize both Theorem 1.2 and Theorem 1.3.

The study of isoperimetric inequalities is very important in geometry and mathematical physics. It is also useful in analysis, particularly, differential equations. For instance, the famous Faber-Krahn inequality showed that the classical isoperimetric inequality for simple closed plane curves is equivalent to the physical isoperimetric inequality which is concerned with the first eigenvalue of the Dirichlet problem ([3, 11]). This inequality has important consequence in physics. Thus, we expect that our new isoperimetric inequalities will have useful applications as well.

2. **Pseudo-Polygons.** In this section we shall introduce the concept of pseudo-polygon in the plane so that for a polygon P_n in M we can construct pseudo-polygons $\hat{P}_n$ in $\mathbf{R}^2$. We will show that the areas $A_k(P_n)$ and $\tilde{A}_k(P_n)$ can be computed from the areas of $\hat{P}_n$'s. We shall use these results to prove formulas (12) and (13).

To simplify the notations, we shall denote $\sin_k$, $\cos_k$ and $\tan_k$ simply by s_k, c_k and t_k respectively. Since $\angle OB_iA_i = \frac{\pi}{2}$, $1 \leq i \leq n$, the following formulas are well-known (*cf.* [10]), where $k = p, h, e$,

$$\sin \alpha_i = s_k(a_i)/s_k(r), \tag{14}$$

$$\cos \alpha_i = t_k(b_i)/t_k(r), \tag{15}$$

$$\tan \alpha_i = t_k(a_i)/s_k(b_i). \tag{16}$$

If $k = h$ and e, we have (Pythagorean Theorem, *cf.* [10])

$$\cos_k(r) = \cos_k(a_i)\ \cos_k(b_i). \tag{17}$$

Let us observe that $0 < F_i < \pi/2$ for $i = 1, 2, \ldots, n$ by Gauss-Bonnet theorem because $F_i = \pi/2 - (\alpha_i + \beta_i)$ if $k = h$, and $\alpha_i + \beta_i = \pi/2 + F_i < \pi$ if $k = e$ (for $r < 1$).

LEMMA 2.1. *Let $k = h, e$. Then for $1 \le i \le n$, we have*

(*a*)
$$\sin F_i = \frac{s_k(a_i)s_k(b_i)}{1 + c_k(r)}.$$

(*b*)
$$\cos F_i = \frac{c_k(a_i) + c_k(b_i)}{1 + c_k(r)}.$$

PROOF. Let $\varepsilon = 1$ if $k = h$ and $\varepsilon = -1$ if $k = e$.

(a)
$$\begin{aligned}
\sin F_i &= \varepsilon \cdot \cos(\alpha_i + \beta_i) \\
&= \varepsilon \left\{ \frac{t_k(a_i)}{t_k(r)} \frac{t_k(b_i)}{t_k(r)} - \frac{s_k(a_i)}{s_k(r)} \frac{s_k(b_i)}{s_k(r)} \right\} && \text{(by (14), (15))} \\
&= \varepsilon \left\{ \frac{s_k(a_i)s_k(b_i)}{s_k^2(r)} \big[c_k(r) - 1\big] \right\} && \text{(by (17))} \\
&= \frac{s_k(a_i)s_k(b_i)}{c_k^2(r) - 1} \big[c_k(r) - 1\big] \\
&= \frac{s_k(a_i)s_k(b_i)}{1 + c_k(r)}.
\end{aligned}$$

(b)
$$\begin{aligned}
\cos F_i &= \sin\left(\frac{\pi}{2} \pm F_i\right) \\
&= \sin(\alpha_i + \beta_i) \\
&= \frac{s_k^2(a_i)c_k(b_i) + s_k^2(b_i)c_k(a_i)}{s_k^2(r)} && \text{(by (14), (15), (17))} \\
&= \varepsilon \frac{\{c_k^2(a_i) - 1\}c_k(b_i) + \{c_k^2(b_i) - 1\}c_k(a_i)}{s_k^2(r)} \\
&= \varepsilon \frac{\{c_k(a_i)c_k(b_i) - 1\}\{c_k(a_i) + c_k(b_i)\}}{c_k^2(r) - 1} \\
&= \frac{c_k(a_i) + c_k(b_i)}{1 + c_k(r)} && \text{(by (17))}.
\end{aligned}$$

DEFINITION 2.2 (PSEUDO-POLYGON). Let $\{\hat{A}_i : 1 \le i \le n + 1\}$ be a set of points on the circle with center at the origin O in $\mathbf{R}^2$ with radius $\hat{r}$ such that if we identify $\mathbf{R}^2$ with the complex plane, then

$$\hat{A}_i = \hat{r} \exp(\sqrt{-1}\theta_i), \quad 1 \le i \le n + 1$$

where $\theta_1 < \theta_2 < \cdots < \theta_{n+1}$ and $0 < \theta_{i+1} - \theta_i < \pi$, $1 \le i \le n$. A *pseudo-polygon* $\hat{P}_n$ is the polygonal path that joins the successive points $\hat{A}_n$'s.

For simplicity of statement, we identify $\hat{P}_n$ with the set $\{\hat{r}, (\hat{\alpha}_i, \hat{a}_i) : 1 \le i \le n\}$, where $2\hat{\alpha}_i = \theta_{i+1} - \theta_i$ and $2\hat{a}_i =$ length of $\hat{A}_i\hat{A}_{i+1}$. Notice that if $\sum_{i=1}^n \hat{\alpha}_i = m\pi$ for some

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positive integer m, then $\hat{P}_n$ is a polygon with $\hat{A}_{n+1} = \hat{A}_1$ which is simple and convex if $m = 1$. Moreover, if $m \geq 2$, n and m coprime, and $\hat{\alpha}_1 = \cdots = \hat{\alpha}_n = m\pi/n$, then $\hat{P}_n$ is a "star polygon" (*cf.* [4, p. 93]). Define the area $A(\hat{P}_n)$ and length $L(\hat{P}_n)$ of $\hat{P}_n$ by

$$A(\hat{P}_n) = \sum_{i=1}^{n} \text{area of } \triangle \hat{A}_i O \hat{A}_{i+1}, \tag{18}$$

and

$$L(\hat{P}_n) = 2\sum_{i=1}^{n} \hat{a}_i. \tag{19}$$

Let us set

$$\tilde{\mathbf{R}}^n_+ = \{\mathbf{\Theta} = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbf{R}^n : L_n(\mathbf{\Theta}) - 4\theta_i > 0, \text{ and } \theta_i > 0, 1 \leq i \leq n\},$$

where $L_n(\mathbf{\Theta}) = 2\sum_{j=1}^n \theta_j$. Then we have the following basic property for a pseudo-polygon.

LEMMA 2.3. *For a pseudo-polygon* $\hat{P}_n$, *if* $\sum_{i=1}^n \hat{\alpha}_i \geq \pi$, *then* $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n) \in \tilde{\mathbf{R}}^n_+$.

PROOF. We need to show that

$$\sum_{i=1}^{n} \hat{a}_i - 2\hat{a}_j > 0, \quad j = 1, 2, \ldots, n. \tag{20}$$

This is true if $\hat{P}_n$ is a polygon. Assume that $\hat{P}_n$ is not a polygon, hence $\sum_{i=1}^n \hat{\alpha}_i > \pi$ and $\hat{A}_{n+1} \neq \hat{A}_1$. Case (i). $j \neq 1, n$. It suffices to verify that $\hat{A}_j\hat{A}_{j+1}$ is an edge of a polygon which is a subset of the curve $\hat{P}_n$. Observe that there exist integers x and y, $2 \leq x \leq j$ and $j+1 \leq y \leq n$, so that the edge $\hat{A}_{x-1}\hat{A}_x$ and the edge $\hat{A}_y\hat{A}_{y+1}$ intersect at a point, say $\hat{Q}$. Otherwise we would have $\sum_{i=u}^{j-1} \hat{\alpha}_i + \hat{\alpha}_j + \sum_{i=j+1}^{v} \hat{\alpha}_i < \pi$, for all u, v where $1 \leq u < j$, and $j+1 \leq v \leq n$. However, this contradicts the hypothesis $\sum_{i=1}^n \hat{\alpha}_i > \pi$. Thus $\hat{A}_j\hat{A}_{j+1}$ is an edge of the polygon with vertices $\{\hat{Q}, \hat{A}_x, \ldots, \hat{A}_j, \hat{A}_{j+1}, \ldots, \hat{A}_y\}$. Case (ii). $j = 1, n$. Let us prove the case $j = 1$. The proof for $j = n$ is similar. If $\hat{A}_1 = \hat{A}_q$ for some $q \neq 1$, then clearly (20) holds. Thus, $\hat{A}_1 \neq \hat{A}_q$, $q \neq 1$. Let $s \geq 3$ be such that $\sum_{i=1}^{s-1} \hat{\alpha}_i < \pi$ and $\sum_{i=1}^{s} \hat{\alpha}_i > \pi$. Then $\hat{A}_s$ and $\hat{A}_{s+1}$ are on the right and left of $\hat{A}_1$ respectively, and so $\hat{\alpha}_s = \angle\hat{A}_s O \hat{A}_{s+1} > \angle\hat{A}_s O \hat{A}_1$, because $\pi = \sum_{i=1}^{s-1} \hat{\alpha}_i + \angle\hat{A}_s O \hat{A}_1$. This will imply the following:

$$2\hat{a}_s = \text{length of } \hat{A}_s\hat{A}_{s+1} > \text{length of } \hat{A}_s\hat{A}_1.$$

There is a polygon $\hat{P}$ (as a curve) such that

$$\hat{A}_1\hat{A}_2 \cup \hat{A}_s\hat{A}_1 \subset \hat{P} \subset \hat{P}_n \cup \hat{A}_s\hat{A}_1, \quad \text{and} \quad \hat{A}_s\hat{A}_{s+1} \not\subset \hat{P},$$

that is, $\hat{A}_1\hat{A}_2$ and $\hat{A}_s\hat{A}_1$ are two edges of $\hat{P}$, and $\hat{A}_s\hat{A}_{s+1}$ is not an edge of $\hat{P}$. This will give the desired inequality (20).

DEFINITION 2.4. Let P_n be an n-sided polygon in M as in Section 1. Since

$$\sin\alpha_i = s_k(a_i)/s_k(r), \qquad \sin(\pi/2-\alpha_i) = t_k(b_i)/t_k(r),$$
$$\sin\beta_i = s_k(b_i)/s_k(r), \qquad \sin(\pi/2-\beta_i) = t_k(a_i)/t_k(r),$$

and by Lemma 2.1

$$\sin F_i = s_k(a_i)s_k(b_i)/\big\{1+c_k(r)\big\}, \qquad (k=h,e)$$
$$\sin(\frac{\pi}{2}-F_i) = \big\{c_k(a_i)+c_k(b_i)\big\}/\big\{1+c_k(r)\big\}, \qquad (k=h,e)$$

hence we can construct n-sided pseudo-polygons $P_{n,\alpha}(k)$, $P_{n,F}(k)$, *etc.*, as follows:

$$P_{n,\alpha}(k) = \Big\{s_k(r), \big(\alpha_i, s_k(a_i)\big) : 1\le i\le n\Big\}, \qquad (k=p,h,e)$$
$$\tilde P_{n,\alpha}(k) = \Big\{t_k(r), \big(\frac{\pi}{2}-\alpha_i, t_k(b_i)\big) : 1\le i\le n\Big\}, \qquad (k=p,h,e)$$
$$P_{n,\beta}(k) = \Big\{s_k(r), \big(\beta_i, s_k(b_i)\big) : 1\le i\le n\Big\}, \qquad (k=p,h,e)$$
$$\tilde P_{n,\beta}(k) = \Big\{t_k(r), \big(\frac{\pi}{2}-\beta_i, t_k(a_i)\big) : 1\le i\le n\Big\}, \qquad (k=p,h,e)$$
$$P_{n,F}(k) = \Big\{1+c_k(r), \big(F_i, s_k(a_i)s_k(b_i)\big) : 1\le i\le n\Big\}, \qquad (k=h,e)$$
$$\tilde P_{n,F}(k) = \Big\{1+c_k(r), \big(\frac{\pi}{2}-F_i, c_k(a_i)+c_k(b_i)\big) : 1\le i\le n\Big\}, \qquad (k=h,e).$$

LEMMA 2.5. *$\tilde P_{n,\alpha}(k)$, n even and $P_{n,\alpha}(k)$ are polygons for $k=p,h,e$, moreover,*

$$P_{n,\beta}(p) = \tilde P_{n,\alpha}(p) \quad \textit{and} \quad \tilde P_{n,\beta}(p) = P_{n,\alpha}(p), \quad n \textit{ even.}$$

For any real valued function S defined on some interval in $\mathbf{R}$, and for any $\mathbf{\Theta}\in\tilde{\mathbf{R}}^n_+$, we shall set

$$S(\mathbf{\Theta}) = \big(S(\theta_1), S(\theta_2), \ldots, S(\theta_n)\big).$$

Thus if we let

$$\mathbf{a}=(a_1,a_2,\ldots,a_n), \qquad \mathbf{b}=(b_1,b_2,\ldots,b_n), \qquad \alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n),$$
$$\beta=(\beta_1,\beta_2,\ldots,\beta_n), \qquad \mathbf{F}=(F_1,F_2,\ldots,F_n), \quad \text{then}$$
$$s_k(\mathbf{a}) = \big(s_k(a_1), s_k(a_2), \ldots, s_k(a_n)\big), \quad \text{and} \quad \sin(\alpha) = (\sin\alpha_1,\ldots,\sin\alpha_n), \quad \textit{etc.}$$

LEMMA 2.6.
(a) $s_k(\mathbf{a}), \sin(\alpha)\in\tilde{\mathbf{R}}^n_+$, $\;n\ge 3$, $\;k=p,h,e$.
(b) $t_k(\mathbf{b}), \cos(\alpha)\in\tilde{\mathbf{R}}^n_+$, $\;n\ge 4$, $\;k=p,h,e$.
(c) $s_k(\mathbf{b}), \sin(\beta)\in\tilde{\mathbf{R}}^n_+$, $\;n\ge 4$, $\;k=p,e$; *or* $k=h$ *and* $(n-4)\pi\ge A(P_n)$.
(d) $t_k(\mathbf{a}), \cos(\beta)\in\tilde{\mathbf{R}}^n_+$, $\;n\ge 3$, $\;k=h$.
(e) $\big(c_k(a_1)+c_k(b_1),\ldots,c_k(a_n)+c_k(b_n)\big), \cos(\mathbf{F})\in\tilde{\mathbf{R}}^n_+$, *if* $k=h$; *or* $k=e$ *and* $(n-2)\pi\ge 2\sum_{i=1}^n\beta_i$.

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PROOF. The results are immediate consequences of Lemma 2.1 and Lemma 2.3, since we have

(a) $\sum_{i=1}^{n} \alpha_i = \pi$, and $s_k(\mathbf{a}) = s_k(r)\sin(\alpha)$.

(b) $\sum_{i=1}^{n}(\frac{\pi}{2} - \alpha_i) \geq \pi$ if $n \geq 4$, and $t_k(\mathbf{b}) = t_k(r)\cos(\alpha)$.

(c) $\sum_{i=1}^{n} \beta_i \geq \pi$, if $n \geq 4$, $k = p, e$; or $k = h$, and $(n-4)\pi \geq A(P_n)$, and $s_k(\mathbf{b}) = s_k(r)\sin(\beta)$.

(d) $\sum_{i=1}^{n}(\frac{\pi}{2} - \beta_i) > \pi$, if $k = h$, and $t_k(\mathbf{b}) = t_k(r)\cos(\beta)$.

(e) $\sum_{i=1}^{n}(\frac{\pi}{2} - F_i) \geq \pi$ if $k = h$; or $k = e$ and $(n-2)\pi \geq 2\sum_{i=1}^{n} \beta_i$.

3. Elliptic and Hyperbolic Areas.

PROPOSITION 3.1. *If $k = h$ and e,*

(a) $A_k(P_n) = \frac{1}{2}\sum_{i=1}^{n} s_k(2a_i)s_k(b_i)$,

(b) $\tilde{A}_k(P_n) = \frac{1}{2}\sum_{i=1}^{n} s_k(a_i)s_k(2b_i)$,

(c) $\mathcal{A}_k(P_n) = \sum_{i=1}^{n} s_k(a_i)s_k(b_i)\{c_k(a_i) + c_k(b_i)\}$.

PROOF. It follows from the definition of $A_k(P_n)$ and Lemma 2.1 that

$$\begin{aligned} A_k(P_n) &= \left\{1 + c_k(r)\right\}\sum_{i=1}^{n} c_k(a_i)\frac{s_k(a_i)s_k(b_i)}{1 + c_k(r)} \\ &= \sum_{i=1}^{n} s_k(a_i)c_k(a_i)s_k(b_i) \\ &= \frac{1}{2}\sum_{i=1}^{n} s_k(2a_i)s_k(b_i). \end{aligned}$$

The proof of (b) is similar, and (c) follows from (a) and (b).

Now we shall give geometric interpretations of the areas $A_k(P_n)$, $\tilde{A}_k(P_n)$ and $\mathcal{A}_k(P_n)$. Set

$$\varepsilon_k = \begin{cases} -1, & \text{if } k = h \\ 1, & \text{if } k = p, e. \end{cases}$$

THEOREM 3.2. *For $n \geq 3$,*

(a) $A_k(P_n) = A\big(P_{n,\alpha}(k)\big) = c_k^2(r)A\big(\tilde{P}_{n,\alpha}(k)\big)$, $\qquad (k = p, h, e)$

(b) $\tilde{A}_k(P_n) = A\big(P_{n,\beta}(k)\big) = c_k^2(r)A\big(\tilde{P}_{n,\beta}(k)\big)$, $\qquad (k = p, h, e)$

(c) $\mathcal{A}_k(P_n) = A\big(P_{n,F}(k)\big) = A\big(\tilde{P}_{n,F}(k)\big) = A\big(P_{n,\alpha}(k)\big) + A\big(P_{n,\beta}(k)\big)$, $\qquad (k = h, e)$.

PROOF. Let $k = h, e$. For $k = p$, the proof is easy.

(a)
$$\begin{aligned} A\big(P_{n,\alpha}(k)\big) &= \sum_{i=1}^{n} s_k(a_i)\left\{s_k^2(r) - s_k^2(a_i)\right\}^{1/2} \\ &= \sum_{i=1}^{n} s_k(a_i)\left\{\varepsilon_k\left[1 - c_k^2(r)\right] - s_k^2(a_i)\right\}^{1/2} \\ &= \sum_{i=1}^{n} s_k(a_i)\left\{\varepsilon_k\left[c_k^2(a_i) - c_k^2(a_i)c_k^2(b_i)\right]\right\}^{1/2} \qquad \text{(by (17))} \\ &= \sum_{i=1}^{n} s_k(a_i)c_k(a_i)s_k(b_i) \\ &= A_k(P_n). \qquad \text{(by Prop. 2.2)} \end{aligned}$$

Also,

$$\begin{aligned} c_k^2(r)A\big(\tilde{P}_{n,\alpha}(k)\big) &= c_k^2(r)\sum_{i=1}^{n} t_k(b_i)\big\{t_k^2(r)-t_k^2(b_i)\big\}^{1/2}\\ &= c_k(r)\sum_{i=1}^{n} t_k(b_i)\big\{s_k^2(r)-c_k^2(a_i)s_k^2(b_i)\big\}^{1/2}\\ &= c_k(r)\sum_{i=1}^{n} t_k(b_i)\Big\{\varepsilon_k\big[1-c_k^2(r)\big]-c_k^2(a_i)s_k^2(b_i)\Big\}^{1/2}\\ &= c_k(r)\sum_{i=1}^{n} t_k(b_i)\Big\{\varepsilon_k-c_k^2(a_i)\big[\varepsilon_k c_k^2(b_i)+s_k^2(b_i)\big]\Big\}^{1/2}\\ &= c_k(r)\sum_{i=1}^{n} t_k(b_i)\Big\{\varepsilon_k\big[1-c_k^2(a_i)\big]\Big\}^{1/2}\\ &= c_k(r)\sum_{i=1}^{n} \frac{s_k(b_i)}{c_k(b_i)}s_k(a_i)\\ &= A_k(P_n). \end{aligned}$$

(*b*)
$$A\big(P_{n,\beta}(k)\big)=\sum_{i=1}^{n} s_k(b_i)\big\{s_k^2(r)-s_k^2(b_i)\big\}^{1/2}=\tilde{A}_k(P_n).$$

Similarly,

$$c_k^2(r)A\big(\tilde{P}_{n,\beta}(k)\big)=c_k^2(r)\sum_{i=1}^{n} t_k(a_i)\big\{t_k^2(r)-t_k^2(b_i)\big\}^{1/2}=\tilde{A}_k(P_n).$$

(c) A simple calculation will give

$$\big\{1+c_k(r)\big\}^2-s_k^2(a_i)s_k^2(b_i)=\big\{c_k(a_i)+c_k(b_i)\big\}^2.$$

Hence

$$A\big(P_{n,F}(k)\big)=\sum_{i=1}^{n} s_k(a_i)s_k(b_i)\Big\{\big[1+c_k(r)\big]^2-s_k^2(a_i)s_k^2(b_i)\Big\}^{1/2}=A_k(P_n).$$

Likewise,

$$A\big(\tilde{P}_{n,F}(k)\big)=\sum_{i=1}^{n}\big\{c_k(a_i)+c_k(b_i)\big\}\Big\{\big[1+c_k(r)\big]^2-\big[c_k(a_i)+c_k(b_i)\big]^2\Big\}^{1/2}=A_k(P_n).$$

As immediate corollaries we obtain the following new Heron's type formulae.

THEOREM 3.3 (HERON FORMULA). *Let* $s=s_k(a_1)+s_k(a_2)+s_k(a_3)$, $k=p,h,e$. *Then*

(21)
$$A_k(P_3)=\Big\{s\big[s-2s_k(a_1)\big]\big[s-2s_k(a_2)\big]\big[s-2s_k(a_3)\big]\Big\}^{1/2}.$$

PROOF. Apply the Heron formula (6) and Theorem 3.2 to the triangle $P_{3,\alpha}(k)$.

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LEMMA 3.4 (BRAHMAGUPTA [2], [6]). *Let a, b, c and d denote the lengths of the sides of a cyclic quadrilateral P_4 in $\mathbf{R}^2$. Then*

$$A(P_4) = \Big\{(s-a)(s-b)(s-c)(s-d)\Big\}^{1/2}, \tag{22}$$

where $s = \frac{1}{2}(a+b+c+d)$.

Since $P_{4,\alpha}(k)$ and $\tilde{P}_{4,\beta}(p)$ are cyclic quadrilaterals in $\mathbf{R}^2$, it follows from Theorem 3.2 and Lemma 3.4 that we have

THEOREM 3.5.

(*a*) $$A_k(P_4) = \Big\{\big[s-2s_k(a_1)\big]\big[s-2s_k(a_2)\big]\big[s-2s_k(a_3)\big]\big[s-2s_k(a_4)\big]\Big\}^{1/2},$$

where $s = \sum_{i=1}^4 s_k(a_i)$, *and* $k = p, h, e$.

(*b*) $$A_p(P_4) = \Big\{(s-2b_1)(s-2b_2)(s-2b_3)(s-2b_4)\Big\}^{1/2},$$

where $s = \sum_{i=1}^4 b_i$.

We might expect that similar formulae for $A_k(P_n)$ exist for cyclic polygons P_n, $n \geq 5$. This is not true in general. For instance, if $k = p$, $n = 5$ (resp. 6), Robbins [12] has proved that if we let $u = 16A_p^2(P_n)$, then u satisfies a monic polynomial equation of degree 7, and if $n = 7$, u satisfies a monic polynomial equation of degree 38 with some of the coefficients the solution of a system of linear equations with 143,307 unknowns. Yet, we are able to find formulas for $A_k(P_n)$ under some restrictions as follows.

Now, let $n = 2m$ ($m \geq 2$), let $P_n = P_{m,m}(k)$ be a cyclic $2m$-gon in M with m sides of length $2a_1$ and remaining m sides of length $2a_2$. Say,

$$a_1 = a_3 = \cdots = a_{2m-1}, \quad \text{and} \quad a_2 = a_4 = \cdots = a_{2m}.$$

THEOREM 3.6. *Let* $n = 2m \geq 2$, *and* $k = p, h, e$. *Then*

$$A_k\big(P_{m,m}(k)\big) = \frac{m}{\sin\left(\frac{\pi}{m}\right)}\Big\{\big[s_k^2(a_1)+s_k^2(a_2)\big]\cos\frac{\pi}{m} + 2s_k(a_1)s_k(a_2)\Big\}. \tag{23}$$

PROOF. In [8], MacNab has proved the following:

$$A_p\big(P_{m,m}(p)\big) = \frac{m}{\sin\left(\frac{\pi}{m}\right)}\Big\{(a_1^2+a_2^2)\cos\frac{\pi}{m} + 2a_1a_2\Big\}. \tag{24}$$

Thus, we can apply Theorem 3.2 and (24) to the polygon $P_{n,\alpha}(k)$ to obtain the result.

Let us observe that if we apply the inequality (1) to the polygon $P_{n,\alpha}(k)$, then Theorem 1.2(a) follows immediately from Theorem 3.2(a). An alternate proof will be given in Section 5.

4. **Pseudo-Perimeters and Isoperimetric Inequalities.** Let $\triangle(n)$ denote the triangle in $\mathbf{R}^3$ with vertices $(1,0,0)$, $\left(0,\frac{1}{n-1},0\right)$ and $\left(0,0,\frac{2}{n}\right)$, $n \geq 3$. Hence if $(x,y,z) \in \triangle(n)$, we have $x \geq 0$, $y \geq 0$, $z \geq 0$ and

$$2x + 2(n-1)y + nz = 2. \tag{25}$$

For $(x,y,z) \in \triangle(n)$, we have introduced the concept of pseudo-perimeter $L_n[x,(n-1)y,nz/2]$ in [7] which is a positive function (homogeneous of degree 1)

$$L_n\left[x,(n-1)y,\frac{nz}{2}\right] : \tilde{\mathbf{R}}^n_+ \longrightarrow \mathbf{R} \qquad \text{defined by}$$

$$\begin{aligned} L_n^2\left[x,(n-1)y,\frac{nz}{2}\right](\boldsymbol{\Theta}) \\ = \left(\frac{n}{n-2}\right)^{n(y+z)} \left\{L_n(\boldsymbol{\Theta})\right\}^{2x+(n-2)y} \left\{\left[L_n(\boldsymbol{\Theta}) - 4\theta_1\right]\cdots\left[L_n(\boldsymbol{\Theta}) - 4\theta_n\right]\right\}^{y+z}. \end{aligned} \tag{26}$$

It follows from (26) that

$$L_n(\boldsymbol{\Theta}) = L_n[1,0,0](\boldsymbol{\Theta}). \tag{27}$$

For (x,y,z) and (x',y',z') in $\triangle(n)$, define

$$(x,y,z) \succ (x',y',z') \quad \text{if } x \geq x' \text{ and } z' \geq z. \tag{28}$$

We have the following fundamental inequality.

THEOREM 4.1. [7]. *Suppose* $(x,y,z) \succ (x',y',z')$ *for* (x,y,z) *and* (x',y',z') *in* $\triangle(n)$. *Then*

$$L_n\left[x,(n-1)y,nz/2\right](\boldsymbol{\Theta}) \geq L_n\left[x',(n-1)y',nz'/2\right](\boldsymbol{\Theta}) \tag{29}$$

for any $\boldsymbol{\Theta} \in \tilde{\mathbf{R}}^n_+$, *and if* $(x,y,z) \neq (x',y',z')$, *equality holds if and only if* $\theta_1 = \theta_2 = \cdots = \theta_n$.

DEFINITION 4.2. Let $(x,y,z) \in \triangle(n)$, and $k = p, h, e$. We define various pseudo-perimeters of P_n as follows.

$$L_k\left[x,(n-1)y,nz/2\right](P_n) = L_n\left[x,(n-1)y,nz/2\right]\left(s_k(\mathbf{a})\right). \tag{30}$$

$$\tilde{L}_k\left[x,(n-1)y,nz/2\right](P_n) = L_n\left[x,(n-1)y,nz/2\right]\left(s_k(\mathbf{b})\right). \tag{31}$$

$$L_k\left[x,(n-1)y,nz/2\right](P_n;c) = L_n\left[x,(n-1)y,nz/2\right]\left(c_k(\mathbf{a})+c_k(\mathbf{b})\right), \tag{32}$$

where $k = h, e$; $c_k(\mathbf{a})+c_k(\mathbf{b}) = \left(c_k(a_1)+c_k(b_1),\dots,c_k(a_n)+c_k(b_n)\right)$, and $c_k(\mathbf{a})+c_k(\mathbf{b}) \in \tilde{\mathbf{R}}^n_+$ if $(x,y,z) \neq (1,0,0)$.

$$L_k\left[x,(n-1)y,nz/2\right](P_n;u,v) = L_n\left[x,(n-1)y,nz/2\right]\left(\omega_k(u,v)\right), \tag{33}$$

where $\omega_k(u,v) = \left(u_k(a_1)v_k(b_1),\dots,u_k(a_n)v_k(b_n)\right)$, $u, v = \sin, \cos$, *etc*., and $\omega_k(u,v) \in \tilde{\mathbf{R}}^n_+$ if $(x,y,z) \neq (1,0,0)$.

These pseudo-perimeters are well-defined by Lemma 2.6 (with some restrictions in some cases). In terms of pseudo-perimeters we can restate Theorem 3.3 and Theorem 3.5 as follows.

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THEOREM 4.3.

$$L_k^2[0,1,0](P_3) = 4d_3A_k(P_3). \qquad (k=p,h,e) \tag{34}$$

THEOREM 4.4.

$$L_k^2[0,0,1](P_4) = 4d_4A_k(P_4). \qquad (k=p,h,e) \tag{35}$$

$$\tilde{L}_p^2[0,0,1](P_4) = 4\delta_4A_p(P_4). \tag{36}$$

As immediate consequences of Theorems 4.1, 4.3 and 4.4 we have the following general isoperimetric inequalities. These inequalities generalize Theorem 1.2(a) for $n=3$, and 4.

THEOREM 4.5. *For any* $(x,y,0)\in\triangle(3)$, $(x,y,0)\succ(0,1/2,0)$.

$$L_k^2[x,2y,0](P_3)\geq 4d_3A_k(P_3), \qquad (k=p,\ h,\ e).$$

If $(x,y,0)\neq(0,1/2,0)$, *equality holds if and only if* P_3 *is regular.*

THEOREM 4.6. *For any* $(x,y,z)\in\triangle(4)$,

(*a*) $$L_k^2[x,3y,2z](P_4)\geq 4d_4A_k(P_4) \qquad (k=p,h,e).$$

(*b*) $$\tilde{L}_p^2[x,3y,2z](P_4)\geq 4\delta_4A_p(P_4).$$

If $(x,y,z)\neq(0,0,1/2)$, *equality holds in (a) (resp. (b)) if and only if* P_4 *is regular.*

Theorem 1.2(a), Theorem 4.5 and Theorem 4.6 suggest the following conjecture.

CONJECTURE 4.7. Let $(x,y,z)\in\triangle(n)$, $(x,y,z)\neq(1,0,0)$, $n\geq 5$. Then

$$L_k^2\big[x,(n-1)y,nz/2\big](P_n)\geq 4d_nA_k(P_n), \qquad (k=p,h,e).$$

Equality holds if and only if P_n is regular.

Now we establish Conjecture 4.7 in some special cases.

THEOREM 4.8. *For any* $(x,y,z)\in\triangle(n)$, $n=2m\geq 6$, *and any* $P_n=P_{m,m}(k)$, $k=p$, *h*, *e*, *we have*

$$\begin{aligned} 1+\beta\left(1-\cos\frac{\pi}{m}\right) &\geq \frac{L_k^2[x,(n-1)y,nz/2](P_n)}{4d_nA_k(P_n)} \\ &\geq 1+\beta\left[\left(1-\cos\frac{\pi}{m}\right)-\frac{1+\cos\frac{\pi}{m}}{(m-1)^2}\right], \end{aligned} \tag{37}$$

where

$$\beta=\frac{\{s_k(a_1)-s_k(a_2)\}^2}{2\left\{[s_k^2(a_1)+s_k^2(a_2)]\cos\frac{\pi}{m}+2s_k(a_1)s_k(a_2)\right\}},$$

and

$$1-\cos\frac{\pi}{m}>\frac{1+\cos\frac{\pi}{m}}{(m-1)^2}. \tag{38}$$

Hence Conjecture 4.7 holds for $P_n=P_{m,m}(k)$.

PROOF. Since $m \geq 3$ and $\cos x = \sum_{i=0}^{\infty}(-1)^i \frac{x^{2i}}{(2i)!}$, a simple calculation yields

$$\frac{m(m-2)}{(m-1)^2} > \frac{2\cos\frac{\pi}{m}}{1+\cos\frac{\pi}{m}}. \tag{39}$$

This implies (38). Applying Theorem 3.6 we see that

$$\frac{L_k^2[1,0,0](P_n)}{4d_nA_k(P_n)} = 1 + \beta\left(1 - \cos\frac{\pi}{m}\right). \tag{40}$$

By definition

$$\begin{aligned} L_k^2[0,0,1](P_n) \\ = \frac{4m^2}{(m-1)^2}\Big\{m(m-2)\big[s_k^2(a_1)+s_k^2(a_2)\big] + \big[m^2+(m-2)^2\big]s_k(a_1)s_k(a_2)\Big\}. \end{aligned} \tag{41}$$

Again, by using Theorem 3.6, it follows from (41) that

$$\frac{L_k^2[0,0,1](P_n)}{4d_nA_k(P_n)} = 1 + \beta\left[\left(1-\cos\frac{\pi}{m}\right) - \frac{1+\cos\frac{\pi}{m}}{(m-1)^2}\right]. \tag{42}$$

Thus, we can use Theorem 4.1 to complete the proof of the theorem.

Conjecture 4.7 for $P_n = P_{m,m}(p)$ was also proved in [7].

REMARK. $L_k^2(P_n) - 4d_nA_k(P_n)$ is called the *isoperimetric deficit* of the polygon P_n. From (40) we have

$$L_k^2(P_n) - 4d_nA_k(P_n) = 4d_n\beta\left(1-\cos\frac{\pi}{m}\right)A_k(P_n). \tag{43}$$

5. **Analytic and Geometric Isoperimetric Inequalities.** In this section we shall study geometric isoperimetric inequalities via analytic isoperimetric inequalities. In particular, we prove both Theorem 1.2 and Theorem 1.3.

For a given constant $\sigma > 0$, define

$$H_n(\sigma) = \Big\{\boldsymbol{\Theta} = (\theta_1,\ldots\theta_n) \in \mathbf{R}^n : \sum_{i=1}^n \theta_i = n\sigma, \theta_i > 0\Big\},$$

and if $0 < \sigma < \frac{\pi}{2}$, set

$$d_n(\sigma) = n\tan\sigma, \qquad \delta_n(\sigma) = n\cot\sigma,$$

and

$$H_n(\sigma,\pi/2) = \Big\{\boldsymbol{\Theta} \in H_n(\sigma) : 0 < \theta_i < \frac{\pi}{2},\ 1 \leq i \leq n\Big\}.$$

Now for $\Theta = (\theta_1,\ldots,\theta_n) \in H_n(\sigma,\pi/2)$, we may assume (by rearranging subscripts if necessary) that $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$. Let p_i, $1 \leq i \leq n$, be constants with $p_1 \geq p_2 \geq \cdots \geq p_n > 0$, and set $\eta = \sum_{i=1}^n p_i$ and $\mu = \sum_{i=1}^n p_i\theta_i/\eta$.

THEOREM 5.1. *Let* $\Theta \in H_n(\sigma, \pi/2)$. *Then*

$$(a) \qquad \Big(\sum_{i=1}^{n} p_i \sin\theta_i\Big)^2 \geq \eta \tan\sigma \sum_{i=1}^{n} p_i \sin\theta_i \cos\theta_i + \Big\{\eta \sin\mu - \sum_{i=1}^{n} p_i \sin\theta_i\Big\}^2.$$

$$(b) \qquad \Big(\sum_{i=1}^{n} p_i \cos\theta_i\Big)^2 \geq \eta \cot\sigma \sum_{i=1}^{n} p_i \sin\theta_i \cos\theta_i + \Big\{\eta \cos\mu - \sum_{i=1}^{n} p_i \cos\theta_i\Big\}^2.$$

Equality holds in (a) (resp. (b)) if and only if $\theta_1 = \theta_2 = \cdots = \theta_n = \sigma = \mu$.

PROOF. Set $\theta_i(t) = t\mu + (1-t)\theta_i$, $1 \leq i \leq n$. Then $0 < \theta_i(t) < \pi/2$ for $0 \leq t \leq 1$. Notice that if $\mu > \theta_i$ (resp. $\mu < \theta_i$), then $\mu > \theta_i(t)$ (resp. $\mu < \theta_i(t)$) for $0 \leq t < 1$, and so

$$(\mu - \theta_i)\big(\cos\mu - \cos\theta_i(t)\big) < 0 \quad \text{if } \mu \neq \theta_i.$$

Observe that $\mu \geq \sigma$ because

$$\mu - \sigma = \frac{1}{n\eta} \sum_{i<j} (p_i - p_j)(\theta_i - \theta_j) \geq 0.$$

Hence to prove (a) it suffices to prove the following inequality

(c)

$$\Big(\sum_{i=1}^{n} p_i \sin\theta_i\Big)^2 \geq \eta \tan\mu \sum_{i=1}^{n} p_i \sin\theta_i \cos\theta_i + \Big\{\eta \sin\mu - \sum_{i=1}^{n} p_i \sin\theta_i\Big\}^2,$$

with equality if and only if $\theta_1 = \theta_2 = \cdots = \theta_n = \sigma = \mu$.

To prove this inequality, let us consider the function

$$F(t) = \Big\{\sum_{i=1}^{n} p_i \sin\theta_i(t)\Big\}^2 - \eta \tan\mu \sum_{i=1}^{n} p_i \sin\theta_i(t) \cos\theta_i(t) - \Big\{\eta \sin\mu - \sum_{i=1}^{n} p_i \sin\theta_i(t)\Big\}^2, \quad 0 \leq t \leq 1.$$

We need to verify that $F(0) \geq 0$, with equality if and only if $\theta_1 = \theta_2 = \cdots = \theta_n = \sigma = \mu$. Since $F(1) = 0$, it suffices to verify that $F'(t) < 0$, $0 \leq t < 1$ if $\Theta \neq (\sigma, \ldots, \sigma)$. Clearly

$$\sum_{i=1}^{n} p_i(\mu - \theta_i) \cos 2\theta_i(t) = 2 \sum_{i=1}^{n} p_i(\mu - \theta_i) \cos^2\theta_i(t)$$

because $\sum_{i=1}^{n} p_i(\mu - \theta_i) = 0$. Hence

$$\begin{aligned} F'(t) &= 2\eta \sin\mu \sum_{i=1}^{n} p_i(\mu - \theta_i) \cos\theta_i(t) - \eta \tan\mu \sum_{i=1}^{n} p_i(\mu - \theta_i) \cos 2\theta_i(t) \\ &= 2\eta \tan\mu \sum_{i=1}^{n} p_i(\mu - \theta_i)\big(\cos\mu - \cos\theta_i(t)\big) \cos\theta_i(t) < 0 \end{aligned}$$

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for $0 \le t < 1$ as desired.

Now set $\tilde{\sigma} = \frac{\pi}{2} - \sigma$. Then (b) follows immediately by applying (a) to $\left(\frac{\pi}{2} - \theta_1, \ldots, \frac{\pi}{2} - \theta_n\right) \in H_n\left(\tilde{\sigma}, \frac{\pi}{2}\right)$.

If $p_1 = p_2 = \cdots = p_n = 1$, Theorem 5.1 was proved by Zhang in [15]. The proofs in [14] and [7] of the following corollary contain gaps. Thus, Theorem 5.1 gives a new proof of these two inequalities.

COROLLARY 5.2 (TANG [14]). *Let $\Theta \in H_n\left(\sigma, \frac{\pi}{2}\right)$. Then*

(*a*) $$\Big(\sum_{i=1}^{n} \sin\theta_i\Big)^2 \ge d_n(\sigma) \sum_{i=1}^{n} \sin\theta_i \cos\theta_i.$$

(*b*) $$\Big(\sum_{i=1}^{n} \cos\theta_i\Big)^2 \ge \delta_n(\sigma) \sum_{i=1}^{n} \sin\theta_i \cos\theta_i.$$

Equality holds in (a) (resp. (b)) if and only if $\theta_1 = \theta_2 = \cdots = \theta_n$.

We are ready to prove Theorem 1.2 and Theorem 1.3.

PROOF OF THEOREM 1.2. Let $k = h, e$.

(a) $$\begin{aligned} L_k^2(P_n) &= 4\Big\{\sum_{i=1}^{n} \sin_k(a_i)\Big\}^2 \\ &= 4s_k^2(r)\Big\{\sum_{i=1}^{n} \sin\alpha_i\Big\}^2 \qquad \text{(by (14))} \\ &\ge 4d_n s_k^2(r) \sum_{i=1}^{n} \sin\alpha_i \cos\alpha_i \qquad \text{(by Corollary 5.2)} \\ &= 4d_n s_k^2(r) \sum_{i=1}^{n} \frac{s_k(a_i)t_k(b_i)}{s_k(r)t_k(r)} \\ &= 4d_n \sum_{i=1}^{n} s_k(a_i)c_k(a_i)s_k(b_i) \\ &= 4d_n A_k(P_n), \qquad \text{(by Prop. 3.1)} \end{aligned}$$

where $d_n = d_n(\pi/n)$. If $k = p$, the proof is similar.

(b) By (9), $0 < Q_n^k < \pi/2$ and $\sum_{i=1}^{n} \beta_i = nQ_n^k$. Hence by Corollary 5.2(a)

$$\begin{aligned} \tilde{L}_k^2(P_n) &= 4\Big\{\sum_{i=1}^{n} \sin_k(b_i)\Big\}^2 \\ &= 4s_k^2(r)\Big\{\sum_{i=1}^{n} \sin\beta_i\Big\}^2 \\ &\ge 4s_k^2(r) d_n^k(P_n) \sum_{i=1}^{n} \sin\beta_i \cos\beta_i \\ &= 4d_n^k(P_n) \sum_{i=1}^{n} s_k(a_i)s_k(b_i)c_k(b_i) \\ &= 4d_n^k(P_n)\tilde{A}_k(P_n). \end{aligned}$$

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(c) $\sum_{i=1}^n F_i = n\sigma$, $0 < \sigma = A(P_n)/2n < \pi/n$ by (9). Thus,

$$\begin{aligned}
L_k^2(P_n;\sin,\sin) &= 4\Big\{\sum_{i=1}^n \sin_k(a_i)\sin_k(b_i)\Big\}\\
&= \big\{1+c_k(r)\big\}^2\Big\{\sum_{i=1}^n \sin F_i\Big\}^2 \qquad \text{(by Lemma 2.1)}\\
&\geq 4\big\{1+c_k(r)\big\}^2 d_n(P_n)\sum_{i=1}^n \sin F_i\ \cos F_i\\
&= 4d_n(P_n)\sum_{i=1}^n s_k(a_i)s_k(b_i)\big\{c_k(a_i)+c_k(b_i)\big\} \qquad \text{(by Lemma 2.1)}\\
&= 4d_n(P_n)A_k(P_n). \qquad \text{(by Prop. 3.1)}
\end{aligned}$$

PROOF OF THEOREM 1.3. The proof is almost identical with the proof of Theorem 1.2, hence we give only the proof of (c). Again, using Lemma 2.1,

$$\begin{aligned}
L_k^2(P_n;c) &= \big\{1+\cos(r)\big\}^2\Big\{\sum_{i=1}^n \cos F_i\Big\}^2\\
&\geq 4\big\{1+\cos(r)\big\}^2\delta_n(P_n)\sum_{i=1}^n \sin F_i\ \cos F_i\\
&= 4\delta_n(P_n)A_k(P_n).
\end{aligned}$$

Now we shall generalize both Theorems 1.2 and 1.3. For $\zeta = (x,y,z) \in \triangle(n)$, if $\zeta \neq (1,0,0)$ we shall always assume that $S(\mathbf{\Theta}) \in \tilde{\mathbf{R}}_+^n$ where $S = \sin$ or $\cos$, and $\mathbf{\Theta} \in \tilde{H}_n(\sigma,\pi/2)$, where

$$\tilde{H}_n(\sigma,\pi/2) = H_n(\sigma,\pi/2)\cap\tilde{\mathbf{R}}_+^n.$$

Set

$$\mu_n(\sigma) = \begin{cases} d_n(\sigma), & \text{if } S=\sin\\ \delta_n(\sigma), & \text{if } S=\cos.\end{cases}$$

Let us consider the following analytic isoperimetric inequality:

$$(*)\qquad \begin{cases} L_n^2\big[x,(n-1)y,nz/2\big]\big(S(\mathbf{\Theta})\big) \geq 4\mu_n(\sigma)\sum_{i=1}^n \sin\theta_i\ \cos\theta_i,\\ \text{with equality if only if } \theta_1=\theta_2=\cdots=\theta_n,\end{cases}$$

where $\mathbf{\Theta} \in H_n(\sigma,\pi/2)$. The inequality $(*)$ depends on n, S, ζ, σ and $\mathbf{\Theta}$, hence it will be simply denoted by $\mathrm{AInq}_n(S;\zeta,\sigma,\mathbf{\Theta})$. From now on, we shall assume that $k = p,h,e$ if $\sigma = \frac{\pi}{n}$ and $A(P_n)/2n$, and $k = h,e$ if $\sigma = Q_n^k$.

THEOREM 5.3. *Suppose that* $\mathrm{AInq}_n(\sin;\zeta,\sigma,\mathbf{\Theta})$ *holds for given* σ *(*$\sigma = \pi/2n$, Q_n^k *in (44) and (45) respectively) and* $\mathbf{\Theta} \in \tilde{H}_n(\sigma,\pi/2)$. *Then the following geometric isoperimetric inequalities hold:*

$$L_k^2\big[x,(n-1)y,nz/2\big](P_n) \geq 4d_nA_k(P_n) \qquad \big(S(\mathbf{\Theta})=\sin(\alpha)\big); \tag{44}$$

$$\tilde{L}_k^2\big[x,(n-1)y,nz/2\big](P_n) \geq 4d_n^k(P_n)\tilde{A}_k(P_n) \qquad \big(S(\mathbf{\Theta})=\sin(\beta)\big). \tag{45}$$

Equality holds in either of (44) and (45) if and only if P_n *is regular.*

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PROOF. The proof is identical with the proof of Theorem 1.2. We use $\mathrm{AInq}_n(\sin;\zeta,\sigma,\mathbf{\Theta})$ instead of Corollary 5.2(a). Let us verify (44): since $\sigma=\frac{\pi}{n}$, $d_n(\sigma)=d_n$, and so,

$$\begin{aligned} L_k^2\big[x,(n-1)y,nz/2\big](P_n) &= s_k^2(r)L_n^2\big[x,(n-1)y,nz/2\big]\big(\sin(\alpha)\big)\\ &\geq 4d_n s_k^2(r)\sum_{i=1}^n \sin\alpha_i\cdot\cos\alpha_i\\ &= 4d_nA_k(P_n).\end{aligned}$$

If $\zeta=(1,0,0)$, $\mathrm{AInq}_n(\sin;\zeta,\pi/n,\mathbf{\Theta})$ is exactly Corollary 5.2(a). Hence this theorem generalizes Theorem 1.2. We also have the following generalization of Theorem 1.3.

THEOREM 5.4. *Suppose that* $\mathrm{AInq}_n(\cos;\zeta,\sigma,\mathbf{\Theta})$ *holds for given* σ *(*$\sigma=\pi/n$, Q_n^k *and* $A(P_n)/2n$ *in (46), (47) and (48) respectively) and* $\mathbf{\Theta}\in\tilde{H}_n(\sigma,\pi/2)$. *Then*

$$L_k^2\big[x,(n-1)y,nz/2\big](P_n;\cos,\sin)\geq 4\delta_nA_k(P_n),\qquad \big(S(\mathbf{\Theta})=\cos(\alpha)\big); \tag{46}$$

$$L_k^2\big[x,(n-1)y,nz/2\big](P_n;\sin,\cos)\geq 4\delta_n^k(P_n)\tilde{A}_k(P_n),\qquad \big(S(\mathbf{\Theta})=\cos(\beta)\big); \tag{47}$$

$$L_k^2\big[x,(n-1)y,nz/2\big](P_n;c)\geq 4\delta_n(P_n)\mathcal{A}_n(P_n),\qquad \big(S(\mathbf{\Theta})=\cos(\mathbf{F})\big). \tag{48}$$

Equality holds in any of (46), (47) and (48) if and only if P_n *is regular.*

Next we shall establish $\mathrm{AInq}_n(S;\zeta,\sigma,\mathbf{\Theta})$ for some special cases. These new analytic isoperimetric inequalities will give new geometric isoperimetric inequalities of types (44)–(48) by Theorems 5.3 and 5.4.

Let $n=2m$ and

$$K_n(\sigma,\pi/2)=\left\{\mathbf{\Theta}\in\tilde{H}_n(\sigma,\pi/2)\,\middle|\,\begin{array}{l}\theta_1+\theta_2=2\sigma,\quad \theta_1=\theta_3=\cdots=\theta_{2m-1}\\ \text{and } \theta_2=\theta_4=\cdots=\theta_{2m}\end{array}\right\}$$

Observe that $K_n(\sigma,\pi/2)$ is an open subset of the 1-dimensional hypersurface

$$\{\Theta\in\mathbf{R}^{2m}:\theta_1+\theta_2=2\sigma,\quad \theta_1=\theta_3=\cdots=\theta_{2m-1},\quad \text{and}\quad \theta_2=\theta_4=\cdots=\theta_{2m}\}.$$

THEOREM 5.5. *For any* $\zeta=(x,y,z)\in\triangle(n)$, $n=2m\geq 6$, *and constant* σ.

(a) If $0<\sigma\leq\frac{\pi}{2}-\frac{\pi}{n}$, $\mathrm{AInq}_n(\cos;\zeta,\sigma,\mathbf{\Theta})$ *holds for* $\mathbf{\Theta}\in K_n\left(\sigma,\frac{\pi}{2}\right)$.

(b) If $\pi/n\leq\sigma<\pi/2$, $\mathrm{AInq}_n(\sin;\zeta,\sigma,\mathbf{\Theta})$ *holds for* $\mathbf{\Theta}\in K_n\left(\sigma,\frac{\pi}{2}\right)$.

PROOF. (a) Let $S=\cos$. It suffices to prove the theorem for $\zeta=\left(0,0,\frac{2}{n}\right)$ by Theorem 4.1. For $\mathbf{\Theta}\in K_n\left(\sigma,\frac{\pi}{2}\right)$, set

$$A(\mathbf{\Theta})=A(\theta_1,\theta_2)=\sum_{i=1}^n\sin\theta_i\cos\theta_i,\quad\text{and}\quad B(\mathbf{\Theta})=B(\theta_1,\theta_2)=L_n^2[0,0,1]\big(S(\mathbf{\Theta})\big).$$

Then

$$\begin{aligned} A(\mathbf{\Theta}) &= \frac{m}{2}(\sin 2\theta_1+\sin 2\theta_2)\\ \text{and}\quad B(\mathbf{\Theta}) &= \alpha\big[m(m-2)(\cos^2\theta_1+\cos^2\theta_2)+2(m^2-2m+2)\cos\theta_1\,\cos\theta_2\big],\end{aligned}$$

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where $\alpha = \frac{4m^2}{(m-1)^2}$. Let

$$F(\theta_1, \theta_2) = B(\theta_1, \theta_2)/A(\theta_1, \theta_2),$$

and $G(\theta_1, \theta_2) = \theta_1 + \theta_2 - 2\sigma$. We shall verify that the function $F(\theta_1, \theta_2)$ under the constraint $G(\theta_1, \theta_2) = 0$ has a unique critical point at $\theta_1 = \theta_2 = \sigma$ by the method of Lagrange multipliers. If $\eta = (\eta_1, \eta_2)$ is a critical point, there is a real number λ such that

$$\nabla F(\eta_1, \eta_2) = \lambda \nabla G(\eta_1, \eta_2)$$

where ∇F denotes the gradient of the function F. Hence

$$\frac{\alpha A(\eta)\{m(m-2)\sin 2\eta_i + 2(m^2-2m+2)\sin\eta_i \ \cos\eta_j\} + mB(\eta)\cos 2\eta_i}{A^2(\eta)} = -\lambda,$$

where $j = 1$, or 2, $j \neq i$. If $\eta_1 \neq \eta_2$,

$$\alpha A(\eta)\Big\{m(m-2)[\sin 2\eta_1 - \sin 2\eta_2] + 2(m^2-2m+2)\sin(\eta_1-\eta_2)\Big\}$$
$$= -mB(\eta)(\cos 2\eta_1 - \cos 2\eta_2).$$

Since $\eta_1 + \eta_2 = 2\sigma$, $A(\eta) = m\sin 2\sigma\cos(\eta_1-\eta_2)$ and $\sin(\eta_1-\eta_2) \neq 0$. Hence

(49) $$\alpha\Big\{(m^2-2m+2) + m(m-2)\cos 2\sigma\Big\}\cos(\eta_1-\eta_2) = B(\eta).$$

As $\cos^2\eta_1 + \cos^2\eta_2 = 1 + \cos 2\sigma\cos(\eta_1-\eta_2)$, it follows from (49) that

$$\cos 2\sigma = \cos(\eta_1+\eta_2) = -\frac{m(m-2)}{m^2-2m+2}.$$

But by (39) we have

$$\cos\frac{\pi}{m} < \frac{m(m-2)}{m^2-2m+2},$$

and so

$$\cos 2\sigma < -\cos\frac{\pi}{m} = \cos\left(\pi - \frac{\pi}{m}\right).$$

This contradicts the hypothesis that $0 < \sigma \leq \frac{\pi}{2} - \frac{\pi}{n}$. This proves that $\eta = (\sigma, \sigma)$ is the only critical point. Set

$$\beta = \frac{\partial^2 F}{\partial\theta_i^2}\Big|_{(\sigma,\sigma)} \quad \text{and} \quad \gamma = \frac{\partial^2 F}{\partial\theta_1\partial\theta_2}\Big|_{(\sigma,\sigma)}, \quad i = 1, 2.$$

As $\frac{\partial G}{\partial\theta_i} = 1$ for $i = 1, 2$, hence at $(\theta_1, \theta_2) = (\sigma, \sigma)$ we have

$$\begin{vmatrix} 0 & \partial G/\partial\theta_1 & \partial G/\partial\theta_2 \\ \partial G/\partial\theta_1 & \partial^2 F/\partial\theta_1^2 & \partial^2 F/\partial\theta_1\partial\theta_2 \\ \partial G/\partial\theta_2 & \partial^2 F/\partial\theta_2\partial\theta_1 & \partial^2 F/\partial\theta_2^2 \end{vmatrix} = 2(\gamma - \beta).$$

A simple calculation gives

$$\gamma - \beta = -\frac{2\alpha m\sin 2\sigma}{A^2(\sigma,\sigma)}\Big[(m^2-2m+2)\cos 2\sigma + m^2 - 2m\Big] < 0.$$

Hence $F(\theta_1, \theta_2)$ has its minimum at (σ, σ) (*cf.* [9]). But $F(\sigma, \sigma) = 4\delta_n(\sigma)$. Therefore, we see that $\mathrm{AInq}_n(\cos; \zeta, \sigma, \boldsymbol{\Theta})$ holds for $0 < \sigma \le \frac{\pi}{2} - \frac{\pi}{n}$.

(b) Set $\tilde{\boldsymbol{\Theta}} = (\frac{\pi}{2} - \theta_1, \ldots, \frac{\pi}{2} - \theta_n)$ and $\tilde{\sigma} = \frac{\pi}{2} - \sigma$. Since $\frac{\pi}{n} \le \sigma < \frac{\pi}{2}$, we have $0 < \tilde{\sigma} \le \frac{\pi}{2} - \frac{\pi}{n}$. Hence we have the inequality $\mathrm{AInq}_n(\cos; \zeta, \tilde{\sigma}, \tilde{\boldsymbol{\Theta}})$ by (a).

Observe that $\sin(\boldsymbol{\Theta}) = \cos(\tilde{\boldsymbol{\Theta}}) \in \tilde{\mathbf{R}}^n_+$ and $d_n(\sigma) = \delta_n(\tilde{\sigma})$. Thus

$$\mathrm{AInq}_n(\sin; \zeta, \sigma, \boldsymbol{\Theta}) = \mathrm{AInq}_n(\cos; \zeta, \tilde{\sigma}, \tilde{\boldsymbol{\Theta}}).$$

This completes the proof of (b), and hence the theorem.

REMARK. Theorem 5.5(a) for $\sigma = \frac{\pi}{n}$ was also proved in [7].

THEOREM 5.6. *Let $P_n = P_{m,m}(k)$ be a cyclic n-gon with m sides of length $2a_1$ and the remaining m sides of length $2a_2$, $n = 2m \ge 6$. Let $\zeta = (x, y, z) \in \triangle(n)$. Then*

(a) (44) and (46) hold for $k = p, h, e$.

(b) (45) holds if $k = h$ and $(n-4)\pi \ge A(P_n)$; or $k = e$.

(c) (47) holds if $k = h$;

(d) (48) holds if $k = h$.

PROOF. By Lemma 2.6, $S(\boldsymbol{\Theta}) \in \tilde{\mathbf{R}}^n_+$ for $S(\boldsymbol{\Theta}) = \sin(\alpha)$, $\sin(\beta)$, $\cos(\alpha)$, $\cos(\beta)$ and $\cos(\mathbf{F})$. Hence the various pseudo-perimeters in (44), (45), (46), (47) and (48) are well-defined. From the definitions we obtain,

(i) $\pi/n \le \sigma$ for $S = \sin$ under the hypotheses of (44) (resp. (45)).

(ii) $\sigma < \pi/2 - \pi/n$ for $S = \cos$ under the hypotheses of (46) ((47) and (48) respectively).

That is, the hypotheses of Theorem 5.5 are satisfied for these cases. Hence we can apply Theorems 5.3 and 5.4 to complete the proof.

REMARKS 5.7. (a) Theorem 5.6 (44) gives a different proof of the Conjecture 4.7 for polygons $P_n = P_{m,m}(k)$.

(b) We proved in [7] that for any σ, $0 < \sigma < \pi/2$, there exists a neighborhood $N(\sigma)$ of $(\sigma, \ldots, \sigma)$ in $\tilde{H}_n(\sigma, \pi/2)$ such that $\mathrm{AInq}_n(\sin; \zeta, \sigma, \boldsymbol{\Theta})$ holds for any $\zeta \in \triangle(n)$ and $\boldsymbol{\Theta} \in N(\sigma)$. By using the arguments of the proof of Theorem 5.5(b), $\mathrm{AInq}_n(\cos; \zeta, \sigma, \boldsymbol{\Theta})$ also holds for any $\zeta \in \triangle(n)$ and $\boldsymbol{\Theta} \in N(\sigma)$. Thus, we can apply both Theorems 5.3 and 5.4 to polygons P_n with $s_k(\mathbf{a})$, $s_k(\mathbf{b})$, *etc.*, in $N(\sigma)$, $\sigma = \pi/2$, $A(P_n)/2n$ and Q^k_n.

Suppose now that $M = \mathbf{H}^2(-1)$ and define

$$\hat{L}_h(P_n) = \sum_{i=1}^{n} \sinh 2a_i. \tag{50}$$

THEOREM 5.8.

$$\hat{L}_h(P_n) \ge \frac{2d_n}{n} \coth \frac{L(P_n)}{2n} A_h(P_n),$$

with equality if and only if P_n is regular.

PROOF. We proved the following inequality in [7]: Let $\sum_{i=1}^{n}\theta_i = n\sigma$, $\sigma > 0$, constant, $\theta_i > 0$, $1 \le i \le n$. Then

$$\Big(\sum_{i=1}^{n}\sinh\theta_i\Big)^2 \le n\tanh\sigma\sum_{i=1}^{n}\sinh\theta_i\cosh\theta_i, \tag{51}$$

with equality if and only if $\theta_1 = \cdots = \theta_n$. Again, the proof of (51) in [7] contains gaps. But this can be proved by the method used in the proof of Theorem 5.1 above.

Applying (51) to $\theta_i = a_i$, $1 \le i \le n$, we have

$$L_h^2(P_n) \le 2n\tanh\frac{L(P_n)}{2n}\hat{L}_h(P_n). \tag{52}$$

It follows from (52) and Theorem 1.2 that

$$\hat{L}_h(P_n) \ge \frac{1}{2n}\coth\frac{L(P_n)}{2n}4d_nA_h(P_n) = \frac{2d_n}{n}\coth\frac{L(P_n)}{2n}A_h(P_n),$$

with equality if and only if P_n is regular.

THEOREM 5.9.

$$\hat{L}_h^2(P_n) > 4d_nA_h(P_n).$$

PROOF. Let $f(x) = x\coth x$, $x > 0$. Then $f(x)$ is increasing. Notice that $\lim_{x\to 0} f(x) = 1$, and so

$$\coth\frac{L(P_n)}{2n} > \frac{2n}{L(P_n)}, \tag{53}$$

moreover, $\hat{L}_h(P_n) > L(P_n)$ because $\sinh x > x$ for $x > 0$. Now, we apply Theorem 5.8 to conclude the proof.

To conclude this section, many results in this paper can be stated more generally by using Theorem 5.1 instead of Corollary 5.2. As an example, we shall give another generalization of Theorem 1.2(a).

Let P_n be a cyclic polygon in M as above with $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_n$, and $p_1 \ge p_2 \ge \cdots \ge p_n > 0$, p_i's are constants. Define the *weighted length* and *area* of P_n by

$$L_k(P_n;p_1,\ldots,p_n) = 2\sum_{i=1}^{n}p_is_k(a_i) \quad \text{and}$$
$$A_k(P_n;p_1,\ldots,p_n) = \frac{1}{2}\sum_{i=1}^{n}p_i\ \sin_k(2a_i)\ \sin_k(b_i),$$

where $k = p, h, e$. Then we have

THEOREM 5.10.

$$L_k^2(P_n;p_1,\ldots,p_n) \ge 4d_nA_k(P_n;p_1,\ldots,p_n),$$

with equality if and only if P_n is regular.

More generally we have the following isoperimetric inequality:

$$L_k^2(P_n;p_1,\ldots,p_n) \ge 4d_nA_k(P_n;p_1,\ldots,p_n) + \big\{2\eta s_k(r)\sin\mu - L_k(P_n;p_1,\ldots,p_n)\big\}^2,$$

with equality if and only if P_n is regular, where $\mu = \sum_{i=1}^{n}p_i\alpha_i/\eta$. This inequality with $M = \mathbf{R}^2$ and $p_1 = \cdots = p_n = 1$ was also proved in [15].

6**. Geometric Inequalities for Triangles.** Let P_3 be a geodesic triangle (not necessarily cyclic) in M, $M = \mathbf{R}^2, \mathbf{H}^2(-1)$, or $\mathbf{S}^2(1)$ (*i.e.*, $k = p, h, e$) with vertices A, B and C. In the rest of this section, we shall denote the angles of P_3 at vertices A, B, and C by α, β, and γ respectively, and assume that,

$$0 < \alpha, \beta, \gamma < \pi/2.$$

We shall denote the lengths of the three sides of P_3, BC, CA and AB by a, b and c respectively. By Gauss-Bonnet theorem $\alpha + \beta + \gamma = 3\sigma_k$, where

$$\sigma_k = \begin{cases} \pi/3, & \text{if } k = p, \\ \left\{\pi - A(P_3)\right\}/3, & \text{if } k = h, \\ \left\{\pi + A(P_3)\right\}/3, & \text{if } k = e. \end{cases}$$

Notice that $\sigma_k = 2Q_3^k$ if $k = h, e$. Set

(54) $$\Phi_k(P_3) = s_k(a)\cos\alpha + s_k(b)\cos\beta + s_k(c)\cos\gamma. \quad (k = p, h, e)$$

THEOREM 6.1. *For $k = p, h, e$, we have the following geometric inequalities:*
(a)

$$\sin\alpha + \sin\beta + \sin\gamma \geq \frac{d_3(\sigma_k)\Phi_k(P_3)}{s_k(a) + s_k(b) + s_k(c)}.$$

(b)

$$\cos\alpha + \cos\beta + \cos\gamma \geq \frac{3\Phi_k(P_3)}{s_k(a) + s_k(b) + s_k(c)}.$$

Equality holds in (a) (resp. (b)) if and only if P_3 is regular.

PROOF. According to the law of sines (5), if we let $\eta_k = \sin\alpha / s_k(a)$, then

(55) $$\sin\alpha = \eta_k s_k(a), \qquad \sin\beta = \eta_k s_k(b), \qquad \sin\gamma = \eta_k s_k(c)$$
$$\text{and} \quad \eta_k = (\sin\alpha + \sin\beta + \sin\gamma)/\left\{s_k(a) + s_k(b) + s_k(c)\right\}.$$

Hence it follows from Corollary 5.2(a) that

$$\begin{aligned}(\sin\alpha + \sin\beta + \sin\gamma)^2 &\geq d_3(\sigma_k)(\sin\alpha\cos\alpha + \sin\beta\cos\beta + \sin\gamma\cos\gamma) \\ &= \eta_k d_3(\sigma_k)\Phi_k(P_3) \\ &= \frac{d_3(\sigma_k)\Phi_k(P_3)(\sin\alpha + \sin\beta + \sin\gamma)}{s_k(a) + s_k(b) + s_k(c)}.\end{aligned}$$

This proves (a). Similarly, from Corollary 5.2(b) and(a) we have

$$\begin{aligned}(\cos\alpha + \cos\beta + \cos\gamma)^2 &\geq \delta_3(\sigma_k)(\sin\alpha\cos\alpha + \sin\beta\cos\beta + \sin\gamma\cos\gamma) \\ &= \eta_k\delta_3(\sigma_k)\Phi_k(P_3) \\ &= \frac{\delta_3(\sigma_k)\Phi_k(P_3)(\sin\alpha + \sin\beta + \sin\gamma)}{s_k(a) + s_k(b) + s_k(c)} \\ &\geq \frac{9\Phi_k^2(P_3)}{\{s_k(a) + s_k(b) + s_k(c)\}^2}.\end{aligned}$$

The proof is complete.

If $M = \mathbf{R}^2$, we can generalize Theorem 6.1. Here $d_3(\sigma_p) = d_3$.

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THEOREM 6.2. *For any* $(x,y,0) \in \triangle(3)$, $(x,y,0) \succ (0,1/2,0)$, *we have*

(*a*) $$\sin\alpha + \sin\beta + \sin\gamma \geq \frac{2d_3\Phi_k(P_3)}{L_3[x,2y,0](a,b,c)}.$$

(*b*) $$\cos\alpha + \cos\beta + \cos\gamma \geq \frac{6\Phi_k(P_3)}{L_3[x,2y,0](a,b,c)}.$$

If $(x,y,0) \neq (0,1/2,0)$, *equality holds in (a) (resp. (b)) if and only if* P_3 *is regular.*

PROOF. Because of the relations (55), there is a pseudo-triangle $\hat{P}_3$ in the plane (which is a triangle) given by

$$\hat{P}_3 = \left\{\frac{1}{\eta_p}, (\alpha, a), (\beta, b), (\gamma, c)\right\}.$$

It is not difficult to see that

$$\begin{aligned} A(\hat{P}_3) &= a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma \\ &= \frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{\eta_p} \\ &= \Phi_p(P_3)/\eta_p. \end{aligned}$$

Applying the Heron's formula Theorem 4.3 and Theorem 4.1 to the triangle $\hat{P}_3$,

$$\begin{aligned} L_3^2[x,2y,0](a,b,c) &\geq L_3^2[0,1,0](a,b,c) \\ &= 4d_3A(\hat{P}_3) \\ &= 4d_3\Phi_p(P_3)/\eta_p. \end{aligned}$$

Hence
$$\frac{\sin\alpha + \sin\beta + \sin\gamma}{a+b+c} = \eta_p \geq \frac{4d_3\Phi_p(P_3)}{L_3^2[x,2y,0](a,b,c)}.$$

Again, by Theorem 4.1,

$$\sin\alpha + \sin\beta + \sin\gamma \geq \frac{2d_3\Phi_p(P_3)L_3[1,0,0](a,b,c)}{L_3^2[x,2y,0](a,b,c)} \geq \frac{2d_3\Phi_p(P_3)}{L_3[x,2y,0](a,b,c)}$$

because $L_3[1,0,0](a,b,c) = 2(a+b+c)$.

The proof of (b) is similar to the proof of Theorem 6.1(b) by using Theorem 6.2(a).

COROLLARY 6.3. *Let* $M = \mathbf{R}^2$. *Then*
(a) $A(P_3) \geq \frac{\sqrt{3}}{4}\Psi^2(P_3)$.
(b) $L(P_3) \geq 3\Psi(P_3)$,
where
$$\Psi(P_3) = \frac{a^2(b^2+c^2-a^2) + b^2(c^2+a^2-b^2) + c^2(a^2+b^2-c^2)}{a(b^2+c^2-a^2) + b(c^2+a^2-b^2) + c(a^2+b^2-c^2)}.$$
Equality holds in (a) (resp. (b)) if and only if P_3 *is regular.*

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PROOF. (a) Since $A(\hat{P}_3) = 4A(P_3)$, $L_3^2[0,1,0](a,b,c) = 16d_3A(P_3)$, hence by Theorem 6.2(b),

$$(\cos\alpha + \cos\beta + \cos\gamma)^2 \geq \frac{9\Phi_p^2(P_3)}{4d_3A(P_3)}.$$

This implies inequality (a) by the law of cosines.

(b) By (3) and (a).

If $M = \mathbf{H}^2(-1)$, we also have

THEOREM 6.4. *If $M = \mathbf{H}^2(-1)$, set*

$$\tilde{\Phi}_h(P_3) = \sin\alpha\cosh a + \sin\beta\cosh b + \sin\gamma\cosh c;$$

then

$$\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3\tilde{\Phi}_h(P_3)}{\cosh a + \cosh b + \cosh c}, \tag{56}$$

with equality if and only if P_3 is regular.

PROOF. From [7, Theorem 3.3] we obtain

$$(\cosh a + \cosh b + \cosh c)^2 \leq 3\coth\frac{L(P_3)}{3}(\sinh a\cosh a + \sinh b\cosh b + \sinh c\cosh c), \tag{57}$$

with equality if and only if P_3 is regular.

Hence by (55) we get

$$(\cosh a + \cosh b + \cosh c)^2 \leq \left\{3\tilde{\Phi}_h(P_3)\coth L(P_3)/3\right\}/\eta_h. \tag{58}$$

Similarly, from (51) and (55) we have

$$(\sinh a + \sinh b + \sinh c)^2 \leq \left\{3\tilde{\Phi}_h(P_3)\tanh L(P_3)/3\right\}/\eta_h. \tag{59}$$

Thus, by (55), (58) and (59),

$$(\sinh a+\sinh b+\sinh c)^2\cdot(\cosh a+\cosh b+\cosh c)^2 \leq \frac{9\tilde{\Phi}_h^2(P_3)(\sinh a + \sinh b + \sinh c)^2}{(\sin\alpha + \sin\beta + \sin\gamma)^2}$$

which implies (56).

Combining Theorem 6.1(a) and Theorem 6.4 we get

COROLLARY 6.5.

$$\frac{\sinh a + \sinh b + \sinh c}{\cosh a + \cosh b + \cosh c} \geq \frac{\Phi_h(P_3)}{\tilde{\Phi}_h(P_3)}\tan\frac{\alpha+\beta+\gamma}{3},$$

with equality if and only if P_3 is regular.

According to the law of cosines

$$\cos\alpha = (\cosh b\cosh c - \cosh a)/\sinh b\sinh c,$$

hence $\Phi_h(P_3)$ and $\tilde{\Phi}_h(P_3)$ depend only on a, b and c.

THEOREM 6.6. *Let M' be a Riemannian manifold with $K_{M'} \leq K_M = -1$, and $P'_3 = \triangle A'B'C'$ be a geodesic triangle in M' with angles α', β', γ', and the lengths of three sides*

$$B'C' = a, \quad C'A' = b, \quad A'B' = c.$$

Set $\Phi_h(P'_3) = \Phi_h(P_3)$ and $\tilde{\Phi}_h(P'_3) = \tilde{\Phi}_h(P_3)$. Then

$$(a) \qquad \sin\alpha' + \sin\beta' + \sin\gamma' \leq \frac{3\tilde{\Phi}_h(P'_3)}{\cosh a + \cosh b + \cosh c};$$

$$(b) \qquad \cos\alpha' + \cos\beta' + \cos\gamma' \geq \frac{3\Phi_h(P'_3)}{\sinh a + \sinh b + \sinh c}.$$

Equality holds in (a) (resp. (b)) if and only if $\alpha' = \beta' = \gamma'$ and $a = b = c$.

PROOF. Since by Toponogov Comparison Theorem (*cf.* [13, p. 38]),

$$\alpha' \leq \alpha, \quad \beta' \leq \beta \quad \text{and} \quad \gamma' \leq \gamma,$$

hence the theorem is an immediate consequence of Theorem 6.4 and Theorem 6.1(b).

From Corollary 6.5 we have

THEOREM 6.7. *Under the hypotheses of Theorem 6.6,*

$$\frac{\sinh a + \sinh b + \sinh c}{\cosh a + \cosh b + \cosh c} \geq \frac{\Phi_h(P'_3)}{\tilde{\Phi}_h(P'_3)} \tan \frac{\alpha' + \beta' + \gamma'}{3}.$$

Equality holds if and only if $\alpha' = \beta' = \gamma'$ and $a = b = c$.

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Department of Mathematics and Statistics
University of Massachusetts
Amherst, MA
U.S.A.

Department of Mathematics and Statistics
University of South Alabama
Mobile, AL
U.S.A.