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The opportunity-threat theory of decision-making under risk

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Abstract

A new theory of decision-making under risk, the Opportunity-Threat Theory is proposed. Analysis of risk into opportunity and threat components allows description of behavior as a combination of opportunity seeking and threat aversion. Expected utility is a special case of this model. The final evaluation is an integration of the impacts of opportunity and threat with this expectation. The model can account for basic results as well as several "new paradoxes" that refuted cumulative prospect theory in favor of configural weight models. The discussion notes similarities and differences of this model to the configural weight TAX model, which can also account for the new paradoxes.

Keywords: decision, risk, opportunity, threat, expected utility, prospect theory, transfer of attention exchange, behavior

1 Introduction

Expected Utility Theory (EUT) (Von Neumann & Morgenstern, 1944) is the most widely accepted normative theory of decision-making under risk. However, as demonstrated by the Allais Paradox (Allais, 1953), EUT does not accurately describe how people decide when presented choices between risky prospects. Many theories have been proposed to account for the Allais paradoxes. Two classes of models that have been the focus of recent experimental work are original and cumulative prospect theory (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992) and configural weight models (Birnbaum, 1974; Birnbaum & Stegner, 1979), including the Transfer of Attention Exchange (TAX) model of Birnbaum & Chavez (1997). This paper proposes an alternative, called the Opportunity-Threat Theory (OTT) and shows how a simplified special model of OTT (SSOT) can explain the classic fourfold pattern of risk attitude as well as key features of the "new paradoxes" (Birnbaum, 2008) that refute the prospect models in favor of the configural weight models. In particular, the new model can account for event-splitting effect, violation of stochastic dominance and violation of restricted-branch independence.

The intuitions behind OTT are rather simple. People are influenced not only by the expected results of their actions, but also are affected by two components of risk, opportunity and threat. A model in which expectation, opportunity, and threat aggregate to form the evaluation of a risky prospect will be presented first in a simple form, to show that it can account for empirical phenomena. Comparisons with alternative approaches and ways in which the simple model might be generalized will be taken up in the discussion.

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2 A Special OTT (SOT) Model of Risky Decision Making

Let M refer to a risky gamble of the form $M = (x_1, p_1; x_2, p_2 \ldots x_i, p_i \ldots x_n, p_n)$, which represents a lottery in which there are exactly n possible mutually exclusive and exhaustive consequences. Consequence x_i occurs with probability p_i and the sum of the probabilities is 1. The expected value of such a gamble, EV, is given by Equation 1:

$$EV = \sum_{i=1}^{n} p_i x_i \tag{1}$$

Expected utility theory (EUT) allows that utilities (subjective values) of the consequences may be a nonlinear function of the monetary values of the consequences, $u_i = u(x_i)$. The expected utility of the gamble is μ:

$$\mu = \sum_{i=1}^{n} p_i u_i \tag{2.1}$$

Under SOT, $\mu = EU$ can be viewed as a *reference*. With at least two unequal consequences, there will be at least one that will be preferred to μ, and at least one over which μ will be preferred. The present model adds two risk components to expected utility; the first of these components is a risk factor due to asymmetry and the second is a risk factor due to variation.

It is useful to define the average utility of the consequences in a gamble, $\bar{u}$ computed as if each consequence is equally likely:

$$\bar{u} = \frac{\sum_{i=1}^{n} u_i}{n} \tag{2.2}$$

When there are two or more consequences, the difference between the average utility of the consequences and the expected utility of consequences is used to define the asymmetry component of the model, θ, as follows:

$$\theta = \frac{\bar{u} - \mu}{(n-1)/n} \tag{2.3}$$

A negative deviation is perceived as *threat* (the possibility of landing below the *reference*). A positive deviation is perceived as *opportunity* (the possibility of landing above the *reference*).

The variation component of the model, ψ, is defined as follows:

$$\psi = \{\sigma^2 + (\alpha\theta)^2\}^{1/2} \tag{2.4}$$

where α is a parameter representing the weight of the θ component and σ is the standard deviation,

$$\sigma = \left\{\sum_{i=1}^{n} p_i(u_i - \mu)^2\right\}^{1/2} \tag{2.5}$$

It has been found that people are typically risk averse for positive valued gambles and typically risk seeking for negative valued gambles; therefore, to account for this empirical finding, a multiplier of ψ, b, reflecting the sign of μ is used, as follows:

$$b = +1, if\, \mu < 0, else - 1 \tag{2.6}$$

When all the u values are non-negative ($\mu \geq 0$), spread is perceived as *threat* (the possibility of landing below the *reference*); $b = -1$. On the other hand, when all the u values are negative ($\mu < 0$), spread is perceived as *opportunity* (the possibility of landing above the *reference*): $b = +1$. For mixed cases with both positive and negative outcomes, sign of μ determines the sign of b.

The overall evaluation of a gamble, V, for gambles with two or more possible consequences is a linear combination of all three components, as follows:

$$V = \mu + \alpha\theta + \beta b\psi + \varepsilon \tag{2.7}$$

where coefficients α and β represent psychological weights assigned to θ and ψ respectively; and ε is an error term. For the case of $n = 1$, $V = u$. The SOT model is idempotent, reducing to $V = u$, when all outcomes are equal. When given a choice between two gambles, the decision maker chooses the option with the higher evaluation, V, apart from error.

Equation 2.7 shows that SOT reduces to EU when $\alpha = \beta = 0$, or when $\alpha\theta + \beta b\psi = 0$. When $\theta = 0$, $V = \mu + \beta b\sigma + \varepsilon$, which is a special case of the TAX model when $n = 2$.[1]

[1]The Appendix shows the derivation of θ and ψ.

2.1 Simplified special opportunity-threat (SSOT) model

For some situations, it may be possible to use a simplified version of SOT. Consider cases where outcomes are monetary and within a relatively narrow range allowing $u(x) = x$. Further, assume that values of p are non-extreme allowing the approximation, $\psi = \sigma$.[2] For simplicity, it will be assumed that there are no errors. Equation 3 represents the simplified SOT (SSOT) model

$$V = \mu + \alpha\theta + \beta b\sigma \tag{3}$$

For simplification, it is assumed that coefficients α and β do not change due to change in domains (positive to negative or vice versa). Examples in this paper use this SSOT model unless otherwise mentioned. It is noted that SSOT model reduces to Expected Value when $\alpha = \beta = 0$, or when $\alpha\theta + \beta b\sigma = 0$.

2.2 Constraints and coefficients

Consider the case of binary gambles $(x, p; 0, 1-p)$, with $x > 0$; here, $n = 2$ and $\mu = px$. From Equation (2.3), $\theta = \frac{\frac{x}{2} - px}{\frac{2-1}{2}} = (1-2p)x$ and from Equation (2.5), $\sigma^2 = p(x - xp)^2 + (1-p)(0 - xp)^2 = \{p(1-p)\}x^2$. Since μ is non-negative, $b = -1$. Therefore,

$$V = px + \alpha(1-2p)x - \beta\{p(1-p)\}^{1/2}x \tag{4.1}$$

transforming to:

$$\frac{V}{x} = p + \alpha(1-2p) - \beta\{p(1-p)\}^{1/2} \tag{4.2}$$

Now, constraints $0 < \frac{V}{x} < 1$ and $0 < p < 1$ are applied. They set the boundary conditions such that even for the smallest probability of smallest positive value of x, V does not reduce to zero. Further, even for the smallest probability of x not obtaining, magnitude of V remains positive under certain x. Then, at $p = \frac{1}{2}$, from Equation 4.2, $\frac{V}{x} = \frac{1}{2} - \frac{\beta}{2}$. Thus, $0 < \frac{1}{2} - \frac{\beta}{2} < 1$, which implies $-1 < \beta < 1$. Also, for $p \sim 0$, $\frac{V}{x} \approx \alpha$, thus $0 < \alpha < 1$.

For estimation of β, consider a mixed outcome experiment with only two possible outcomes with equal probabilities $(x_1, \frac{1}{2}; -x_2, \frac{1}{2})$ and observed $V = 0$. Given symmetric distribution, $\theta = 0$, giving, $V = 0 = \mu - \beta\sigma$, or, $\beta = \mu/\sigma$. Now, $\mu = \frac{1}{2}x_1 - \frac{1}{2}x_2$. Also, $\sigma^2 = \frac{1}{2}(x_1 - \mu)^2 + \frac{1}{2}(-x_2 - \mu)^2 = \frac{1}{2}(x_1 - \frac{1}{2}x_1 + \frac{1}{2}x_2)^2 + \frac{1}{2}(-x_2 - \frac{1}{2}x_1 + \frac{1}{2}x_2)^2$. That yields,

[2]For simple binary gambles $(x, p; 0, 1-p)$, with $x > 0$ and typical $\alpha = \frac{1}{8}$, $(\frac{\alpha\theta}{\sigma})^2 = \frac{(1-2p)^2}{64p(1-p)}$ has its minimum value at $p = \frac{1}{2}$, where it equals 0. The expression $(\frac{\alpha\theta}{\sigma})^2$ increases in value as p moves towards 0 or 1. Even at $p = 0.05 (or, 0.95)$, $(\frac{\alpha\theta}{\sigma})^2 = 0.26$, only.

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TABLE 1: Fourfold pattern shown with experimental dataset for gambles of type $(x, p; 0, 1-p)$ from Tversky & Kahneman (1992). SSOT, CPT and TAX, all three are able to explain the fourfold pattern. However, SSOT analyzes risk into *opportunity* and *threat* components.

Gamble	Observed	SSOT components			Calculated cash equivalents (V)		
x,p	cash equivalent	μ	$\alpha\theta$	$b\beta\sigma$	Prior SSOT	Prior CPT	Prior TAX
−\$100,0.95	−84	−95	11	7	−77	−83	−59
−\$100,0.05	−8	−5	−11	7	−9	−8	−8
\$100,0.95	78	95	−11	−7	77	77	59
\$100,0.05	14	5	11	−7	9	10	8

Note: Data presented only to demonstrate the pattern and not to show accuracy of prediction. OTT does not preclude the coefficients from taking different values for gains and losses domain. However, it is not necessary to explain the pattern here. Observed and calculated cash equivalents are in \$, in this table as well as the subsequent tables.

$\sigma = \frac{1}{2}x_1 + \frac{1}{2}x_2$. Thus, $\beta = \mu/\sigma = \frac{x_1 - x_2}{x_1 + x_2}$. Tversky & Kahneman (1992) reported four problems of this kind that yield $\beta = 0.42, 0.34, 0.34, 0.30$. An average is taken and converted to an equivalent fraction for convenience, giving $\beta = \frac{1}{3}$, which is in the range established above.

Now, with an estimate of β at hand, α can be estimated as follows. Assume that there exists a point where, $\frac{V}{x} = p$. Then, from Equation 4, $0 = \alpha(1-2p) - \beta\{p(1-p)\}^{1/2}$ yielding, $\alpha = \frac{\beta\{p(1-p)\}^{1/2}}{1-2p}$. Tversky & Kahneman (1992) considered $p \leq 0.1$ as low. Following that, $p = 0.1$ is taken as the point of transition from low probability to moderate probability. Data from the same study shows that $\frac{V}{x} \sim p$ at $= 0.1$. At that point, with $\beta = \frac{1}{3}$, $\alpha = \frac{1}{8}$ is obtained, which is in the range established above.

It must be recognized that α and β represent a psychological weighting process and as such are likely to vary with individual differences and experimental factors. The rest of this paper uses $\alpha = \frac{1}{8}$ and $\beta = \frac{1}{3}$ as "prior" parameters for purpose of calculations to illustrate how the SSOT model functions. These rough parameters are not intended to be used for comparison of accuracy or predictive power of various models.

3 Results

It has been well argued in Birnbaum (2008) as to how the so-called "new paradoxes" refute Cumulative Prospect Theory, Rank-Dependent Utility, and Rank-and Sign-Dependent Utility Theories in favor of a class of models that includes the transfer of attention exchange (TAX) model. Here, it is examined if SSOT is also capable of explaining key drivers of these new paradoxes, viz., event-splitting, stochastic dominance and restricted branch independence. It will be shown first that SSOT can reproduce some basic behavioral observations in decision-making under risk, including the "fourfold pattern".

3.1 Fourfold pattern

Tversky & Kahneman (1992) described a "fourfold pattern": risk aversion for gains and risk seeking for losses of high probability; risk seeking for gains and risk aversion for losses of low probability. Table 1 shows selected results illustrating this fourfold pattern and shows how SSOT can account for them.

Consider $(100, 5\%; 0, 95\%)$. (i) The *reference* $\mu = 100 * 5\% = 5$. (ii) $\theta = (1-2*5\%)*100 = 90$. It is multiplied with coefficient $\alpha = \frac{1}{8}$. Thus, the impact $\alpha\theta = \frac{1}{8} * 90 \approx 11$. This is positive and is taken as *opportunity*. (iii) The standard deviation, $\sigma = \{5\% * 95\%\}^{\frac{1}{2}} 100 = 22$. Since the gamble is in gains domain, $b = -1$. Thus, $b\sigma = -22$. With coefficient $\beta = \frac{1}{3}$ we have $\beta b\sigma = \frac{1}{3}(-22) \approx -7$. This is negative and is taken as *threat*. The final *value*, $V = 5 + 11 - 7 = 9 > 5$(expected value). This is in line with the reported relationship.

As p increases, θ decreases, crossing 0 when $p = \frac{1}{2}$, and turning negative after that. In the case of $(100, 95\%; 0, 5\%)$, $\mu = 95$. $\theta = (1-2*95\%)*100 = -90$, leading to negative impact of $\alpha\theta = \frac{1}{8} * (-90) \approx -11$. Standard deviation and b do not change so $V = 95 - 11 - 7 = 77 < 95$(expected value).

For $(-100, 5\%; 0, 95\%)$, $\mu = -5$. $\theta = (1-2*5\%)(-100) = -90$, leading to negative impact of $\alpha\theta = \frac{1}{8} * (-90) \approx -11$. Due to domain change, $b = +1$, hence, $\beta b\sigma = \frac{1}{3} * (+1) * 22 \approx 7$,. Therefore, $V = -5 - 11 + 7 = -9 < -5$(expected value).

TABLE 2: Event-splitting problems 1.1 (row 1) and 1.2 (row 2) from Birnbaum (2008). Prior CPT does not, but prior SSOT and prior TAX predict preference reversal due to event-splitting of gambles in 1.2.

Choice		% Choosing	Calculated cash equivalents					
			Prior SSOT		Prior TAX		Prior CPT	
First gamble	Second gamble	second gamble	First	Second	First	Second	First	Second
A: 85 to win \$100 10 to win \$50 05 to win \$50	B: 85 to win \$100 10 to win \$100 05 to win \$7	62	81.7	83.7	68.4	69.7	82.2	79.0
A': 85 to win \$100 15 to win \$50	B': 95 to win \$100 05 to win \$7	26	82.2	78.1	75.7	62.0	82.2	79.0

For $(-100, 95\%; 0, 5\%)$, we have, $\mu = -95$, $\theta = (1 - 2 * 95\%)(-100) = 90$, $\alpha\theta = \frac{1}{8} * 90 \approx 11$ and $b = +1$, hence, $\beta b\sigma = \frac{1}{3} * (+1) * 22 \approx 7$. Therefore, $V = -95 + 11 + 7 = -77 > -95$(expected value). Thus, the model reproduces the fourfold pattern of Table 1.

More generally, consider gambles of type $(x, p; 0, 1 - p)$. In the gains domain, spread is perceived as *threat*. θ, which equals $(1 - 2p)x$, is perceived as an *opportunity* for $p \leq \frac{1}{2}$. For any given x, this factor becomes stronger as p reduces. At low p, it overrides the *threat* factor when $\alpha(1 - 2p) > \beta\{p(1 - p)\}^{1/2}$. In the losses domain, spread is perceived as *opportunity*. θ, which equals $(1 - 2p)(-x)$, is perceived as *threat* for $p \leq \frac{1}{2}$. Thus, for low p, it is *threat* aversion and otherwise it is *opportunity* seeking. In this model, decision-makers can be simultaneously *opportunity* seeking and *threat* averse.

3.2 Event-splitting effect

A simple case of event splitting: $A(x, p; 0, 1 - p)$ split to $B(x, p - r; x, r; 0, 1 - p)$. Obviously, there is no difference in μ, since, $(p - r)x + rx = px$. There is no difference in σ either as $p(x - px)^2 + (1 - p)(0 - px)^2 = (p - r)(x - px)^2 + r(x - px)^2 + (1 - p)(0 - px)^2$.[3] However, there is change in θ. $\theta_A = \frac{\frac{x}{2} - px}{\frac{2-1}{2}} = (1 - 2p)x$ and $\theta_B = \frac{\frac{2x}{3} - px}{\frac{3-1}{3}} = (1 - 1.5p)x$.

Take any gamble $G_{base}(x_1, p_1; x_2, p_2 \ldots x_i, p_i \ldots x_n, p_n)$ with $x_i > 0$, pick its element k , (x_k, p_k) and split it to generate elements $(x_k, p_k - r)$ and (x_k, r) for another gamble G_{split}. Then, from Equation 2.3, $\theta_{base} = \frac{n}{n-1}(\sum \frac{x_i}{n} - \mu)$ and $\theta_{split} = \frac{n+1}{n}(\sum \frac{x_i}{n+1} + \frac{x_k}{n+1} - \mu)$. Thus, $\theta_{split} - \theta_{base} = \frac{x_k}{n} - \frac{\sum x_i - \mu}{n(n-1)}$. Therefore, $V_{split} > V_{base}$ if,$x_k > \frac{\sum x_i - \mu}{(n-1)}$.

For a binary gamble $(x, p; 0, 1 - p)$, $x > 0$, splitting of higher branch satisfies this condition ($x > \frac{x - px}{(2-1)}$) and leads to increase in value. Splitting of lower branch ($0 < \frac{x - px}{(2-1)}$) leads to decrease in value. Because splitting in this model can either increase or decrease the value of a gamble, this model violates the property of branch-splitting independence identified and tested by Birnbaum (2007).

An example of event-splitting non-independence from Birnbaum (2008) is illustrated in Table 2. 1.1A is produced by splitting lower branch of 1.1A' (causing small decrease in value) and 1.1B is produced by splitting higher branch of 1.1B'(causing significant increase in value). The majority prefers A to B, and a majority prefers B' to A'.

3.3 Violation of stochastic dominance

That SSOT predicts violation of stochastic dominance is demonstrated in this section through a recipe simplified from Birnbaum (2008). Take a base gamble $G_0(x, p; 0, 1 - p)$, with $x > 0$. Now, modify it to generate a stochastically dominating gamble $G_+(x, p; y, q; 0, 1 - p - q)$ where y is a small positive quantity and q is a relatively small probability. Next, generate a stochastically dominated gamble $G_-(x, p - s; z, s; 0, 1 - p)$ where z is a positive quantity slightly lower in value to x and s is a relatively small probability.

From Equations 2, for $G_0(x, p; 0, 1 - p)$:
$\mu_0 = px$, $\theta_0 = \frac{x/2 - px}{(2-1)/2} = x - 2px = (1 - 2p)x$, and
$\sigma_0 = p(x - px)^2 + (1 - p)(0 - px)^2 = p(1 - p)x^2$.

For $G_+(x, p; y, q; 0, 1 - p - q)$:
$\mu_+ = px + qy$, $\theta_+ = \frac{\frac{x+y}{3} - (px+qy)}{\frac{3-1}{3}} = \frac{x+y}{2} - (\frac{3}{2})(px + qy) = \frac{(1-3p)}{2}x + \frac{(1-3q)}{2}y$, and
$\sigma_+ = p(x - \mu_+)^2 + q(y - \mu_+)^2 + (1 - p - q)(0 - \mu_+)^2 = px^2 + qy^2 - \mu_+^2 = p(1 - p)x^2 + q(1 - q)y^2 - 2pqxy$.

[3] It is important to note that none of traditional measures of higher moments will change either, since for a moment of order m, $p(x - px)^m + (1 - p)(0 - px)^m = (p - r)(x - px)^m + r(x - px)^m + (1 - p)(0 - px)^m$. Thus, no moments-only model (for example, mean-variance or mean-variance-skewness models) will be able to explain change in value due to event-splitting.

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Table 3: Violation of stochastic dominance in Birnbaum (2008) problem 3.1.Values $\alpha(\theta_+ - \theta) \approx -\alpha\frac{(1-p)x}{2} = -0.6$ and $\alpha(\theta - \theta_-) \approx -\alpha\frac{px}{2} = -5.4$ predict slight reduction in value moving from G_0 to G_+ and relatively higher increase in value moving from G_0 to G_-. Prior CPT is not, but prior SSOT and prior TAX are consistent with the observed data.

Choice		% Choosing	Calculated cash equivalents					
			Prior SSOT		Prior TAX		Prior CPT	
First gamble	Second gamble	second gamble	First	Second	First	Second	First	Second
G_+: 90 to win \$96 05 to win \$14 05 to win \$12	G_-: 85 to win \$96 05 to win \$90 10 to win \$12	73	70.6	74.9	45.8	63.1	70.3	69.7
G_0 : 90 to win \$96 10 to win \$12	–	–	70.8	–	58.1	–	70.1	–

For $G_-(x, p-s; z, s; 0, 1-p)$, substituting $r = p - s$ (replaced back in the last step):
$\mu_- = rx + sz$, $\theta_- = \frac{\frac{x+z}{3} - (rx+sz)}{\frac{3-1}{3}} = \frac{x+z}{2} - (\frac{3}{2})(rx+sz) = \frac{(1-3r)}{2}x + \frac{(1-3s)}{2}z$ and $\sigma_- = r(x-\mu_-)^2 + s(z-\mu_-)^2 + (1-p)(0-\mu_-)^2 = rx^2 + sz^2 - \mu_-^2 = r(1-r)x^2 + s(1-s)z^2 - 2rsxz = p(1-p)x^2 + (2sp - s - s^2)x^2 + s(1-s)z^2 - 2(p-s)sxz$.

The impact on mean is straightforward: $\mu_+ - \mu = qy \approx 0$ and, $\mu - \mu_- = s(x-z) \approx 0$. The difference in σ is as follows: $\sigma_+^2 - \sigma^2 = q(1-q)y^2 - 2pqxy = qy((1-q)y - 2px) \approx 0$,since, $qy \approx 0$ and $\sigma^2 - \sigma_-^2 = -(2sp - s - s^2)x^2 - s(1-s)z^2 + 2(p-s)sxz = s(-2px^2 + x^2 + sx^2 - (1-s)z^2 + 2(p-s)xz) = s(x^2 - z^2 + sx^2 + sz^2 - 2sxz - 2px^2 + 2pxz) = s(x-z)(x + z + s(x-z) - 2px) \approx 0$, since, $s(x-z) \approx 0$.

The differences in θ, which are decisive, can be calculated: $\theta_+ - \theta = \frac{(1-3p)}{2}x + \frac{(1-3q)}{2}y - (1-2p)x = \frac{-1+p}{2}x + \frac{(1-3q)}{2}y \approx -\frac{(1-p)x}{2}$ and $\theta - \theta_- = (1-2p)x - \{\frac{(1-3r)}{2}x + \frac{(1-3s)}{2}z\} = \frac{(-p)}{2}x + \frac{(1-3s)}{2}(x-z) \approx -\frac{px}{2}$. Negative signs in both these expressions imply violation of stochastic dominance. In summary, similar to event-splitting effect, violation of stochastic dominance in cases such as the one described here is driven by θ. G_0 to G_+ is splitting of the lower branch leading to reduction in value; while G_0 to G_- is splitting of the higher branch leading to increase in value.

For illustration, calculations for one experimental set from Birnbaum (2008) are shown in Table 3. It is reported that a majority (73%) of subjects preferred G_- over G_+ despite expected values being 87.3 and 87.7 respectively. SSOT values these prospects at 74.9 and 70.6, consistent with the majority preference, driven by change in θ while deltas in μ and σ are very small.

3.4 Violation of restricted branch independence

Consider two gambles with the same number of branches and the same probability distribution, $S = (x_1, p_1; x_2, p_2 \ldots x_i, p_i \ldots x_n, p_n)$ and $R = (y_1, p_1; y_2, p_2 \ldots y_i, p_i \ldots y_n, p_n)$ having a common branch such that $x_n = y_n = z$. Restricted branch independence assumes that a change in z will not change the preference relationship between S and R. Suppose S is preferred over R, then, under SSOT, $V_S > V_R$. Then, if $\frac{\partial V_s}{\partial z} \geq \frac{\partial V_R}{\partial z}$, $V_S > V_R$ for all z. Otherwise, with increase in z, the gap in values will close and preference may get switched. A standard case is analyzed to understand how this derivative function works. For convenient tracking, label $p_n = r$. Assume, $x_i \geq 0$, for all i, such that $\mu \geq 0$ and $b = -1$. Also, introduce an additional constraint $0 < \beta < 1$ to model a typical spread-averse agent.

Differentiating Equation 3 w.r.t. z, $\frac{\partial V}{\partial z} = \frac{\partial \mu}{\partial z} + \alpha\frac{\partial \theta}{\partial z} - \beta\frac{\partial \sigma}{\partial z}$. Now, $\frac{\partial \mu}{\partial z} = r$, $\frac{\partial \theta}{\partial z} = \frac{n}{n-1}(\frac{1}{n} - r)$ and $\frac{\partial \sigma}{\partial z} = \frac{1}{2\sigma}\frac{\partial \sigma^2}{\partial z} = \frac{r(z-\mu)}{\sigma}$. Therefore, $\frac{\partial V}{\partial z} = r + \alpha\frac{n}{n-1}(\frac{1}{n} - r) - \beta\frac{r(z-\mu)}{\sigma}$. Thus, if, $\frac{\partial V_s}{\partial z} \geq \frac{\partial V_R}{\partial z}$, then $-\beta\frac{r(z-\mu_S)}{\sigma_S} \geq -\beta\frac{r(z-\mu_R)}{\sigma_R}$. That is, $\frac{(z-\mu_S)}{\sigma_S} \leq \frac{(z-\mu_R)}{\sigma_R}$ or $z \leq \frac{\mu_S\sigma_R - \mu_R\sigma_S}{\sigma_R - \sigma_S}$. This is not guaranteed and will depend on all the constituents of S and R. Assume that for z' this condition is not satisfied. Then, the corresponding second gamble (R') will be preferred over the corresponding first gamble (S'), that is, a violation of type SR' will be observed. Note that this condition ($z \leq \frac{\mu_S\sigma_R - \mu_R\sigma_S}{\sigma_R - \sigma_S}$) is not dependent on θ, α or β . Thus, the same pattern applies to the gambles with uniform probabilities as well and also for any value of the coefficients and the only type of violation predicted by SSOT is SR'. In summary, violation of restricted branch independence is driven by the change in σ.

Table 4: Violation of restricted branch independence in Birnbaum (2008) problems 13.1 (row 1) and 13.2 (row 2). While, R is preferred over S in 13.1, the preference switches in 13.2. Prior CPT is not, but prior SSOT and prior TAX are consistent with the observed data.

Choice		% Choosing	Calculated cash equivalents					
			Prior SSOT		Prior TAX		Prior CPT	
First gamble	Second gamble	second gamble	First	Second	First	Second	First	Second
S: 25 to win \$44 25 to win \$40 50 to win \$5	R: 25 to win \$98 25 to win \$10 50 to win \$5	40	18.5	17.8	20.0	19.2	19.8	29.8
S': 50 to win \$111 25 to win \$44 25 to win \$40	R': 50 to win \$111 25 to win \$98 25 to win \$10	62	62.8	66.7	57.2	60.7	69.4	62.3

As an example from Birnbaum (2008, Table 4), for Problem 13.1, R is preferred over S and $(z = 5) \leq (\frac{\mu_S\sigma_R - \mu_R\sigma_S}{\sigma_R - \sigma_S} = 18)$. Thus, this preference holds as long as $z \leq \frac{\mu_S\sigma_R - \mu_R\sigma_S}{\sigma_R - \sigma_S}$ holds. However, for Problem 13.2, $(z = 111) > (\frac{\mu_S\sigma_R - \mu_R\sigma_S}{\sigma_R - \sigma_S} = 49)$, and the preference reversal is predicted.

4 Discussion

As the previous section shows, SSOT is capable of explaining several empirical phenomena explained by CPT and TAX both (fourfold pattern) or by TAX only (event-splitting effect, violations of stochastic dominance and violation of restricted branch independence).

Event-splitting effect and violations of stochastic dominance in the examples described above are related and are explained by the θ component under SSOT.

Violations of restricted branch independence are explained by the σ component.

4.1 The Opportunity-Threat Theory

Let S be a set of future states ($s \in S$) of the world of which exactly one state will obtain. Assume that it is unknown to the decision-maker as to which state will obtain. Further, assume that S can be mapped to a consequence set X through some function $\phi(s) = (x, p)$ where the first term (x) denotes the objective magnitude of the consequence and the second term (p) denotes the objective probability of that consequence. These objective probabilities sum to 1. In addition, assume that the outcome set X can be mapped by the decision-maker to a mental set M through some functions $u(x) = u$ and $\pi(p) = \pi$ corresponding to subjective or implied utilities and subjective or implied probabilities, respectively. These subjective or implied probabilities also sum to 1. Also, assume that the decision-maker is able to map this mental set M to a decision-making single number V (called *value*) such that $V(M) = V$.

Two key assumptions underlie this (M to V) mapping process:

- First, the decision-maker has a referencing algorithm that generates *reference* with which M can be divided into two mutually exclusive and collectively exhaustive subsets M_{up} (the *upside* set, containing elements considered more or equally preferred compared to *reference*) and M_{dn} (the *downside* set, containing elements considered less preferred compared to *reference*).
- Second, that there is available a netting algorithm that maps M_{up} and M_{dn} into Y_{up}, the *relative upside* set, containing elements of M_{up} net of *reference* and into Y_{dn}, the *relative downside* set, containing elements of M_{dn} net of *reference*), respectively.

With these assumptions, the following definitions are stated:

1. $Y_{agg} = agg(Y_{up}, Y_{dn})$, where agg is a function that measures direction and impact of aggregation of Y_{up} and Y_{dn}.
2. $Y_{dis} = dis(Y_{up}, Y_{dn})$, where dis is a function that measures direction and impact of distance between Y_{up} and Y_{dn}.
3. $opportunity = Y_{agg} > 0, Y_{dis} > 0$.
4. $threat = Y_{agg} < 0, Y_{dis} < 0$.

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Then, the Opportunity-Threat Theory (OTT) asserts that

$$V = f(reference, opportunity, threat) \quad (5)$$

and that the decision-maker will prefer or be indifferent to M_1 compared to M_2 iff $V(M_1) \geq V(M_2)$, except by way of error.

To illustrate, assume a roll of dice that pays \$1, if rolled 1, \$2, if rolled 2 etc. Thus, M = (1,1/6; 2,1/6; 3,1/6; 4,1/6; 5,1/6; 6,1/6). Further, assume an obvious reference point that is the average (3.5). Outcomes higher or equal to average are mapped to M_{up} = (4,1/6; 5,1/6; 6,1/6) and rest are mapped to M_{dn} = (1,1/6; 2,1/6; 3,1/6). Now, assume a simple netting algorithm that is to subtract the average from each of the outcomes. Thus, the relative sets are constructed as Y_{up} = (0.5,1/6; 1.5,1/6, 2.5,1/6) and Y_{dn} = (−2.5,1/6, −1.5,1/6, −0.5,1/6). To complete the illustration, assume $agg(.) = EV(Y_{dn}) + EV(Y_{up})$ and $dis(.) = EV(Y_{dn}) - EV(Y_{up})$ (so that it accounts for spread aversion). Then, these numbers are respectively, $Y_{agg} = 0$ (note that it was a symmetric gamble) and $Y_{dis} = -1.5$. Thus, opportunity = 0 and threat = −1.5. Assuming V(.) = reference + opportunity + threat, $V = 3.5 + 0 - 1.5 = 2$. Thus, this gamble will be valued at \$2.00 instead of expected value of \$3.50.

In general, OTT applies to cases involving monetary outcomes or otherwise. It allows incorporation of different measures of *opportunity* and *threat*. It also permits different attitudes to opportunity and threat. It does not require symmetry in attitudes in gains and losses domains. In moral situations, *opportunity* may lie in domain of morally correct and *threat* may lie in domain of morally incorrect. In such cases attitudes may be modelled with binary parameters.

Several parameters that can serve as *reference* have been discussed in the literature, for example, status quo (Thaler, 1980; Samuelson & Zeckhauser, 1988), omission (Baron & Ritov, 1994) or aspiration (van de Ven & Diecidue, 2008). OTT allows incorporation of different points of reference and even multiple points of reference. In a special case, the point of reference happens to be Expected Utility. In that case, if either the agent is neutral to both *opportunity* and *threat* or net impact of *opportunity* and *threat* cancel out, the model reduces to Expected Utility.

4.2 A structural comparison of SSOT, CPT and TAX

To structurally compare SSOT with CPT and TAX, a working model of OTT was developed. Consider, $X = (x_1, p_1; x_2, p_2 \ldots x_n, p_n)$, with n denoting the number of exhaustive and mutually exclusive future states and $\sum p_i = 1$. Assume $X_{up} = (x_1, p_1; x_2, p_2 \ldots x_i, p_i \ldots x_k, p_k)$ and $X_{dn} = (x_{k+1}, p_{k+1}; x_{k+2}, p_{k+2} \ldots x_j, p_j \ldots x_n, p_n)$, such that the upside set has k elements and the downside set has $n-k$ elements. Define, $V(X_{up}) = \sum(x_i - \mu)$, $i = 1 \ldots k$ and $V(X_{dn}) = \sum(x_j - \mu), j = k+1 \ldots n$. Further, define $V(O) = V(X_{up}) + V(X_{dn})$ analogous to θ and $V(T) = |V(X_{up}) - V(X_{dn})|$ analogous to σ (using range as a measure of spread instead of standard deviation). Then, analogous to Equation 2,

$$V = \mu + \alpha' V(O) + \beta' b V(T) \quad (6.1)$$

where $\mu = \sum_{i=1}^{n} p_i x_i$ and α' and β' are coefficients for respective terms.

Now, assuming non-negative domain ($b = -1$) and for simplicity $V(X_O) - V(X_T) > 0$, expansion of all the terms in Equation 5 yields,

$$V = \sum_{i=1}^{k} w_i x_i + \sum_{j=k+1}^{n} w_j x_j \quad (6.2)$$

where $w_i = (\alpha' - \beta') + \{1 - (\alpha' - \beta')k - (\alpha' + \beta')(n-k)\}p_i$ and $w_j = (\alpha' + \beta') + \{1 - (\alpha' - \beta')k - (\alpha' + \beta')(n-k)\}p_j$.

Equation 6.2 reveals that there is an implied rank-order. However, there are only two ranks — greater than expected value and lower than expected value. Also, the weights carry a component that is not a product of p. Thus, the SSOT model is significantly different from CPT. Now, Equation 6.2 is also analogous to a two-branch gamble. Clearly, there is transfer of weight from the upper branch to the lower branch. SSOT thus has similarities with TAX. However, in this formulation, all gambles reduce to a two-branch gamble. Therefore, potentially, there could be differences from TAX that will need to be explored.

SSOT, in this paper is shown to explain certain key phenomena underlying the new paradoxes. TAX has been very extensively studied and explains a wide range of empirical findings. SSOT may or may not be able to explain all those findings. Also, TAX and SSOT treat probabilities in a fundamentally different manner (non-linear vs. linear, respectively). I thus speculate that TAX cannot be a special case of SSOT, although under limiting conditions (for example, binary gambles with equal probabilities) SSOT and special TAX appear identical.

4.3 Areas of future research and conclusion

Further, areas of future research should include exploration of nature and stability of the coefficients α and β used in SOT. In SSOT, a single value of α and a single value of β was assumed for convenience. One question is whether these coefficients should have different values in different domains (positive, negative and mixed). Moreover, SOT should be extended to a stochastic model and to probability distributions that may not be discrete. The nature and impact of error should also be explored. Another important area will be to understand how this model can be applied in ambiguous (Ellsberg, 1961; Camerer & Weber, 1992) situations. Implications forinvestment decisions (Markowitz, 1952), in particular, also remain to be explored.

In conclusion, a new theory of decision-making under risk is proposed. This Opportunity-Threat theory relies on analyzing risk into its components. A simplified special model (SSOT) of this theory is able to explain a range of empirical phenomena that are explained by TAX but not by CPT.

References

Allais, M. (1953). L'extension des théories de l'équilibre économique général et du rendement social au cas du risque. Econometrica, *Journal of the Econometric Society*, 269-290.

Baron, J., & Ritov, I. (1994). Reference points and omission bias. *Organizational Behavior and Human Decision Processes*, *59*, 475-498.

Birnbaum, M. H. (1974). The nonadditivity of personality impressions. *Journal of Experimental Psychology Monograph*, *102*, 543-561.

Birnbaum, M. H. (2007). Tests of branch splitting and branch-splitting independence in Allais paradoxes with positive and mixed consequences. *Organizational Behavior and Human Decision Processes*, *102*, 154-173.

Birnbaum, M. H. (2008). New paradoxes of risky decision making. *Psychological Review*, *115*(2), 463-501.

Birnbaum, M. H., & Chavez, A. (1997). Tests of theories of decision making: Violations of branch independence and distribution independence. *Organizational Behavior and Human Decision Processes*, *71*(2), 161-194.

Birnbaum, M. H., & Stegner, S. E. (1979). Source credibility in social judgment: Bias, expertise, and the judge's point of view. *Journal of Personality and Social Psychology*, *37*(1), 48-74.

Camerer, C., & Weber, M. (1992). Recent developments in modeling preferences: Uncertainty and ambiguity. *Journal of Risk and Uncertainty*, *5*, 325-370.

Ellsberg, D. (1961). Risk, ambiguity, and the Savage axioms. *The Quarterly Journal of Economics*, 643-669.

Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, *47*, 263-291.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, *7*(1), 77-91.

Samuelson, W., & Zeckhauser, R. (1988). Status quo bias in decision making. *Journal of Risk and Uncertainty*, *1*(1), 7-59.

Thaler, R. (1980). Toward a positive theory of consumer choice. *Journal of Economic Behavior & Organization*, *1*(1), 39-60.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, *5*(4), 297-323.

van de Ven, J., & Diecidue, E. (2008). Aspiration level, probability of success and failure, and expected utility. *International Economic Review*, *49*, 683-700.

Von Neumann, J., & Morgenstern, O. (1944). *Game theory and economic behavior*. Princeton, Princeton University.

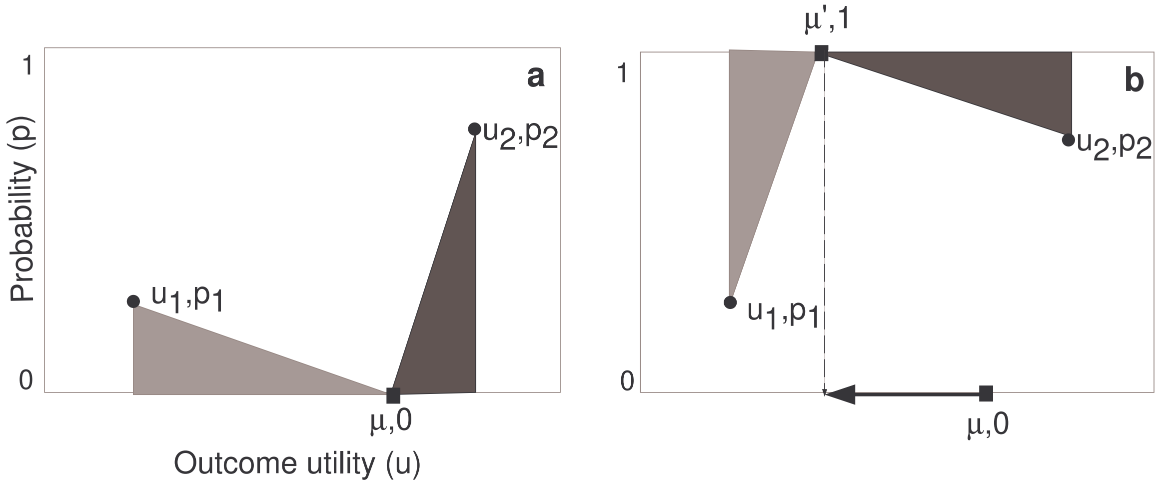

FIGURE 1: Consider a simple binary gamble with utility, probability pairs $(u_1, p_1; u_2, p_2)$, where $u_2 > u_1 > 0$ and $p_1 + p_2 = 1$. Panel **a** shows the expected value μ that is analogous to the center of gravity balancing the weights of the shaded areas. Imagine that the decision-maker wants to find a value equivalent μ' that is certain ($p = 1$). This then becomes the decision-enabling center of gravity (μ', 1) shown in panel **b**.

Appendix: Derivation of θ and ψ parameters

Special Opportunity-Threat (SOT) model introduces parameters θ and ψ, which are derived as shown in Figure 1:

Following Figure 1, characterize the linear shift (on x-axis) as an aggregation parameter, net of upside and downside, $\theta = \mu' - \mu$. Now, if μ'plays the role of center of gravity, $\sum_i^n (1 - p_i)(u_i - \mu') = 0$. Simple expansion leads to $\sum_i^n (u_i - \mu') - \sum_i^n p_i(u_i - \mu') = n\bar{u} - n\mu' - \mu + \mu' = 0$, where $\bar{u} = \frac{\sum_{i=1}^n u_i}{n}$. That gives, $\mu' = \frac{n\bar{u}-\mu}{n-1}$. Thus, $\theta = \mu' - \mu = \frac{n\bar{u}-\mu}{n-1} - \mu = \frac{n(\bar{u}-\mu)}{(n-1)}$. Alternatively, θ can be viewed as sum of $(u_i - \mu)$ residuals divided by the degree of freedom $(n - 1)$.

Now, only a fraction of this factor is actually incorporated in the decision-making process depending on psychological weighting. Thus, the center of gravity ends up at $\mu'' = \mu + \alpha\theta$ (μ'' is not shown in the figure). Finally, distance parameter is calculated as standard deviation of outcome values around this new point. $\psi^2 = \sum_{i=1}^n p_i(u_i - \mu'')^2$. Now, $\sum_{i=1}^n p_i(u_i - \mu'')^2 = \sum_{i=1}^n p_i(u_i - \mu - \alpha.\theta)^2 = \sum_{i=1}^n p_i(u_i - \mu)^2 + \sum_{i=1}^n p_i(\alpha.\theta)^2$. Thus, $\psi = \{\sigma^2 + (\alpha\theta)^2\}^{1/2}$.