EXPLICIT SOLUTIONS FOR SURVIVAL PROBABILITIES IN THE CLASSICAL RISK MODEL

BY

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ABSTRACT

The purpose of this paper is to show that, for the classical risk model, explicit expressions for survival probabilities in a finite time horizon can be obtained through the inversion of the double Laplace transform of the distribution of time to ruin. To do this, we consider Gerber and Shiu (1998) and a particular value for their *penalty function*. Although other methods to address the problem exist, we find this approach, perhaps, more direct and simple. For the analytic inversion, we have applied twice, after some algebra, the Laplace complex inversion formula.

KEYWORDS

Classical risk model; double Laplace transform; Time to ruin; Ruin probabilities.

1. INTRODUCTION

In this paper we assume that the counting claims process is a homogeneous Poisson process and that the Laplace transform of the density function of the single amounts is known. In the following, we shall use the model and notation of Bowers *et al.* (*1987*, Chapter *12*), although with slight modifications. Thus, we consider that the company has an initial reserve of $U(0) = u$, whose surplus at time t is

$$U(t) = u + ct - Y(t), \qquad t \geq 0, \tag{1}$$

where c is the constant premium income per unit of time and $\{Y(t)\}_{t \geq 0}$ is a compound Poisson process with parameter λ, where $Y(t)$ is the aggregate claims amount up to time t and $Y(0) \equiv 0$.

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Let $g(t)$ be the density function of the time between two consecutive claims, so that $g(t) = \lambda \exp(-\lambda t)$.

A single claim amount is denoted by the random variable X with distribution function $F(x)$ and density function $f(x) = F'(x)$.

The distribution and density functions of the aggregate claims amount up to time t are denoted by $F_t(x)$ and $f_t(x)$, respectively.

The Laplace transform of a non-negative function $h(y)$ will be denoted by $\hat{h}(s)$, defined in the complex plane by

$$\hat{h}(s) = \int_0^\infty \exp(-sy)\, h(y)\, dy. \tag{2}$$

Unless otherwise specified, the abscissa of convergence will be denoted by σ.

If the function has two independent variables (x,y), we will maintain the previous definition with respect to each of them, that is,

$$\hat{h}(x,s) = \int_0^\infty \exp(-sy)\, h(x,y)\, dy,$$

and

$$\hat{h}(\delta,y) = \int_0^\infty \exp(-\delta x)\, h(x,y)\, dx.$$

The double Laplace transform will be defined by

$$\hat{\hat{h}}(\delta,s) = \int_0^\infty \int_0^\infty \exp(-\delta x - sy)\, h(x,y)\, dxdy.$$

The finite time ruin probability of the company, up to time t, considering an initial reserve u, will be denoted by $\psi(u,t)$ and the corresponding survival probability by $\sigma(u,t) = 1 - \psi(u,t)$. The ultimate ruin probability will then be $\psi(u) = \psi(u,\infty)$ and the non-ruin probability $\sigma(u) = 1 - \psi(u)$.

Following Gerber and Shiu (1998) and Dickson and Hipp (2001), we denote by T the time to ruin and we define a function ϕ by

$$\phi(u) = E[e^{-\delta T} I(T < \infty) \,|\, U(0) = u], \tag{3}$$

where I denotes the usual indicator function and δ is a parameter defined in the complex plane with a non-negative real part. $\phi(u)$ can then be considered the Laplace transform corresponding to the random variable T. For $\delta = 0$, we get $\phi(u) = \psi(u)$.

We remark that ϕ can be written as

$$\phi(u) = \int_0^\infty e^{-\delta t} \frac{\partial}{\partial t} \psi(u,t)\, dt = \psi(u,0) + \delta \hat{\psi}(u,\delta). \tag{4}$$

The purpose of our work is to find formulae for finite time survival probabilities, by means of the complex inversion of the corresponding Laplace transforms.

We start the research by capturing the double Laplace transform given by Gerber and Shiu (1998), which is the basis of our future developments. To help the reader and keep the sequence of the text we derive in a simple way the referred formula, taking the opportunity to introduce and explain some aspects we will need later.

Part of the present work was presented by the author at the I:ME conference in Lisbon in 2002, namely our formulae for the exponential and Erlang(2) cases. Since then we added some other developments, like considering the mixed exponential amount distribution case and other authors have also done additional research on the matter. Namely, Drekic and Willmot (2003) considered the density of the time to ruin for exponential distributed claims, Dickson *et al.* (2003) consider a Sparre Andersen model and derive an expression for the density of the time to ruin for exponential claims and Dickson and Willmot (2004) invert the Laplace transform of the time to ruin in the classical model and give a formula which can be used to calculate some of the results we present.

In Section 2 of this paper the Laplace transforms of $\phi(u)$ and $\sigma(u,t)$ are derived. In Section 3 we then work the complex inversion of the double transform $\hat{\hat{\sigma}}(\delta,s)$, and explicit results for exponential, Erlang(2) and mixed exponential distributed claims are achieved. For the complex inversion, we suppose that the reader is familiar with the *residue* theorem and with the Laurent series expansion for complex functions. At the end of the section we present some numerical results and, for comparison purposes, we compute the exact formulae obtained and the well known Seal's formula,

$$\sigma(u,t) = F_t(u+ct) - c\int_0^t f_\tau(u+c\tau)\,\sigma(0,t-\tau)\,d\tau,$$

where

$$\sigma(0,y) = \frac{1}{cy}\int_0^{cy} F_y(x)\,dx.$$

The numerical evaluation of the integrals has been done using the *dichotomic* algorithm considered by Lima *et al.* (2002).

2. The Laplace transforms of ϕ and $\sigma(u,t)$

Considering the instant and the amount of the first claim we may write

$$\begin{aligned}\phi(u) = &\int_0^\infty g(t)[1-F(u+ct)]\,e^{-\delta t}\,dt \\ &+ \int_0^\infty g(t)\,e^{-\delta t}\int_0^{u+ct} f(x)\,\phi(u+ct-x)\,dxdt.\end{aligned} \tag{5}$$

Changing the integration variable and differentiating with respect to u we get the relation

$$c\phi'(u) = (\lambda+\delta)\phi(u) - \lambda[1-F(u)] - \lambda\int_0^u f(x)\,\phi(u-x)\,dx, \tag{6}$$

which is equation (2.16) of Gerber and Shiu (1998) with their $\omega(x, y) = 1$ for all x and y.

Taking the Laplace transform of both sides of (6) and simplifying we get

$$\hat{\phi}(s) = \frac{c\phi(0) + \frac{\lambda}{s}\left[\hat{f}(s) - 1\right]}{cs - \lambda - \delta + \lambda\hat{f}(s)}. \tag{7}$$

Note that the zeroes of the denominator of (7) are the solutions of the Lundberg's fundamental equation

$$\lambda + \delta - \mathrm{cs} = \lambda\hat{f}(s). \tag{8}$$

Following Gerber and Shiu (1998), we verify that (8) has a non-negative root ρ, which is zero for $\delta = 0$ and, for a large class of claim amount distributions, also has a negative root denoted by R. We note that for $\delta = 0$ the absolute value of R is Lundberg's adjustment coefficient. Considering in (7) $s = \rho$ the denominator vanishes, so that ρ must also be a zero of the numerator and

$$\phi(0) = \frac{\lambda\left[1 - \hat{f}(\rho)\right]}{c\rho} \tag{9}$$

and

$$\hat{\phi}(s) = \frac{\frac{\lambda}{s}\left[\hat{f}(s) - 1\right] - \frac{\lambda}{\rho}\left[\hat{f}(\rho) - 1\right]}{cs - \lambda - \delta + \lambda\hat{f}(s)}. \tag{10}$$

Consider now the defective density under expression (4). We may write

$$\sigma(u,t) = 1 - \psi(u,t) = 1 - \int_0^t \frac{\partial}{\partial v}\psi(u,v)\,dv, \tag{11}$$

so that, considering the properties of Laplace transforms, the double transform of $\sigma(u,t)$ can be written as

$$\hat{\hat{\sigma}}(s,\delta) = \frac{1}{\delta s} - \frac{\hat{\phi}(s)}{\delta}. \tag{12}$$

Replacing (7) in (12) and simplifying we get

$$\hat{\hat{\sigma}}(s,\delta) = -\frac{-cs + \delta + cs\phi(0)}{s\delta\left(cs - \lambda - \delta + \lambda\hat{f}(s)\right)}. \tag{13}$$

Taking the same arguments used for expression (9), we get

$$\phi(0) = 1 - \frac{\delta}{c\rho}. \tag{14}$$

The substitution of (14) into (13) gives

$$\hat{\hat{\sigma}}(s,\delta) = \frac{s-\rho}{\rho s\left(cs-\lambda-\delta+\lambda\hat{f}(s)\right)}, \tag{15}$$

an expression obtained by Gerber and Shiu (1998), using a different approach.

3. The complex inversion of $\hat{\hat{\sigma}}$

3.1. Exponential distributed claims

If the single claim amount distribution is exponential with mean $1/\alpha$, we have

$$\hat{f}(s) = \frac{\alpha}{\alpha+s}, \qquad \mathrm{Re}(s) > -\alpha,$$

the Lundberg equation becomes

$$\lambda + \delta - sc = \alpha\frac{\lambda}{s+\alpha}, \tag{16}$$

and

$$\hat{\hat{\sigma}}(s,\delta) = \frac{(-\rho+s)(\alpha+s)}{\rho s\left(-\delta a-\delta s-\lambda s+cs\alpha+cs^2\right)}. \tag{17}$$

The first inversion $\hat{\hat{\sigma}}(s,\delta)$ with respect to s (the counterpart of u) is simple and $\hat{\sigma}(u,\delta)$ can be determined by the complex inversion formula for Laplace transforms, so that,

$$\hat{\sigma}(u,\delta) = \sum \text{ (residues of } e^{us}\hat{\hat{\sigma}}(s,\delta) \text{ at each of its singularities in } C), \tag{18}$$

(Marsden 1999, page 471). We can see that the implicit integrand function in (18) is analytic, with respect to s, except for the poles $s = 0$ and $s = R$. Note that $s = \rho$ is a removable singularity. The residues $r(s)$ will then be evaluated by the relation

$$r(s) = e^{us}\frac{(-\rho+s)(\alpha+s)}{-\rho\delta\alpha-2\rho\delta s-2\rho\lambda s+2\rho cs\alpha+3\rho cs^2},$$

giving

$$\hat{\sigma}(u,\delta) = \frac{1}{\delta} + e^{uR}\frac{(-\rho+R)(\alpha+R)}{-\rho\delta\alpha-2\rho\delta R-2\rho\lambda R+2\rho cR\alpha+3\rho cR^2}. \tag{19}$$

The inversion of $\hat{\sigma}(u,\delta)$ is not that simple. The inverse transform of $1/\delta$, the first part of (19), is 1. However, for the second part, we must consider the residues of the function

$$e^{\delta t+uR}\,\frac{(-\rho+R)(\alpha+R)}{-\rho\delta\alpha-2\rho\delta R-2\rho\lambda R+2\rho cR\alpha+3\rho cR^2}. \tag{20}$$

If we substitute R and ρ by their expressions as functions of δ, extracted from Lundberg's equation (16), the integrand function is not analytic in any half complex plane. But, we may change the integration variable δ through its natural relation with R from that equation, that is

$$\delta = cR-\lambda+\alpha\frac{\lambda}{R+\alpha}. \tag{21}$$

If we multiply the new integrand function by $d\delta/dR$ we get

$$-\exp\left(R\,\frac{-t\lambda+t\alpha+tcR+u\alpha+uR}{\alpha+R}\right)\frac{-\lambda\alpha+c\alpha^2+2c\alpha R+cR^2}{\alpha(-\lambda+c\alpha+cR)\,R}, \tag{22}$$

where ρ has been substituted by $-R-(-\delta-\lambda+c\alpha)/c$ (from the relation between the two roots of Lundberg's equation). The new integrand is analytic, except for three singularities: $\{0\}$, $\{r\}$ and $\{-\alpha\}$, which respect the relations

$$-\alpha<r<0.$$

We note that $|r|$ is Lundberg's adjustment coefficient.

The first two singularities are simple poles, but $\{-\alpha\}$ is an essential singularity. However, notice that when, under Lundberg's fundamental equation, δ progresses from 0 to $+\infty$, R decreases from r until $-\alpha$. For $\delta=0$, the integral that defines $\hat{\sigma}(u,\delta)$ is divergent, so that this transform and the first inversion already made, is only valid for $\mathrm{Re}(\delta)>0$. We conclude that $\{r\}$ and $\{0\}$ must be excluded for the second inversion. Besides, we can verify that the consideration of any of these poles, would result in an expression that would not give probabilities at all. The analysis of the new contour of integration presents some additional difficulties, since $\{-\alpha\}$ corresponds to $\delta=\infty$. However, it can be avoided, because the limit of the contour can precede the limit of the improper integral considered, so that the singularity $\{-\alpha\}$ shall be considered inside the contour.

To evaluate the residue at $\{-\alpha\}$ it is necessary to develop the integrand function into a Laurent series expanded about the point $R=-\alpha$. Considering $R+\alpha=z$, expression (22) becomes

$$\frac{\lambda\alpha-cz^2}{\alpha(\lambda-cz)(-z+\alpha)}\exp\left((-z+\alpha)\,\frac{t\lambda-tcz-uz}{z}\right),$$

or, using partial fractions, we get

$$\frac{1}{\alpha}\,e^{-(t\lambda+u\alpha+t\alpha)}\left(-1+\frac{\lambda}{\lambda-cz}+\frac{\alpha}{-z+\alpha}\right)e^{(u+tc)z}\,e^{(\frac{1}{z}\alpha t\lambda)}. \tag{23}$$

Considering that

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$$-1+\frac{\lambda}{\lambda-cz}+\frac{\alpha}{-z+\alpha}=1+\left(\frac{1}{\alpha}+\frac{c}{\lambda}\right)z+\left(\frac{1}{\alpha^2}+\frac{c^2}{\lambda^2}\right)z^2+\left(\frac{1}{\alpha^3}+\frac{c^3}{\lambda^3}\right)z^3+\ldots,$$

$$e^{(u+tc)z}=1+(u+tc)z+\frac{(u+tc)^2}{2!}z^2+\frac{(u+tc)^3}{3!}z^3+\ldots,$$

$$e^{\frac{1}{z}\alpha t\lambda}=1+\alpha t\lambda z^{-1}+\frac{(\alpha t\lambda)^2}{2!}z^{-2}+\frac{(\alpha t\lambda)^3}{3!}z^{-3}+\ldots,$$

and taking the product of these series, we obtain the residue at $z = 0$, which is the coefficient of z^{-1} in the result obtained. After simplification, we get

$$\begin{aligned}\sigma(u,t) &= 1+\frac{e^{-[(\lambda+c\alpha)t+\alpha u]}}{\alpha}\sum_{k=0}^{\infty}\frac{(u+ct)^k(\lambda\alpha t)^{k+1}}{k!(k+1)!}\\ &\quad-\frac{e^{-[(\lambda+c\alpha)t+\alpha u]}}{\alpha}\sum_{j=0}^{\infty}\left[\left(\frac{c}{\lambda}\right)^j+\left(\frac{1}{\alpha}\right)^j\right]\sum_{k=0}^{\infty}\frac{(u+ct)^k(\lambda\alpha t)^{j+k+1}}{k!(j+k+1)!}.\end{aligned} \tag{24}$$

3.2. Erlang(2) distributed claims

If a single claim amount has an Erlang(2) distribution, with Laplace transform

$$\hat{f}(s)=\frac{\alpha^2}{(\alpha+s)^2},\qquad \mathrm{Re}(s)>-\alpha, \tag{25}$$

the Lundberg fundamental equation will then be

$$\delta+\lambda-cs=\lambda\frac{\alpha^2}{(\alpha+s)^2}. \tag{26}$$

This equation has three roots – Q, R and ρ respecting the relations

$$Q<-\alpha<R<0\leq\rho.$$

Note that $\rho = 0$ if and only if $\delta = 0$. Substituting (25) in (15) we get

$$\begin{aligned}\hat{\hat{\sigma}}(s,\delta) &= \frac{\rho-s}{\rho s\left(\lambda\left(1-\hat{f}(s)\right)+\delta-cs\right)}\\ &= \frac{(\rho-s)(\alpha+s)^2}{\rho s\left(2\lambda\alpha s+\lambda s^2+\delta\alpha^2+2\delta a s+\delta s^2-cs\alpha^2-2cs^2\alpha-cs^3\right)}.\end{aligned} \tag{27}$$

The first inversion of (27) with respect to s (the counterpart of u in the complex plane), gives the first Laplace transform $\hat{\sigma}(u,\delta)$ of $\sigma(u,t)$ with respect to t and can be obtained through the inversion formula (18) .

We can see that the integrand function in (27) is analytic with respect to s, except for the poles $s = 0$, $s = R$ and $s = Q$. Note that $s = \rho$ is a removable singularity and that Q is outside the domain of $\hat{f}(s)$. The residues should then be evaluated by the relation

$$r(s) = e^{us}\frac{(\rho - s)(\alpha + s)^2}{4\rho\lambda\alpha s + 3\rho\lambda s^2 + \rho\delta\alpha^2 + 4\rho\delta a s + 3\rho\delta s^2 - 2\rho c s\alpha^2 - 6\rho c s^2\alpha - 4\rho c s^3},$$

giving $\hat{\sigma}(u,\delta)$ in the form

$$\frac{1}{\delta} + e^{uR}\frac{(\rho - R)(\alpha + R)^2}{\rho\left(4\lambda\alpha R + 3\lambda R^2 + \delta\alpha^2 + 4\delta a R + 3\delta R^2 - 2cR\alpha^2 - 6cR^2\alpha - 4cR^3\right)}. \tag{28}$$

As in the last example, the inversion of $\hat{\sigma}(u,\delta)$ is not that simple. The inverse transform of $1/\delta$, the first part of (28), is 1. However, for the second part, we must consider the residues of the function

$$e^{\delta t + uR}\frac{(\rho - R)(\alpha + R)^2}{\rho\left(4\lambda\alpha R + 3\lambda R^2 + \delta\alpha^2 + 4\delta a R + 3\delta R^2 - 2cR\alpha^2 - 6cR^2\alpha - 4cR^3\right)}.$$

We note that when δ, the implicit integration variable, progresses from 0 to ∞, R decreases from r until $-\alpha$.

To proceed with the inversion, we must change the integration variable δ through its natural relation with R, that is

$$\delta = -\lambda + cR + \lambda\frac{\alpha^2}{(\alpha + R)^2}, \tag{29}$$

and multiply the result obtained by the derivative of δ with respect to R. After some simplification we get

$$\begin{aligned} &-\exp\left(-R\frac{2t\lambda\alpha + t\lambda R - tc\alpha^2 - 2tcR\alpha - tcR^2 - u\alpha^2 - 2u\alpha R - uR^2}{(\alpha + R)^2}\right)\frac{1}{R} \\ &+\exp\left(-R\frac{2t\lambda\alpha + t\lambda R - tc\alpha^2 - 2tcR\alpha - tcR^2 - u\alpha^2 - 2u\alpha R - uR^2}{(\alpha + R)^2}\right)\frac{1}{\rho}. \end{aligned} \tag{30}$$

Using the same arguments as for the exponential distribution example, we can conclude that, again, $-\alpha$ is the only singular point to be considered. To determine the residue of the first part of (30) let us expand it into a Laurent series of powers of $(R + \alpha)$. For that purpose we take $R = z - \alpha$ obtaining

$$\begin{aligned} &\exp\left(-t\lambda - tc\alpha - u\alpha + ztc + zu + \frac{1}{z^2}t\lambda\alpha^2\right)\frac{1}{-z + \alpha} \\ &= \exp(-t\lambda - tc\alpha - u\alpha)\left\{\exp[(u + ct)z]\exp\left[\left(t\lambda\alpha^2\right)z^{-2}\right]\frac{1}{-z + \alpha}\right\}. \end{aligned} \tag{31}$$

Each of the last three factors can be expanded in a series of powers of z. Multiplying this series and collecting the coefficient of z^{-1} we get

$$\exp(-t\lambda - t\alpha - u\alpha)\sum_{j=1}^{\infty}\left\{\left(\frac{1}{\alpha}\right)^{2j-1}\sum_{k=1}^{\infty}\frac{a^{2k-1}b^{k+j-1}}{(2k-1)!(k+j-1)!}\right.$$
$$\left.+\left(\frac{1}{\alpha}\right)^{2j}\sum_{k=1}^{\infty}\frac{a^{2k-2}b^{k+j-1}}{(2k-2)!(k+j-1)!}\right\}, \tag{32}$$

where a denotes $u + ct$ and b denotes $t\lambda\alpha^2$.

To determine the residue of the second part of (30) it is necessary to express $1/\rho$ as a function of R. Writing Lundberg's fundamental equation (26) in the form

$$s^3 + \left(2\alpha - \frac{1}{c}\delta - \frac{1}{c}\lambda\right)s^2 + \left(\alpha^2 - \frac{2}{c}\alpha\delta - \frac{2}{c}\alpha\lambda\right)s -$$
$$\frac{1}{c}\alpha^2\delta = s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

and considering that the sum of the three roots is $-a_1$ and that its product is $-a_3$, we have

$$\rho = -a_1 - R - R_1 = -a_1 - R + \frac{a_3}{\rho R} = -a_1 - (z-\alpha) + \frac{a_3}{\rho(z-\alpha)}.$$

Considering that δ expressed as a function of z is

$$\delta = -\frac{\lambda z^2 - cz^3 + cz^2\alpha - \lambda\alpha^2}{z^2},$$

we get

$$\frac{1}{\rho} = \frac{\rho c z^2}{-\alpha\left(-\lambda\alpha^2 - \alpha\rho\lambda + cz^2\alpha - \lambda\alpha z + 2\rho c z^2\right)}. \tag{33}$$

Solving the last equation for $1/\rho$, we can see that it has two roots but only one represents $1/\rho$, which is

$$h(z) = \frac{1}{\rho} = \frac{1}{2\left(cz^2 - \lambda z - \alpha\lambda\right)}\,\frac{-2cz^2 + \alpha\lambda - \sqrt{\left(\alpha^2\lambda^2 + 4cz^3\lambda\right)}}{\alpha}. \tag{34}$$

Expanding this expression into a Maclaurin's series, we may write

$$\frac{1}{\rho} = \sum_{k=0}^{\infty} c_k z^k,$$

where

$$c_k = \frac{1}{k!}\left[\frac{d^k}{dz}h(z)\right]_{z=0}.$$

Considering now the second part of (30). It can be written as

$$\exp(-t\lambda - t\alpha - u\alpha)\left\{\exp[(u+ct)z]\exp\left(t\lambda\alpha^2\right)z^{-2}\sum_{k=0}^{\infty}c_k z^k\right\}.$$

Expanding the exponential functions between brackets in powers of z and taking the product of the three series, the coefficient of z^{-1} becomes

$$\sum_{j=1}^{\infty}c_{(2j-2)}\sum_{k=1}^{\infty}\frac{a^{2k-1}b^{k+j-1}}{(2k-1)!(k+j-1)!}+\sum_{j=1}^{\infty}c_{(2j-1)}\sum_{k=1}^{\infty}\frac{a^{2k-2}b^{k+j-1}}{(2k-2)!(k+j-1)!},$$

where $a = u + ct$ and $b = t\lambda\alpha^2$ as before. Finally, for this part of the residue, we get

$$\exp(-t\lambda - t\alpha - u\alpha)\sum_{j=1}^{\infty}\left\{c_{(2j-2)}\sum_{k=1}^{\infty}\frac{a^{2k-1}b^{k+j-1}}{(2k-1)!(k+j-1)!}\right.$$
$$\left.+\sum_{j=1}^{\infty}c_{(2j-1)}\sum_{k=1}^{\infty}\frac{a^{2k-2}b^{k+j-1}}{(2k-2)!(k+j-1)!}\right\}.$$

Putting all together, we have,

$$\sigma(u,t) = 1 - \exp(-t\lambda - t\alpha - u\alpha)\times$$
$$\sum_{j=1}^{\infty}\left\{\begin{array}{l}\left(\frac{1}{\alpha}\right)^{2j-1}\sum_{k=1}^{\infty}\frac{a^{2k-1}b^{k+j-1}}{(2k-1)!(k+j-1)!}+\left(\frac{1}{\alpha}\right)^{2j}\sum_{k=1}^{\infty}\frac{a^{2k-2}b^{k+j-1}}{(2k-2)!(k+j-1)!}\\ +c_{(2j-2)}\sum_{k=1}^{\infty}\frac{a^{2k-1}b^{k+j-1}}{(2k-1)!(k+j-1)!}+c_{(2j-1)}\sum_{k=1}^{\infty}\frac{a^{2k-2}b^{k+j-1}}{(2k-2)!(k+j-1)!}\end{array}\right\}.$$

Substituting a and b and simplifying, we get

$$\sigma(u,t) = 1 - \exp(-t\lambda - t\alpha - u\alpha)\times$$
$$\sum_{j=1}^{\infty}\left\{\begin{array}{l}\left[\left(\frac{1}{\alpha}\right)^{2j-1}+c_{(2j-2)}\right]\sum_{k=1}^{\infty}\frac{(u+ct)^{2k-1}\left(t\alpha^2\lambda\right)^{j+k-1}}{(2k-1)!(k+j-1)!}\\ +\left[\left(\frac{1}{\alpha}\right)^{2j}+c_{(2j-1)}\right]\sum_{k=1}^{\infty}\frac{(u+ct)^{2k-2}\left(t\alpha^2\lambda\right)^{j+k-1}}{(2k-2)!(k+j-1)!}\end{array}\right\}. \tag{35}$$

3.3. Mixed exponential distributed claims

If the individual claim amount has a mixed exponential distribution with two weights, b and $(1-b)$, the Laplace transform is

$$\hat{f}(s) = b\frac{\alpha}{\alpha+s}+(1-b)\frac{\beta}{\beta+s}. \tag{36}$$

Supposing that, without loss of generality, $\beta > \alpha$, the transform will be defined only for $s > -\alpha$.

It is interesting to see the graphic corresponding to both sides of Lundberg's equation

$$\delta + \lambda - cs = \lambda \hat{f}(s),$$

Considering $\delta = .5$, $\lambda = 1$, $\alpha = 1/2$, $\beta = 2$, $c = 1.1$ and $b = 1/3$, the graphic has the following shape:

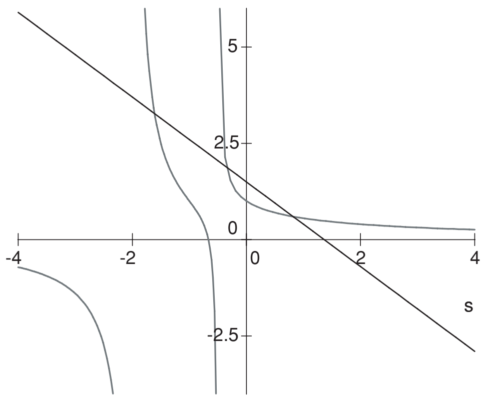

FIGURE 1.

The substitution of $\hat{f}(s)$ in (15) gives, after simplification,

$$\hat{\hat{\sigma}}(s,\delta) = \frac{-(\rho - s)(\alpha + s)(\beta + s)}{h(s)}, \tag{37}$$

Where

$$\frac{h(s)}{\rho s} = cs^3 + (-\lambda - \delta + c\alpha + c\beta)s^2 + [\alpha(b\lambda - \delta - \lambda + c\beta) - b\beta\lambda - \beta\delta]s - \delta\beta\alpha \tag{38}$$

As it can be seen, equation $h(s) = 0$ has a null root and three roots more: Q, R and ρ, each of them depending on δ and respecting the inequalities

$$-\beta < Q < -\alpha < R < 0 \leq \rho. \tag{39}$$

We can see that when δ moves from zero to infinity, ρ also goes from zero to infinity and R varies from right to left starting from r until $-\alpha$. The root Q is outside the domain of $\hat{f}(s)$ and consequently must not be considered for inversion purposes.

The first inversion of expression (37) with respect to s can be obtained through the inversion formula (18), already used for the previous distributions.

We can see that the implicit function defined in relation (37) is analytic with respect to s, except for the singularities $s = 0$ and $s = R$. Notice once again that $s = \rho$ is a removable singularity.

Considering that the above singular points are simple poles, the corresponding residues $r(.)$ can be determined by the relation

$$r(s) = \frac{-(\rho - s)(\alpha + s)(\beta + s)}{h'(s)}. \tag{40}$$

Considering the complex inversion formula we get

$$\hat{\sigma}(u,\delta) = \frac{1}{\delta} + e^{uR} r(R). \tag{41}$$

The inversion of this transform needs the evaluation of an integral with respect to δ of the function

$$e^{\delta t}\hat{\sigma}(u,\delta) = \frac{e^{\delta t}}{\delta} + e^{\delta t + uR} r(R). \tag{42}$$

The inverse transform of $1/\delta$, the first part of (41), is 1.
For the second part we must consider the residues of the function

$$\exp(\delta t + uR)\, r(R).$$

Expressing δ as a function of R, that is,

$$\delta = -\lambda + cR + \lambda \hat{f}(R),$$

multiplying by the derivative of δ with respect to R and simplifying,

$$\exp\left[\left(-\lambda + cR + \lambda \hat{f}(R)\right)t + uR\right] r(R)\delta'(R)$$

becomes

$$\left(\tfrac{1}{\rho} - \tfrac{1}{R}\right)\exp\left[R\,\frac{-t\lambda\alpha - tR\lambda + tc\beta\alpha + tR\alpha\alpha + tRc\beta + tcR^2 + t\lambda b\alpha - t\lambda\beta b + u\beta\alpha + uR\alpha + uR\beta + uR^2}{(\alpha + R)(\beta + R)}\right]. \tag{43}$$

As in the previous examples, the singularities to be considered are in the exponential part of the integrand function and they are two essential singular points: one corresponding to $R = -\alpha$ and the other resulting from $R = -\beta$. As the order does not matter, we will start with $R = -\alpha$ and with the second part of (43)

$$-\tfrac{1}{R}\exp\left[R\,\frac{-t\lambda\alpha - tR\lambda + tc\beta\alpha + tR\alpha\alpha + tRc\beta + tcR^2 + t\lambda b\alpha - t\lambda\beta b + u\beta\alpha + uR\alpha + uR\beta + uR^2}{(\alpha + R)(\beta + R)}\right]$$

If we write $R = z - \alpha$, the above expression becomes

$$-\frac{1}{z - \alpha}\exp\left[(z - \alpha)\,\frac{-t\lambda z - t\alpha\alpha z + tc\beta z + tcz^2 + t\lambda b\alpha - t\lambda\beta b - u\alpha z + u\beta z + uz^2}{z(\beta + z - \alpha)}\right].$$

Expanding the exponent in simple fractions we get

$$-\frac{1}{z-\alpha}\exp(-t\lambda - t\alpha - u\alpha)\exp[(ct+u)z]\exp\left(t\lambda b\frac{\alpha}{z}\right)\exp\left(-\beta t\lambda\frac{-1+b}{\beta+z-\alpha}\right).$$

Expanding each component of this expression in powers of z, we get

$$\begin{aligned}
\frac{-1}{z-\alpha} &= \sum_{k=0}^{\infty}\left(\frac{1}{\alpha}\right)^k z^k,\\
e^{(ct+u)z} &= \sum_{k=0}^{\infty}\frac{(ct+u)^k}{k!}z^k,\\
e^{t\lambda b\frac{\alpha}{z}} &= \sum_{k=0}^{\infty}\frac{(t\lambda b\alpha)^k}{k!}z^{-k},\\
e^{-\beta t\lambda\frac{-1+b}{\beta+z-\alpha}} &= \sum_{k=0}^{\infty}c_k z^k.
\end{aligned}$$

We must note that the last exponential does not have a development as simple as the others, though the coefficients c_k may be obtained through the corresponding derivatives,

$$c_k = \frac{1}{k!}\left[\frac{d^k}{dz^k}\exp\left(-\beta t\lambda\frac{-1+b}{\beta+z-\alpha}\right)\right]_{z=0}.$$

After the product of the four series, the coefficient of z^{-1} is

$$\sum_{j=0}^{\infty}c_j\sum_{k=1}^{\infty}\left(\frac{1}{\alpha}\right)^k\sum_{l=0}^{\infty}\frac{(t\lambda b\alpha)^{j+k+l-1}}{(j+k+l-1)!}\times\frac{(ct+u)^{l-1}}{(l-1)!},$$

so that, the first component of the residue corresponding to the root α is

$$\exp(-t\lambda - t\alpha - u\alpha)\sum_{j=0}^{\infty}c_j\sum_{k=1}^{\infty}\left(\frac{1}{\alpha}\right)^k\sum_{l=1}^{\infty}\frac{(t\lambda b\alpha)^{j+k+l-1}}{(j+k+l-1)!}\times\frac{(ct+u)^{l-1}}{(l-1)!}. \qquad (44)$$

For the other component of (43), we must express $1/\rho$ as a function of R, followed by the development into a Laurent series as we have just made. For the purpose, we start with the Lundberg's fundamental equation written as

$$\begin{aligned}
0 &= s^3 + a_1 s^2 + a_2 s + a_3\\
&= s^3 - \frac{\lambda+\delta-c\alpha-c\beta}{c}s^2 - \frac{\delta\alpha-\lambda b\alpha+\lambda\beta b+\lambda\alpha+\delta\beta-c\alpha\beta}{c}s - \delta\alpha\frac{\beta}{c}.
\end{aligned}$$

Considering that the sum of the three roots is $-a_1$ and its product is $-a_3$, we have

$$\rho = -a_1 - R - R_1 = -a_1 - (z-\alpha) + \frac{a_3}{\rho(z-\alpha)},$$

or,

$$\rho = \frac{\lambda + \delta - c\alpha - c\beta}{c} - (z - \alpha) - \frac{\delta\beta\alpha}{c\rho(z - \alpha)}. \tag{45}$$

Expressing δ as a function of z through the relation

$$\begin{aligned}\delta &= -\lambda + c(z - \alpha) + \lambda\hat{f}(z - \alpha) \\ &= (z - \alpha)\frac{-\lambda z - cz\alpha + cz\beta + cz^2 + \lambda b\alpha - \lambda\beta b}{z(\beta + z - \alpha)},\end{aligned}$$

substituting in (45) and considering $\rho = 1/x$, we obtain a second degree equation in x,

$$b_0 x^2 + b_1 x + b_2 = 0,$$

Where

$$\begin{aligned} b_0 &= \beta\alpha c z^2 + (-\beta\alpha^2 c + \beta^2\alpha c - \beta\alpha\lambda)z + \beta\alpha^2\lambda b - \beta^2\alpha\lambda b \\ b_1 &= (c\beta + c\alpha)z^2 + (c\beta^2 - \lambda\beta + \lambda\beta b - c\alpha^2 - \lambda b\alpha)z + \lambda b\alpha^2 - \alpha\lambda\beta b \\ b_2 &= -cz\alpha + cz\beta + cz^2. \end{aligned}$$

Considering the two roots of the last equation, we conclude that only one represents $1/\rho$, which must be that one that gives $x = 0$ for $z = 0$, once for $R = -\alpha$, we know that $\rho = \infty$. Solving for instance, for the parameters considered in the graphic of Figure 1, we have,

$$x(z) = \frac{-105z + 10 - 110z^2 - \sqrt{(5481z^2 + 3180z + 6908z^3 + 100 + 4356z^4)}}{2(-20 + 44z^2 + 26z)}.$$

Expanding $x(z)$ in a Maclaurin series we get,

$$x(z) = \sum_{k=0}^{\infty} d_k z^k, \text{ where } d_k = \frac{1}{k!}\left[\frac{d^k}{dz^k}x(z)\right]_{z=0},$$

and, as in the first part of the residue, taking the product of the four series and isolating the coefficient of z^{-1}, we get an expression similar to the previous one, just with the changes resulting from the series corresponding to $1/\rho$ instead $1/R$.

After simplification we get

$$\exp(-t\lambda - t\alpha - u\alpha)\sum_{j=0}^{\infty} c_j \sum_{k=1}^{\infty} d_{k-1} \sum_{l=1}^{\infty} \frac{(t\lambda b\alpha)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!}. \tag{46}$$

Putting both components together, the value of the residue corresponding to $-\alpha$ comes:

$$\exp(-t\lambda - tc\alpha - u\alpha)\sum_{j=0}^{\infty} c_j \sum_{k=1}^{\infty}\left[\left(\frac{1}{\alpha}\right)^k + d_{k-1}\right]\sum_{l=1}^{\infty}\frac{(t\lambda b\alpha)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!}. \tag{47}$$

For the root β, taking similar steps we get

$$\exp(-t\lambda - tc\beta - u\beta)\sum_{j=0}^{\infty} e_j \sum_{k=1}^{\infty}\left[\left(\frac{1}{\beta}\right)^k + f_{k-1}\right]\sum_{l=1}^{\infty}\frac{(t\lambda(1-b)\beta)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!}, \tag{48}$$

where the coefficients $\{e_k\}$ are obtained from the following Maclaurin's development:

$$e^{\alpha t\lambda \frac{b}{\alpha+z-\beta}} = \sum_{k=0}^{\infty} e_k z^k,$$

and the coefficients $\{f_k\}$ from the new development of $1/\rho$ in powers of z.

Summing up the three residues we finally have for $\sigma(u,t)$ the expression

$$\begin{aligned} 1 &- e^{-t\lambda - tc\alpha - u\alpha}\sum_{j=0}^{\infty} c_j \sum_{k=1}^{\infty}\left[\left(\frac{1}{\alpha}\right)^k + d_{k-1}\right]\sum_{l=1}^{\infty}\frac{(t\lambda b\alpha)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!} \\ &- e^{-t\lambda - tc\beta - u\beta}\sum_{j=0}^{\infty} e_j \sum_{k=1}^{\infty}\left[\left(\frac{1}{\beta}\right)^k + f_{k-1}\right]\sum_{l=1}^{\infty}\frac{(t\lambda(1-b)\beta)^{j+k+l-1}}{(j+k+l-1)!} \times \frac{(ct+u)^{l-1}}{(l-1)!}. \end{aligned} \tag{49}$$

3.4. Some numerical results

We have computed the explicit formulae for survival probabilities derived in the previous subsections for the three distribution cases presented and also the corresponding Seal's formulae. The results presented consider four different initial reserves (0, 1, 2 and 10) and ten consecutive time intervals (1, 2,...,10).

In the tables below, in column (1) we present the results from explicit formulae and in column (2) the numerical results obtained with Seal's integral formulae.

Some numerical differences between the first column and the second are due, in our opinion, to the complexity of the integral approximation using Seal's formulae and the implicit error committed with this type of approximation. The results obtained from the explicit formulae, should be in principle more accurate, once we have used some auxiliary functions such as Bessel functions and *generalized hypergeometric* functions that simplify significantly the necessary programs and that are built-in functions in the software used (Maple).

TABLE 1

Poisson (1) / Exponential (1) / c = 1.1

	u=0		u=1		u=2		u=10	
t	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
1	0,536599341	0,536599370	0,761944014	0,761944092	0,880294317	0,880294304	0,999691627	0,999691547
2	0,407136174	0,407136102	0,645431014	0,645430961	0,794327577	0,794327496	0,998650012	0,998649994
3	0,344789020	0,344789042	0,574022178	0,574022082	0,731540865	0,731540858	0,906770312	0,996770207
4	0,306693192	0,306693119	0,524715500	0,524715467	0,683592552	0,683592414	0,994104657	0,994104585
5	0,280402460	0,280402236	0,488107054	0,488106927	0,645580747	0,645580699	0,990767006	0,990766857
6	0,260881492	0,260881422	0,459570548	0,459570485	0,614551659	0,614551572	0,986885328	0,986885022
7	0,245661758	0,245661567	0,436536063	0,436535882	0,588632685	0,588632504	0,982580343	0,982577390
8	0,233373726	0,233373573	0,417448330	0,417448064	0,566579259	0,566578870	0,977957564	0,977932668
9	0,223188948	0,223188773	0,401304257	0,401304371	0,547530320	0,547530579	0,973105567	0,973070064
10	0,214573156	0,214572985	0,387424252	0,387424120	0,530869718	0,530869574	0,968096976	0,968031681

TABLE 2

Poisson (1) / Erlang (2) / c = 1.1

	u=0		u=1		u=2		u=10	
t	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
1	0,488408329	0,488408329	0,751323376	0,751323571	0,897816146	0,897816210	0,999982332	0,999982321
2	0,364106058	0,364106058	0,635115445	0,635115467	0,814990211	0,814990259	0,999833305	0,999833287
3	0,307657089	0,307657089	0,565020725	0,565020650	0,753321999	0,753321990	0,999407099	0,999407088
4	0,273761441	0,273761441	0,517044703	0,517044698	0,705940773	0,705940754	0,998610508	0,998610486
5	0,250576368	0,250576368	0,481626155	0,481626066	0,668289058	0,668289121	0,997410280	0,997410231
6	0,233457960	0,233457960	0,454130089	0,454129892	0,637525634	0,637525569	0,995815550	0,995815446
7	0,220164930	0,220164930	0,432005863	0,432005642	0,611822732	0,611822549	0,993859724	0,993858203
8	0,209465851	0,209465851	0,413720023	0,413719840	0,589956294	0,589956068	0,991587492	0,991570632
9	0,200620565	0,200620562	0,398288708	0,398288104	0,571075787	0,571075558	0,989046753	0,989029026
10	0,193154096	0,193154049	0,385049822	0,385046060	0,554571624	0,554570268	0,986284040	0,986245934

TABLE 3

POISSON (1) / MIXED EXPONENTIAL ($\alpha = 1/2$, $b = 1/3$, $\beta = 2$) / $c = 1.1$

	u=0		u=1		u=2		u=10	
t	(1)	(2)	(1)	(2)	(1)	(2)	(1)	(2)
1	0,5808459	0,5808459	0,7986258	0,7986259	0,8832519	0,8832520	0,9965719	0,9965719
2	0,4613693	0,4613693	0,6888200	0,6888201	0,7994853	0,7994854	0,9913266	0,9913266
3	0,3976509	0,3976509	0,6172846	0,6172846	0,7373643	0,7373643	0,9848518	0,9848517
4	0,3562979	0,3562979	0,5661029	0,5661030	0,6892756	0,6892756	0,9775631	0,9775625
5	0,3267453	0,3267453	0,5272508	0,5272508	0,6507489	0,6507484	0,9697607	0,9797420
6	0,3043222	0,3043222	0,4965127	0,4965132	0,6190425	0,6190417	0,9616615	0,9616539
7	0,2865857	0,2865857	0,4714373	0,4714368	0,5923864	0,5923853	0,9534211	0,9534167
8	0,2721181	0,2721181	0,4504924	0,4507775	0,5695864	0,5697266	0,9451512	0,9451518
9	0,2600340	0,2600340	0,4326665	0,4329839	0,5498062	0,5500083	0,9369309	0,9369279
10	0,2497496	0,2497496	0,4172624	0,4175560	0,5324411	0,5326497	0,9288161	0,9288266

4. ADDITIONAL REMARKS

In this paper we have shown that, in the classical risk model, it is possible to obtain explicit expressions for survival probabilities in a finite time horizon when the individual claims are mixed exponential or Erlang distributed. The same techniques used could also be extended to other particular claim amount distributions.

The extension of these techniques to other risk models depends, in first place, on the possibility of obtaining an explicit or implicit formula for the double Laplace transform corresponding to formula (13) and, in second place, on the analytic properties of this transform.

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