ON THE MOTION OF THREE VORTICES

J. L. SYNGE

1. Introduction. In a perfect incompressible fluid extending to infinity, the determination of the motion of N parallel rectilinear vortex filaments involves the solution of N non-linear differential equations, each of the first order. The method of Kirchhoff[1] provides certain constants of the motion. If we describe the positions of the vortices by their point-traces on a plane perpendicular to them, the following facts follow from the theory of Kirchhoff:

(1.1) The mean centre of the system is fixed.

$$\sum_{m,n}{}' \kappa_m \kappa_n \log r_{mn} = \text{const.} \tag{1.2}$$

$$\sum_m \kappa_m r^2{}_m = \text{const.} \tag{1.3}$$

Here the summations cover the range $1, 2, \ldots N$; the prime indicates that $m = n$ is omitted; κ_m are the strengths of the vortices; r_{mn} is the distance between the vortices of strengths κ_m and κ_n; r_m is the distance of the vortex of strength κ_m from a fixed point.

In this paper we shall be concerned solely with the configurations of the vortex system, understanding by *configuration* the geometrical figure formed by the vortices, without regard to rigid body displacements of that figure. Thus, if a system of three vortices forms a triangle with sides of fixed lengths throughout the motion, we say that the configuration is fixed.

The following theorems, applicable to a system consisting of any number of vortices, are obvious from the usual equations of vortex motion, and are quoted here for reference.

THEOREM 1: *If, given a configuration, the strengths of all the vortices are suddenly reversed, the system retraces the sequence of configurations through which it has come.*

THEOREM 2: *Given at $t = t_0$ a configuration in which all the vortices are collinear, then the configurations at times $t = t_0 \pm \tau$ are reflections of one another for all values of τ.*

THEOREM 3: *A system cannot pass through more than two distinct collinear configurations; the times required to pass from one collinear configuration to the other are all the same.*

THEOREM 4: *Suppose that there are two systems of vortices, S_1 and S_2, each consisting of the same number of vortices, and the strengths of the vortices in S_2*

Received May 2, 1948.

[1]Cf. Sir H. Lamb, *Hydrodynamics* (Cambridge, 1932), 230; H. Villat, *Leçons sur la théorie des tourbillons* (Paris, 1930), 46.

being those of S_1 all multiplied by the same factor K^2; suppose further that initially the configurations are similar, without reflection, the lengths in S_2 being those in S_1 all multiplied by the factor L. Then the subsequent configuration of S_2 after time t_2 is similar, without reflection, to the configuration of S_1 after time t_1, where $t_2 = t_1(L^2/K^2)$.

As an immediate consequence of (1.2) and (1.3), we have the following result:

THEOREM 5: *If the strengths of all the vortices have the same sign, their mutual distances are bounded above and below for all time, positive and negative.*

No further general results appear to be available, so we turn to special cases. The general case can be specialized in a number of ways. We might specialize the strengths of vortices, perhaps choosing them all of the same strength, or plus and minus one fixed value. On the other hand we might specialize by restricting the number of vortices in the system, and this is in fact the specialization we shall adopt.

Since the case of two vortices is trivial, we turn to the case of three vortices, without imposing any particular *a priori* condition on their strengths. This is precisely the problem discussed by W. Gröbli[2] over seventy years ago. However, he was interested in obtaining formal analytic solutions for the motion, and found it necessary at an early stage to specialize the strengths of the vortices. He seems to have missed the interesting fact that the motions may be classified according to the positive or negative character of the sum of the products of the strengths in pairs, $\kappa_2\kappa_3 + \kappa_3\kappa_1 + \kappa_1\kappa_2$. It seems appropriate therefore to take up this problem again, concentrating on a qualitative classification of all possible motions rather than on the development of analytic solutions. The basic equations (2.5) are the same as those of Gröbli, but are obtained here in a simpler way. The representation of the motions by trilinear coordinates is believed to be new.

2. The equations of motion and their integrals. Let $\kappa_1, \kappa_2, \kappa_3$ be the strengths of the three vortices (i.e. the circulations around them), and R_1, R_2, R_3 the lengths of the sides of the triangle formed by them, R_1 being opposite κ_1, and so on, so that, in the notation of (1.2), $R_1 = r_{23}$, etc. In accordance with the usual convention, we regard a strength as positive when it gives a counter-clockwise circulation. It is convenient to get rid of the factor 2π by defining

$$k_1 = \kappa_1/2\pi,\quad k_2 = \kappa_2/2\pi,\quad k_3 = \kappa_3/2\pi. \tag{2.1}$$

It is assumed that none of the three strengths vanishes.

Consider the rate of increase $R'_1 = dR'_1/dt$ of the side R_1. The motions due to the vortices k_2 and k_3 at its extremities contribute nothing to R'_1. One end of R_1, viz. k_2, has due to k_1 a velocity of magnitude k_1/R_3 perpendicular to R_3, and the other end viz. k_3, has due to k_1 a velocity of magnitude k_1/R_2 per-

[2] *Vierteljahrschrift der naturforschenden Gesellschaft in Zürich*, vol. 22 (1877), 37-81, 129-167. Gröbli also investigated certain cases of symmetry for N vortices.

pendicular to R_2. Let θ_1, θ_2, θ_3 be the angles of the triangle formed by the vortices. Then, on reference to Figure 1, it is seen that

$$R'_1 = \epsilon k_1(R_2^{-1} \sin \theta_3 - R_3^{-1} \sin \theta_2), \tag{2.2}$$

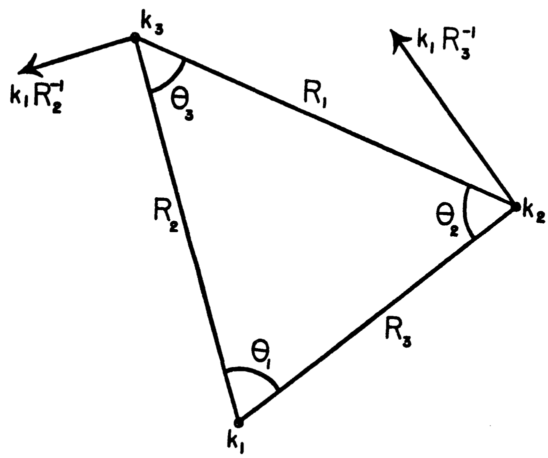

FIG. 1

Rate of growth of a side of the triangle.

where $\epsilon = +1$ or -1 according as the circuit of the triangle in the order $k_1k_2k_3$ is positive or negative respectively (counter-clockwise or clockwise). Let A denote the area of the triangle, prefixed by a plus or minus sign according as the above circuit is positive or negative. Then ϵA is positive, and

$$\epsilon A = \tfrac{1}{2}R_2R_3 \sin \theta_1 = \tfrac{1}{2}R_3R_1 \sin \theta_2 = \tfrac{1}{2}R_1R_2 \sin \theta_3. \tag{2.3}$$

We have also the formula

$$\epsilon A = [s(s - R_1)(s - R_2)(s - R_3)]^{\frac{1}{2}}, \quad s = \tfrac{1}{2}(R_1 + R_2 + R_3). \tag{2.4}$$

If we substitute from (2.3) in (2.2) and the two similar equations, we get

$$\begin{aligned} k_1^{-1}R_1R'_1 &= 2A(R_2^{-2} - R_3^{-2}), \\ k_2^{-1}R_2R'_2 &= 2A(R_3^{-2} - R_1^{-2}), \\ k_3^{-1}R_3R'_3 &= 2A(R_1^{-2} - R_2^{-2}). \end{aligned} \tag{2.5}$$

Adding and integrating, we get

$$k_1^{-1}R_1^2 + k_2^{-1}R_2^2 + k_3^{-1}R_3^2 = a, \tag{2.6}$$

where a is a constant. If we multiply (2.5) in order by R_1^{-2}, R_2^{-2}, R_3^{-2}, add, and integrate, we get

$$k_1^{-1} \log R_1 + k_2^{-1} \log R_2 + k_3^{-1} \log R_3 = b, \tag{2.7}$$

where b is a constant. This is the same as Kirchhoff's equation (1.2), and (2.6) is equivalent to (1.3), but more convenient for our purpose because expressed in terms of the sides of the triangle. The above equations were given by Gröbli (*loc. cit.*).

The differential equations (2.5), with their integrals (2.6) and (2.7), form the basis of our work. To these we shall add another equation, obtained by differentiating (2.4) and then substituting for R'_1, R'_2, R'_3. In this way we get

$$A' = f(R_1, R_2, R_3), \tag{2.8}$$

where

$$\begin{aligned} f(R_1, R_2, R_3) &= \tfrac{1}{2}[\Sigma k_1 R_1^{-1}(R_2^{-2} - R_3^{-2})][(s-R_1)(s-R_2)(s-R_3) + s\Sigma(s-R_2)(s-R_3)] \\ &\quad - s\Sigma k_1 R_1^{-1}(R_2^{-2} - R_3^{-2})(s - R_2)(s - R_3) \,. \end{aligned} \tag{2.9}$$

Here and later, Σ indicates summation over a cyclic permutation of suffixes.

3. Fixed configurations. Let us now seek necessary and sufficient conditions that the configuration of the three vortices remains fixed, so that the motion is a rigid body motion. If the configuration is fixed, then $R'_1 = R'_2 = R'_3 = 0$ and so by (2.5) we must have either $R_1 = R_2 = R_3$ (equilateral configuration), or $A = 0$ (collinear configuration). These are necessary conditions. Any equilateral configuration does remain fixed, as was pointed out by Gröbli (*loc. cit.*), and this is a sufficient condition. But $A = 0$ is not a sufficient condition for fixity. At first sight this appears to be in conflict with (2.5). Suppose we take for R_1, R_2, R_3 any three constant values satisfying one of the equations

$$R_1 = R_2 + R_3, \quad R_2 = R_3 + R_1, \quad R_3 = R_1 + R_2, \tag{3.1}$$

such values make $A = 0$ by (2.4), and hence these values constitute a formal solution of (2.5). However, it is a singular solution, and does not in general satisfy the full set of equations of vortex motion. In order that the collinear configuration may remain fixed, it is further necessary that $A' = 0$, or

$$f(R_1, R_2, R_3) = 0, \tag{3.2}$$

where f is as in (2.9). We may sum up as follows:

THEOREM 6: *Necessary and sufficient conditions for a fixed configuration are either that the initial configuration be equilateral, or that it be collinear, satisfying* (3.2).

4. Variable configurations and the trilinear representation. The values of R_1, R_2, R_3 determine a configuration to within a reflection. Thus we might discuss changes in configuration by following a representative point in a space in which R_1, R_2, R_3 are taken as rectangular Cartesian coordinates. Since these quantities are essentially positive, we would be concerned only with the positive octant. Collision of the representative point with one of the walls of this octant would correspond to a collision of two of the vortices. The motion of the system would correspond to a curve of intersection of surfaces (2.6) and (2.7), the sense in which the curve is described being determined by reference to (2.5), with use of the fact that t increases. But the representative point is further restricted since R_1, R_2, R_3 must always satisfy the triangle inequalities

$$R_1 \leqslant R_2 + R_3, \quad R_2 \leqslant R_3 + R_1, \quad R_3 \leqslant R_1 + R_2. \tag{4.1}$$

In fact, the planes (3.1) form boundaries in the representative space which the representative point is forbidden to cross. If the representative point meets one of the planes (3.1), the configuration becomes collinear. Then, by Theorem 2, the system passes back through the same sequence of configurations but with the orientation reversed; the representative point moves back along the curve by which it came to the collinear configuration.

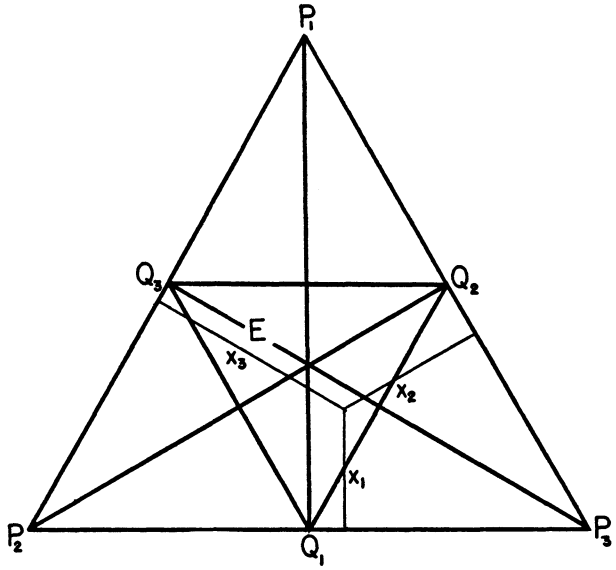

FIG. 2

The trilinear representation.

However, there is another and better representation by trilinear coordinates in a plane, as shown in Figure 2, and that is the representation which will be used in this paper. $P_1P_2P_3$ is an equilateral triangle of unit height, and x_1, x_2, x_3 are trilinear coordinates, i.e. the distances of a general point from the sides of the triangle $P_1P_2P_3$; these values satisfy

$$x_1 + x_2 + x_3 = 1. \tag{4.2}$$

Now put

$$\begin{aligned} x_1 &= R_1(R_1 + R_2 + R_3)^{-1}, \\ x_2 &= R_2(R_1 + R_2 + R_3)^{-1}, \\ x_3 &= R_3(R_1 + R_2 + R_3)^{-1}, \end{aligned} \tag{4.3}$$

and so connect the points of the representative plane with the configurations of the vortex system. To each configuration of the system there corresponds a unique x-point, with one exception: a triple collision ($R_1 = R_2 = R_3 = 0$) is not represented. On the other hand, to a given x-point there corresponds a single infinity of configurations, all similar to one another, together with the reflections of those configurations. The centroid E of the triangle ($x_1 = x_2 = x_3 = 1/3$) corresponds to all equilateral configurations.

Let $Q_1Q_2Q_3$ be the middle points of the sides of the triangle $P_1P_2P_3$. On Q_2Q_3 we have $x_1 = \frac{1}{2}$ and hence $x_1 = x_2 + x_3$ or $R_1 = R_2 + R_3$. In fact, Q_2Q_3 corresponds to the first of (3.1), and the three sides of the triangle $Q_1Q_2Q_3$ correspond to the three planes (3.1) which the representative point is forbidden to cross. Since E is certainly permitted, the permitted region is the interior of the triangle $Q_1Q_2Q_3$. The points $Q_1Q_2Q_3$ correspond to collisions of the vortices, k_2 and k_3 colliding at Q_1, etc.

All points on the sides of the triangle $Q_1Q_2Q_3$ correspond to collinear configurations. Since the configuration can change its orientation only by passing through a collinear configuration, we may use the two sides of the representative plane, all configurations with positive orientation being represented on the front of the plane and all configurations with negative orientation on the back. The sides $Q_1Q_2Q_3$ are then cuts by which the representative point passes from one side of the plane to the other. We might in fact throw away all the diagram except the triangle $Q_1Q_2Q_3$, and allow the representative point to pass round the edges of this triangle.

As the system moves, the representative point describes a curve C. To find the differential equations of C, we differentiate (4.3) and substitute from (2.5). This gives

$$x'_1 = KH_1, \; x'_2 = KH_2, \; x'_3 = KH_3, \tag{4.4}$$

where

$$K = 2AR_1^{-2}R_2^{-2}R_3^{-2}(\Sigma R_1)^2, \tag{4.5}$$

and

$$\begin{aligned} H_1 &= -\, k_1x_1(x_2^2 - x_3^2) + x_1\Sigma k_1x_1(x_2^2 - x_3^2), \\ H_2 &= -\, k_2x_2(x_3^2 - x_1^2) + x_2\Sigma k_1x_1(x_2^2 - x_3^2), \\ H_3 &= -\, k_3x_3(x_1^2 - x_2^2) + x_3\Sigma k_1x_1(x_2^2 - x_3^2). \end{aligned} \tag{4.6}$$

We check that $H_1 + H_2 + H_3 = 0$, as of course it must be, by (4.2).

By (4.4.) we have

$$\frac{dx_1}{H_1} = \frac{dx_2}{H_2} = \frac{dx_3}{H_3} = Kdt. \tag{4.7}$$

The first two of these equations define a congruence of x-curves, and this congruence defines the behaviour of the configuration, except for orientation, rate of change, and scale. However, orientation is determined by the side of the representative plane on which the point lies, and rate of change is given by (4.4). As regards scale, if the shape of the configuration is given, its size may in general be determined by (2.6) or (2.7), the values of the constants a and b being given by the initial configuration. There is, however, one exceptional case, and this we shall now discuss.

The integrals (2.6) and (2.7) may be written

(4.8) $$k_1^{-1}x_1^2 + k_2^{-1}x_2^2 + k_3^{-1}x_3^2 = a(R_1 + R_2 + R_3)^{-2},$$

(4.9) $$k_1^{-1}\log x_1 + k_2^{-1}\log x_2 + k_3^{-1}\log x_3 = b - (k_1^{-1} + k_2^{-1} + k_3^{-1})\log (R_1 + R_2 + R_3).$$

If $a = 0$ and

(4.10) $$k_2k_3 + k_3k_1 + k_1k_2 = 0,$$

then $(R_1 + R_2 + R_3)$ disappears from (4.8) and (4.9). In this exceptional case, the values of x_1, x_2, x_3, a, b fail to determine the values of R_1, R_2, R_3. We may state the following results.

THEOREM 7: *If the strengths of the vortices do not satisfy* (4.10), *and b is known from an initial configuration, then to each x-point there corresponds by* (4.9) *a unique configuration, except for orientation.*

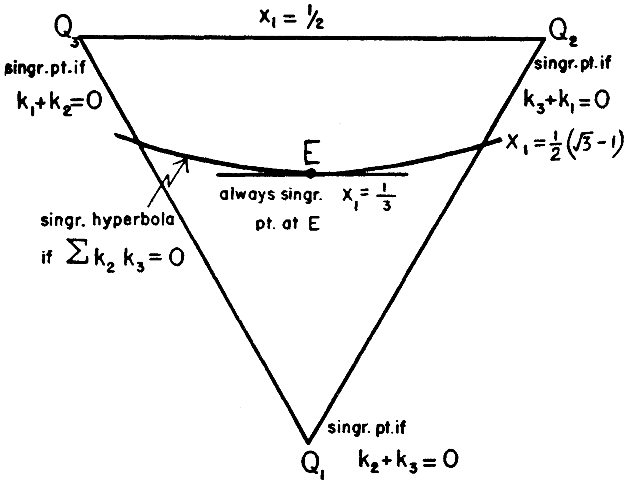

FIG. 3

Singular points.
(Hyperbola drawn for $2k_1 = -k_2 = -k_3$.)

THEOREM 8: *If the strengths of the vortices satisfy* (4.10), *then to each x-point on the conic*

(4.11) $$k_2k_3x_1^2 + k_3k_1x_2^2 + k_1k_2x_3^2 = 0$$

there corresponds a single infinity of similar configurations of both orientations; to each x-point lying off the conic (4.11) *there corresponds by* (4.8) *a unique configuration, except for orientation.*

It is easily seen that, under the condition (4.10), the conic (4.11) is a hyperbola. It passes through the centroid E, and meets two sides of the triangle $Q_1Q_2Q_3$, each in one point. At E the tangent to (4.11) has the direction given by

(4.12) $$dx_1 : dx_2 : dx_3 = k_1(k_2 - k_3) : k_2(k_3 - k_1) : k_3(k_1 - k_2).$$

The hyperbola is shown in Figure 3 for the case

$$2k_1 = -k_2 = -k_3. \tag{4.13}$$

It is important to know that *no curve C can cut a median of the triangle $Q_1Q_2Q_3$ in an infinite number of points.* To show this, we consider the median P_1Q_1, on which we have

$$x_2 = x_3 = \tfrac{1}{2}(1 - x_1).$$

By (4.8) and (4.9) we have at an intersection of a curve C with the median P_1Q_1

$$k_1^{-1}x_1^2 + (k_2^{-1} + k_3^{-1})\,\tfrac{1}{4}\,(1 - x_1)^2 = a/4s^2, \tag{4.14}$$

$$k_1^{-1}\log x_1 + (k_2^{-1} + k_3^{-1})\log\tfrac{1}{2}(1 - x_1) = b - (k_1^{-1} + k_2^{-1} + k_3^{-1})\log 2s,$$

where, as earlier, $2s = R_1 + R_2 + R_3$. If we eliminate s, we get an equation in x_1, a, b; for a given curve C the constants a and b are assigned, and this equation determines the values of x_1 corresponding to the intersections of C with P_1Q_1. It is clear that in the range $0 \leqslant x_1 \leqslant 1$ there can be at most a finite number of solutions, and so the result is proved.

5. Singular points. The most powerful way of studying the congruence (4.7) is through its singular points, at which

$$H_1 = H_2 = H_3 = 0. \tag{5.1}$$

On account of the triangle inequalities (4.1), we are interested only in singular points lying inside the triangle $Q_1Q_2Q_3$ or on its boundary. Let us first examine the points Q_1, Q_2, Q_3, to see if any one of them can be singular.

At Q_1 we have $x_1 = 0$, $x_2 = x_3 = \frac{1}{2}$; hence, by (4.6),

$$H_1 = 0,\ H_2 = -\tfrac{1}{8}k_2 + \tfrac{1}{16}(k_2 - k_3),\ H_3 = \tfrac{1}{8}k_3 + \tfrac{1}{16}(k_2 - k_3). \tag{5.2}$$

These equations are consistent with (5.1) if, and only if,

$$k_2 + k_3 = 0. \tag{5.3}$$

When this condition is satisfied, Q_1 is a singular point. The points Q_2 and Q_3 may of course be discussed in exactly the same way.

For all points in the triangle $Q_1Q_2Q_3$ or on its boundary, other than the vertices Q_1, Q_2, Q_3, we have x_1, x_2, x_3 all different from zero. Then, if we substitute in (5.1) from (4.6), we can divide across by these factors, and obtain

$$x_2^2 - x_3^2 = k_1^{-1}\theta,\ x_3^2 - x_1^2 = k_2^{-1}\theta,\ x_1^2 - x_2^2 = k_3^{-1}\theta, \tag{5.4}$$

$$\theta = \Sigma k_1x_1(x_2^2 - x_3^2).$$

Addition gives

$$\theta\Sigma k_1^{-1} = 0. \tag{5.5}$$

Suppose first that $\theta = 0$; then (5.4) give $x_1 = x_2 = x_3 = 1/3$. Thus the point E is a singular point, as is indeed obvious. On the other hand, if (4.10) is satisfied, then (5.5) is satisfied with $\theta \neq 0$. If we multiply (5.4) in order by x_1^2, x_2^2, x_3^2 and add, we get

$$\Sigma k_1^{-1}x_1^2 = 0 \tag{5.6}$$

which is the same equation as (4.11). All singular points (other than Q_1, Q_2, Q_3, discussed above) must lie on this conic. Moreover it is easy to see that, if (4.10) is satisfied, then every point on the conic (4.11) or (5.6) is a singular point. We have already remarked that this conic is a hyperbola.

Let us sum up our conclusions about singular points as follows.

THEOREM 9: *The singular points of the congruence* (4.7), *inside or on the triangle* $Q_1Q_2Q_3$, *are as follows. If*

(5.7) $$k_2k_3 + k_3k_1 + k_1k_2 \neq 0,$$

and

(5.8) $$k_2 + k_3 \neq 0, \; k_3 + k_1 \neq 0, \; k_1 + k_2 \neq 0,$$

then the only singular point is at E (*equilateral configuration*). *If*

(5.9) $$k_2k_3 + k_3k_1 + k_1k_2 = 0,$$

then (5.8) *are necessarily true*; *the singular points make up the hyperbola* (4.11), *which passes through* E. *If*

(5.10) $$k_2 + k_3 = 0, \; k_3 + k_1 \neq 0, \; k_1 + k_2 \neq 0,$$

then (5.7) *is necessarily true*; *the only singular points are at* E *and* Q_1. *Similar results hold on permuting suffixes in* (5.10). *If*

(5.11) $$k_1 = -k_2 = -k_3,$$

the only singular points are at E, Q_2, Q_3. *Similar results hold on permutation of suffixes.*

These results are shown in Figure 3.

6. Behaviour of representative curves near the point *E*. To explore the curves near the point E, we put

(6.1) $$x_1 = y_1 + 1/3, \; x_2 = y_2 + 1/3, \; x_3 = y_3 + 1/3,$$

so that

(6.2) $$y_1 + y_2 + y_3 = 0.$$

Then (4.6) gives, to the first order in y_1, y_2, y_3,

(6.3) $$\begin{aligned} H_1 &= -\tfrac{2}{9}k_1(y_2 - y_3) + \tfrac{2}{27}\Sigma k_1(y_2 - y_3), \\ H_2 &= -\tfrac{2}{9}k_2(y_3 - y_1) + \tfrac{2}{27}\Sigma k_1(y_2 - y_3), \\ H_3 &= -\tfrac{2}{9}k_3(y_1 - y_2) + \tfrac{2}{27}\Sigma k_1(y_2 - y_3). \end{aligned}$$

As in (4.7) we have, as differential equations of the congruence,

(6.4) $$\frac{dy_1}{H_1} = \frac{dy_2}{H_2} = \frac{dy_3}{H_3} = Kdt.$$

It is convenient to define

(6.5) $$z_1 = y_2 - y_3, \; z_2 = y_3 - y_1, \; z_3 = y_1 - y_2,$$

so that, by (6.2),

(6.6) $$y_1 = -\tfrac{1}{3}(z_2 - z_3), \; y_2 = -\tfrac{1}{3}(z_3 - z_1), \; y_3 = -\tfrac{1}{3}(z_1 - z_2).$$

From (6.4) we obtain

(6.7) $$\frac{dz_1}{L_1} = \frac{dz_2}{L_2} = \frac{dz_3}{L_3},$$

where

$$L_1 = -\tfrac{9}{2}(H_2 - H_3) = k_2 z_2 - k_3 z_3,$$

(6.8)
$$L_2 = -\tfrac{9}{2}(H_3 - H_1) = k_3 z_3 - k_1 z_1,$$

$$L_3 = -\tfrac{9}{2}(H_1 - H_2) = k_1 z_1 - k_2 z_2.$$

If we put each fraction in (6.7) equal to ds, we have the equations

(6.9)
$$\frac{dz_1}{ds} = k_2 z_2 - k_3 z_3,$$
$$\frac{dz_2}{ds} = -k_1 z_1 + k_3 z_3,$$
$$\frac{dz_3}{ds} = k_1 z_1 - k_2 z_2.$$

We have, by (6.5),

(6.10)
$$z_1 + z_2 + z_3 = 0,$$

and so the first two of (6.9) give

(6.11)
$$\frac{dz_1}{ds} = k_3 z_1 + (k_2 + k_3) z_2,$$
$$\frac{dz_2}{ds} = -(k_1 + k_3) z_1 - k_3 z_2.$$

The solutions are of the form exp (λs), where the eigenvalues λ satisfy

(6.12)
$$\begin{vmatrix} k_3 - \lambda & k_2 + k_3 \\ -k_1 - k_3 & -k_3 - \lambda \end{vmatrix} = 0,$$

or

(6.13)
$$\lambda^2 = -\Sigma k_2 k_3.$$

Three cases arise:

Case I: $\Sigma k_2 k_3 > 0$; eigenvalues pure imaginary;

Case II: $\Sigma k_2 k_3 < 0$; eigenvalues real, one positive and one negative;

Case III: $\Sigma k_2 k_3 = 0$; eigenvalues both zero.

7. *Case I*: $k_2 k_3 + k_3 k_1 + k_1 k_2 > 0$.

In Case I the curves (6.9) are closed curves, surrounding the point E. However, (6.9) is only a linear approximation to the curves C, and it does not follow immediately that the curves C are closed. But if a curve C is not closed, then, since it cannot intersect itself, it must cut a median P_1Q_1 in an infinite number of points. This we have shown earlier to be impossible. Hence *all curves C near E are in fact closed curves* (Figure 4). The sense in which such a curve is described depends on the initial orientation of the triangle (cf. (4.4), (4.5)).

If we expand the orbit (which, roughly speaking, means bringing two of the vortices closer together, since Q_1, Q_2, Q_3 correspond to collisions), we shall reach an orbit C_0 which touches the periphery $Q_1Q_2Q_3$ at a point corresponding to a fixed collinear configuration. This configuration will be approached as a limit, not attained in finite time.

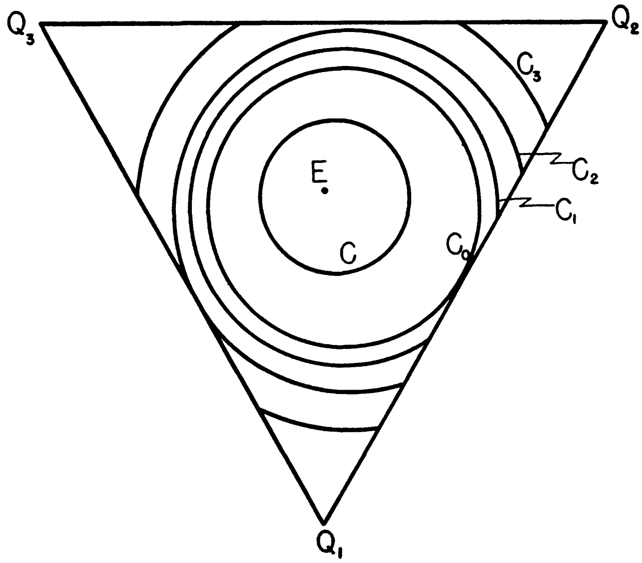

FIG. 4

Representative curves for Case I: $\Sigma k_2k_3 > 0$.

It is interesting to consider here the particular case, $k_1 = k_2 = k_3$, which of course belongs to Case I. Now the figure is symmetric, and C_0 will touch all three sides of $Q_1Q_2Q_3$. Thus the system, if started on such a curve, will oscillate in infinite time between two fixed collinear configurations, these two configurations being different. For three equal vortices, the only fixed collinear configurations are those in which the vortices are equally spaced (Figure 5). Such a configuration, if slightly disturbed, will pass in a long time near to one of the configurations shown in Figure 6. Equation (2.5) tells us the lengths in Figure 6 are the same as those in Figure 5. If the representative curve of the disturbed motion does not meet $Q_1Q_2Q_3$ (i.e. if it belongs to the class C of Figure 4), then all three configurations of Figures 5 and 6 will be approached one after another. By symmetry, the representative curve cannot belong to class C_1 or class C_2. If it is of class C_3, then the motion is an oscillation between a collinear configuration adjacent to that shown in Figure 5 and a collinear configuration adjacent to one of those shown in Figure 6. These oscillations between configurations which differ only through interchange of vortices of equal strength appear rather interesting.

In the general case of unequal strengths, contact will be established first with one side of $Q_1Q_2Q_3$, as for C_0 in Figure 4. When we expand the orbit further to C_1, we get an oscillation, performed in finite time, between two collinear configurations which are actually the same configuration. We may think of the return journey as performed on the back of the representative plane; it has reversed orientation.

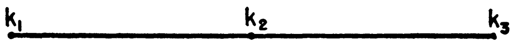

FIG. 5

Fixed collinear configuration.
$(k_1 = k_2 = k_3)$

Further expansion gives us C_2, which cuts one side of $Q_1Q_2Q_3$ and touches another. Here we have an oscillation between two different collinear configurations, one of which is a fixed configuration and is not attained in finite time.

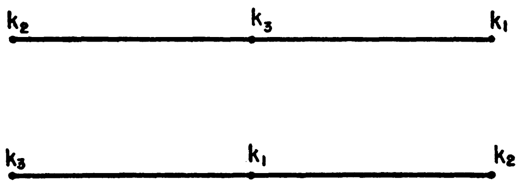

FIG. 6

Transforms of configuration of FIG. 5.

The final stage is C_3, representing an oscillation in finite time between two different collinear configurations.

This exhausts the possibilities in Case I. In this case the equilateral configuration is of course stable for small disturbances.

8. *Case II:* $k_2k_3 + k_3k_1 + k_1k_2 < 0$.

Here the eigenvalues are $\pm\mu$, where

(8.1) $$\mu = (-\Sigma k_2k_3)^{\frac{1}{2}} > 0.$$

The solutions of (6.11) are

(8.2) $$z_1 = A_1e^{\mu s} + B_1e^{-\mu s},$$
$$z_2 = A_2e^{\mu s} + B_2e^{-\mu s},$$

where

(8.3) $$A_1(\mu - k_3) - A_2(k_2 + k_3) = 0,$$
$$B_1(-\mu - k_3) - B_2(k_2 + k_3) = 0.$$

https://doi.org/10.4153/CJM-1949-022-2 Published online by Cambridge University Press

As $s \to \infty$, the curve recedes asymptotically in the direction

$$z_1/z_2 = A_1/A_2 = (k_2 + k_3)/(\mu - k_3), \tag{8.4}$$

and as $s \to -\infty$, we have a curve coming in asymptotically from the direction

$$z_1/z_2 = B_1/B_2 = -(k_2 + k_3)/(\mu + k_3). \tag{8.5}$$

These directions may be expressed symmetrically. They correspond to values of z_1, z_2, z_3 which make

$$\frac{dz_1}{ds} : \frac{dz_2}{ds} : \frac{dz_3}{ds} = z_1 : z_2 : z_3,$$

and so, by (6.7), they satisfy

$$\begin{aligned} \lambda z_1 - k_2 z_2 + k_3 z_3 &= 0, \\ k_1 z_1 + \lambda z_2 - k_3 z_3 &= 0, \\ -k_1 z_1 + k_2 z_2 + \lambda z_3 &= 0, \\ z_1 + z_2 + z_3 &= 0. \end{aligned} \tag{8.6}$$

If we multiply the first three of these equations in order by k_1, k_2, k_3, and add, and then solve with the last of (8.6), we get

$$\begin{aligned} z_1 : z_2 : z_3 = {} & \lambda(k_2 - k_3) + 3k_2k_3 - \Sigma k_2k_3 \\ & : \lambda(k_3 - k_1) + 3k_3k_1 - \Sigma k_2k_3 \\ & : \lambda(k_1 - k_2) + 3k_1k_2 - \Sigma k_2k_3 . \end{aligned} \tag{8.7}$$

We are to put $\lambda = \pm \mu$ to get the two directions. Figure 7 shows such directions (D_1, D_2, D_3, D_4) and the general nature of the curves near E.

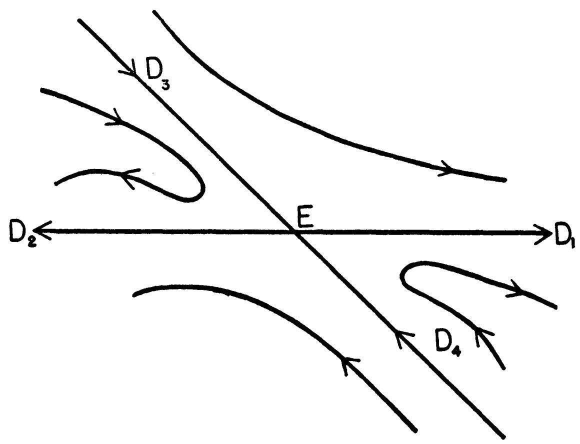

FIG. 7

Representative curves near E.
Case II: $\Sigma k_2k_3 < 0$.

The curves which start from E in the directions D_1, D_2, D_3, D_4 must pass out across the periphery $Q_1Q_2Q_3$ since they cannot cross nor can they cut a median of the triangle an infinite number of times. Similarly all representative curves must cross the periphery $Q_1Q_2Q_3$. The general nature of the pattern is shown in Figure 8.

The curves labelled D_1, D_2, D_3, D_4 represent motions in which the configuration oscillates between the equilateral configuration and a collinear configuration. The time of approach to E, or recession from it, is infinite. The other

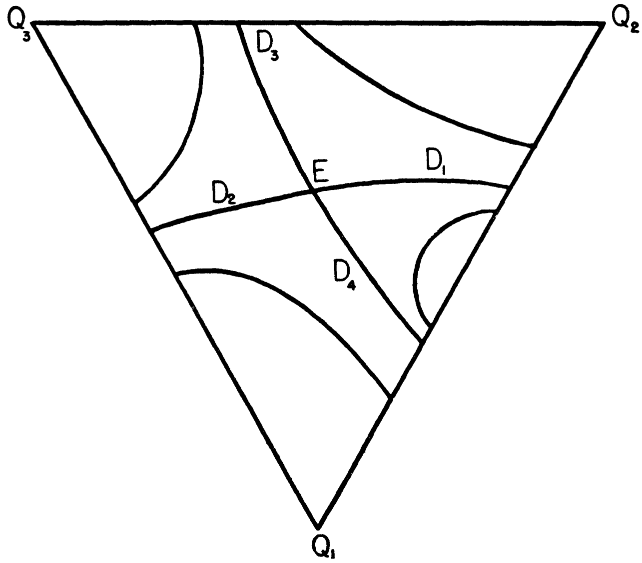

FIG. 8

Representative curves for Case II: $\Sigma k_2 k_3 < 0$.

curves represent oscillations between two collinear configurations, not necessarily distinct. The times involved are finite unless the collinear configuration involved is a fixed configuration. There are no periodic motions which do not include collinear configurations.

The equilateral configuration is unstable in this case for small disturbances.

9. *Case III*: $k_2k_3 + k_3k_1 + k_1k_2 = 0$.

We have already seen in Theorem 8 that in this case there is a hyperbola (4.11) composed of singular points ($H_1 = H_2 = H_3 = 0$). If the initial configuration is represented by a point on this hyperbola, then by (4.4) the representative point remains fixed. Thus the configuration remains fixed in *shape*. To see how it changes its size, we refer to (2.5), in which the right-hand sides are now constants. It is clear that the squares of the sides increase or decrease linearly with time, remaining fixed in length only if the representative point is at E.

If initially the representative point does not lie on the hyperbola (4.11), then both shape and size change. This hyperbola forms a barrier which the representative point cannot cross. Hence the motion consists of an oscillation between collinear configurations.

Institute for Advanced Studies,
Dublin, Eire