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# ON FINITE LINE TRANSITIVE AFFINE PLANES WHOSE COLLINEATION GROUPS CONTAIN NO BAER INVOLUTIONS

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**1. Introduction.** A finite line transitive affine plane $A$ is a finite plane which admits a collineation group $G$ acting transitively on the set of all lines of $A$. Wagner [**11**] has shown that $A$ is a translation plane and Hering [**9**] recently investigated the structure of $A$ under the assumption that $G$ has a composition factor isomorphic to a given nonabelian simple group. The purpose of this paper is to show that if the number of points on a line of $A$ is odd, and if $G$ contains no Baer involutions, then the hypothesis of Hering's Main Theorem holds. In particular, the result we prove is the following:

THEOREM 1. *If $A$ is a finite affine plane of order $n$, $n$ odd, and if a collineation group $G$ of $A$ has no Baer involutions and is transitive on the lines of $A$, then one of the following holds:*

(a) *$A$ is desarguesian.*

(b) *$G$ is isomorphic to a subgroup of $\Gamma L(1, p^{2m})$ where $n = p^m$. $G$ does not contain any non-trivial central collineation.*

(c) $n = p^m = 9$.

(d) $n = p^m = 27$.

Nondesarguesian planes of type (b), (c) and (d) are known to exist. That $n$ is a power of a prime follows from [**11**]. The case where $n$ is even is handled in Hering's Main Theorem if one assumes $G$ contains no Baer involutions. $\Gamma L(1, q)$ is the semi-direct product of the multiplicative group $M$ of the field $F$ of $q$ elements and the group of automorphisms of $F$ as they act on $M$. Since $A$ is a translation plane, there is no loss in assuming $G$ fixes some point $P$ of $A$.

**2. Notation and previously known results.** The group theoretic notation can be found in [**6**] and [**8**]. For groups $G$, $S$, $F$, $C_G(S)$ is the centralizer of $S$ in $G$. $Z(F)$ is the center of $F$. Geometric results and notation can be found in [**8**, Chapter 20] and [**4**]. In this paper, small letters $k$, $l$, $m$, etc. will represent lines and capital letters $L$, $M$, etc. will represent points. $L\ I\ m$ will mean point $L$ is incident with line $m$. $\pi$ will be a projective plane containing the affine plane $A$ and $l$ will be the line at infinity of $\pi$. If $G$ is as in § 1, $G$ is transitive on the points of $l$. If $X$ is any set of collinations of $\pi$, $\bar{X}$ will represent the action of $X$ as a permutation group on the points of $l$. An orbit of a permutation group $P$ will be a set of points on which $P$ acts transitively.

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A homology of $\pi$ is a collineation of $\pi$ that fixes all the points of some line $k$ of $\pi$ and exactly one point not on $k$. An involution is a collineation of order two. A Baer involution is an involution fixing a subplane $\pi'$ or $\pi$ such that the order of $\pi'$ is the square root of the order of $\pi$. A collineation $\alpha$ of $\pi$ is called a shear if there is a line $k \neq l$ of $\pi$ such that $\alpha$ fixes every point of $k$ and every line through $k \cap l$.

Before we quote theorems from [**9**] we need to introduce special notation. $\Phi_{2m}^*(p)$ is defined in [**10**, § 3], where $n = p^m$, $n$ the order of $\pi$. Let $r$ be the largest prime dividing $\Phi_{2m}^*(p)$. Let $R$ be a Sylow $r$-subgroup of $G$, $S$ the normal closure of $R$ in $G$, $L$ the centralizer of $S$ in the endomorphism ring of $V$ and $F$ the Fitting subgroup of $G$. Here we can view $G$ as a group of linear transformations on the $2m$ dimensional vector space over a field of prime order $p$ since $A$ is a translation plane.

THEOREM 2 [**9**, Theorem 4.4].
(a) *$SF/F$ is simple.*
(b) *$G/S$ is isomorphic to a subgroup of the metacylic group $\Gamma L(1, p^{2m})$.*
(c) $C_G(S) = Z(F)$.
(d) *If $G$ is not solvable, then $\Phi_{2m}^*(p) || |SF/S|$ and $C_{G/L}(SF/F) = 1$.*

We give a slightly altered and abbreviated version of the Main Theorem of [**9**]

THEOREM 3 [**9**, Main Theorem]. *Suppose that one of the following conditions is satisfied:*

I. *$G$ contains a composition factor isomorphic to $PSL(2, q)$, where $q$ is a prime power and $q \geqq 4$.*
II. *$G$ contains a composition factor isomorphic to the alternating group of degree $i$, where $i \geqq 5$.*
III. *$G$ contains a nontrivial shear.*
IV. $2 || p^m - 1$.
V. *$G$ is solvable.*

*Then we have one of the following cases:*
(a) *$A$ is desarguesian.*
(b) *$G$ is isomorphic to a subgroup of $\Gamma L(1, p^{2m})$. $G$ does not contain any nontrivial central collineation of $A$.*
(c) $p^m = 9$.
(d) $p^m = 27$.

V is actually not in Hering's Main Theorem, but V implies (b) is proved in [**5**]. The proof of Theorem 1 of this paper follows fairly easily from the Main Theorem of [**9**] and Theorem 1 of [**2**] and Theorem 1 of [**3**]. These latter two theorems we give here combined and in simplified form, taking advantage of the fact that $\Pi$ (and A) is a translation plane.

THEOREM 4. *Let $G$ be a collineation group of the translation plane $\Pi$. Assume*

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*$G$ fixes the line at infinity $l$ of $\Pi$ and a point $P \nmid l$ of $\Pi$. If $G$ contains no Baer involutions then either $G$ is solvable or $G$ contains subgroups $M$ and $N$ such that $G \triangleright M \triangleright N$, $G/M$ and $N$ are solvable and such that $M/N$ is isomorphic to one of the following:*

(a) *$PSL(2, q)$, $q$ a power of an odd prime;*

(b) *$A_7$ the alternating group of seven letters;*

(c) *$L_1 \times L_2$ where $L_i$ is isomorphic to a group of type* (a) *or* (b)*;*

(d) *$A_5$ the alternating group of five letters;*

(e) *$PS_{p^4}(q)$, $q$ a power of an odd prime.*

*Proof.* See [**1**] for a discussion of $PS_{p^4}(q)$. A symplectic geometry consists of a vector space $V$ and a non-singular bilinear form $f$ defined on $V$ such that $f(x, x) = 0$ for all $X$ in $V$. $Sp(4, q)$ is the group of all linear transformations $T$ on $V$ such that $f(T(X), T(Y)) = f(X, Y)$ for all $X, Y \in V$ $PS_{p^4}(q)$ is the quotient group of $Sp(4, q)$ over its center. To prove the theorem we need the following Lemma.

LEMMA. *Let $S$ be a 2-group acting on a finite projective plane $\Pi$ of odd order $n$, and fixing a point $P$ on a line $l$ of $\Pi$, $P \nmid l$. Assume for each point $Q$ and for each line $k$ of $\Pi$ there is at most one involutory homology in $S$ with center $Q$ and axis $k$. Then one of the following holds:*

(a) *$S$ contains at most three involutory homologies.*

(b) *There is a subgroup $T$ of $S$, with $[S:T] = 2, S > T$ where $T$ fixes points $L$ and $M$ on $l$, and an involutory homology $\alpha$ of $S$ such that $L\alpha = M$, $M\alpha = L$ and $S = \alpha, T$. $T$ contains at most three involutory homologies.*

(c) *There is a normal subgroup $T$ of $S$ with $[S:T] = 2$, such that $T$ contains at most one involutory homology, and $s = \langle T, \alpha \rangle$, $\alpha$ an involutory homology acting as an odd permutation on $l$; i.e.*, $2||n - 1$.

*Proof.* Assume $S$ fixes a point $L$ of $l$. Since $l$ has $n + 1$ points and $n$ is odd, $S$ must fix two points $L$ and $M$ on $l$. Let $\alpha$ be an involutory homology in $S$. Then $\alpha$ fixes the points $P$, $L$ and $M$. $P \nmid l$ so one of these points is the center of $\alpha$ and the remaining two lie on the axis of $\alpha$. From the hypothesis of the lemma we see $S$ contains at most three involutory homologies. Thus (a) holds.

We can now assume $S$ fixes no point of $l$; i.e., $S$ has no orbits of length one on $l$. Hence every orbit of $S$ on $l$ has length a power of two. Assume one orbit of $S$ on $l$ has length two and consists of the points $L$ and $M$. Let $T$ be the subgroup of $S$ fixing $L$ and $M$. Then $[S:T] = 2$ and $S \triangleright T$. $T$ contains at most three involutory homologies by the argument of the previous paragraph. Because of (a), we can assume there is an involutory homology $\alpha$ in $S$ such that $\alpha \in S - T$. $S \triangleright T$ and $\alpha \notin T$ so $\alpha$ must interchange $L$ and $M$. Hence (b) follows.

We can now assume every orbit of $S$ on $l$ has length divisible by four. Thus $4|n + 1$ and $2||n - 1$. Because of (a) we can assume there is an involutory homology $\alpha$ acting nontrivially on $l$. $\alpha$ fixes exactly two points of $l$ and $\alpha$ acts as an odd permutation on the remaining $n - 1$ points of $l$ since $2||n - 1$. If we let $T$ be the subgroup of even permutations in $S$, (c) follows and the lemma is proved.

The hypothesis of the lemma is contained in the theorems of [**2**] and [**3**]. (a) is also the hypothesis of Theorem 1 of [**3**] and (b) is the hypothesis of Theorem 1 of [**2**]. (c) is just III of Theorem 3 of this paper and thus Theorem 4 is proved.

We will also need the following well known theorem.

THEOREM 5 [**4**, 3.2.18, p. 145]. *If $\Pi$ is a projective plane of order $n$, and $\Pi'$ is a proper subplane of $\Pi$ of order $m$, then one of the following holds:*
(a) $m^2 = n$;
(b) $m^2 + m \leqq n$.

**3. The proof of Theorem 1.** It should be obvious from Theorems 3 and 4 that we need only show that $G$ does not have a composition factor isomorphic to $PS_{p^4}(q)$, $q = u^v$ for some prime $u$.

Let $S$ and $F$ be as in Theorem 2. As pointed out in some remarks after Theorem 4.4 in [**9**], $S \cap F$ is isomorphic to a subgroup of the Schur multiplier of $SF/F$. We can assume $SF/F \cong PS_{p^4}(q)$ and hence by [**7**], $|S \cap F| \leqq 2$. We are assuming $G$ fixes some point $P$ of $\Pi$. $\Pi$ is a translation plane of odd order and hence there is a unique involutory homology of $\Pi$ with center $P$ and axis $l$, $l$ the line at infinity of $\Pi$. We can, without loss, assume $\alpha \in S$ so that $S/\langle \alpha \rangle \cong PS_{p^4}(q)$. We easily have $S/\langle \alpha \rangle = \bar{S}$, $\bar{S}$ the permutation group representing the action of $S$ on the points of $l$.

There is clearly more than one involution in $S$ and hence there exists an involutory homology $\sigma$ with center $L$, say, on $l$ and axis $MP$, $MIl$. Let $\bar{G}_M$ be the stabilizer of $M$ in $\bar{G}$. If $\bar{\sigma}$ is not normal in $\bar{G}_M$ there exists a conjugate $\bar{\rho}$ of $\sigma$ with axis $MP$. $\bar{H} = \langle \bar{\sigma}, \bar{\rho} \rangle$ is a Frobenius group. All the elements of $H \langle \sigma, \rho \rangle$ fix $MP$ pointwise. If $\bar{\tau}$ is in the Frobenius kernel of $H$ then $\tau$ is a nontrivial elation with axis $MP$. $\tau$ acts as a shear on $A$ and hence Theorem 1 follows from Theorem 3.

From now on we can assume $\bar{G}_M \rhd \langle \bar{\sigma} \rangle$, $\bar{\sigma}$ fixes only the two points $L$ and $M$ of $l$ so $\bar{G}_M = \bar{G}_{LM}$. Also $\bar{S}_M = \bar{S}_{LM}$. Assume $\bar{S}$ has $k$ orbits on $q$. $G \rhd S$ and $\bar{G}$ is transitive on $l$. If $g \in G$ and $\bar{g}$ sends one orbit $O_1$ of $\bar{S}$ onto a second orbit $O_2$, then $\bar{\sigma}' = \bar{g}^1 \bar{\sigma} g$ fixes the same number of points on $O_2$ as $\bar{\sigma}$ does on $O_1$. However, from the structure of $\bar{S} = PS_{p^4}(q)$ we know $\bar{\sigma}$ and $\bar{\sigma}'$ are conjugate in $\bar{S}$; hence $\bar{\sigma}$ fixes the same number of points on $O_2$ as it does on $O_1$. Clearly then, since $G$ is transitive on the orbits of $\bar{S}$, $\bar{\sigma}$ fixes the same number of points on each of the $k$ orbits of $\bar{S}$. Since $\bar{\sigma}$ fixes exactly two points of $l$, $k \leqq 2$.

$q = u^v$, where $u$ is a prime. Let $\bar{\rho} \in C_{\bar{S}}(\bar{\sigma})$ with $|\bar{\rho}| = u$. $u \neq 2$ by Theorem 4, so $\bar{\rho}$ fixes $L$ and $M$. We now proceed to determine exactly how many points $\rho$ fixes on $l$. The number of such points is either the number of cosets $\bar{S}_{LM}\bar{x}$ in $\bar{S}$ such that $\bar{S}_{LM}\overline{x\rho} = \bar{S}_{LM}\bar{x}$ or double this number if $S$ has two orbits on $l$.

Denote a preimage of $\bar{x} \in \bar{S}$ in $Sp(4, q)$ by $\tilde{x}$. In the usual $4 \times 4$ matrix representation of $Sp(4, q)$ over a field of order $q$ we can assume

$$\tilde{\sigma} = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \text{ and } \tilde{\rho} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix},$$

where $A$, 0, $I$ are $2 \times 2$ matrices with $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, 0 the zero matrix and $I$ the $2 \times 2$ identity matrix. Let

$$\tilde{\tau} \in Sp(4, q),\ |\tilde{\tau}| = 2 \quad \text{and} \quad \tilde{\tau}\tilde{\rho}\tilde{\tau} = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}.$$

Since $\pi$ is a translation plane there are exactly two involutions fixing $L$ and $M$ (recall we are excluding Baer involutions from being in $G$) and acting non-trivially on $l$, namely $\sigma$ and $\alpha\bar{\tau}$. The image of $\tau$ in $S$ has order two so $\bar{\tau} \notin \bar{S}_{LM}$. It follows then since the image of $\tilde{\tau}$ in $PS_{p^4}(q)$ centralizes the image of $\sigma$ in $PS_{p^4}(q)$, $\bar{\tau}$ centralizes $\bar{\sigma}$. $|\bar{\tau}| = 2$ and $\bar{\tau} \notin \bar{S}_{LM}$ so $\bar{\tau}$ interchanges $L$ and $M$. Let $\tau$ be a preimage of $\bar{\tau}$ in $S$. Then from the action of $\bar{\tau}$ on $l$, we get $\tau^{-1}\sigma\tau = \sigma\alpha$. Thus it is clear that $S \cong Sp(4, q)$ as opposed to the only alternative, $S = Tx\langle\alpha\rangle$, $T \cong PS_{p^4}(q)$. $\tau \notin S_{LM}$ so $N = N_S(\langle\alpha, \sigma\rangle) = \langle\tau, C_S(\sigma)\rangle$ and $[N:S_{LM}] = 2$. Since $S \cong Sp(4, q)$, $|C_S(\sigma)| = q^2(q^2 - 1)^2$. $C_S(\sigma)$ must fix $L$ and $M$ so $S_{LM} = C_S(\sigma)$. $|Sp(4, q)| = q^4(q^4 - 1)(q^2 - 1)$ so $[\bar{S}:\bar{S}_{LM}] = [Sp(4, q):C_S(\sigma)] = q^2(q^2 + 1)$. This, along with the fact that $\bar{S}$ has at most two orbits on $l$, implies $p^n + 1$ is either $q^2(q^2 + 1)$ or $2q^2(q^2 + 1)$.

If $S_{LM}X$ is a coset in $S_{LM}C_S(\rho)$, then clearly $S_{LM} \times \rho = S_{LM}x$. The number of distinct right cosets of $S_{LM}$ in $S_{LM}C_S(\rho)$ is

$$[C_S(\rho):C_S(\rho) \cap S_{LM}] = [C_S(\rho):C_S(\rho) \cap C_S(\sigma)].$$

By direct computation with the matrix representation of $Sp(4, q)$ one can show $|C_S(\rho)| = 2q^4(q^2 - 1)$ and $|C_S(\rho) \cap C_S(\sigma)| = 2q^2(q^2 - 1)$. Thus $\rho$ fixes at least $2q^4(q^2 - 1)/2q^2(q^2 - 1) = q^2$ points of $l$. $\tau \notin S_{LM}$ so $\rho$ fixes $q^2$ of the cosets in $S_{LM}\tau C_S(\rho)$. From the original definition of $\tau$ we find $\rho$ fixes $q^2$ of the cosets of $S_{LM}\tau C_S(\rho)$. Thus $\rho$ fixes at least $2q^2$ points.

We now show that $\rho$ fixes a subplane $\pi'$ of $\Pi$. If not, $\rho$ can fix only the points $L$ and $P$ of $LP$. Since $|\rho| = u$, and $q = u^v$, $n|p^n - 1$ since $\rho$ acts as a semi-regular permutation on the points of $LP$ excluding $L$ and $P$. But we have seen above that $q^2|p^n + 1$ so $u|p^n + 1$. This gives $u|(p^n + 1) - (p^n - 1)$ so $u = 2$, contradicting Theorem 4(e).

The order of $\pi'$ is at least $2q^2 - 1$ since $\rho$ fixes at least $2q^2$ points of $l$. By Theorem 5, either $(2q^2 - 1)^2 = p^n + 1$ or $(2q^2 - 1)^2 + 2q^2 - 1 \leqq p^n + 1$. Both of these conditions are easily seen to be impossible since we have seen $p^n + 1$ is either $q^2(q^2 + 1)$ or $2q^2(q^2 + 1)$.

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