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INFLECTIONAL CONVEX SPACE CURVES

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Let Φ be a regular closed C^2 curve on a sphere S in Euclidean three-space. Let $H(S)[H(\Phi)]$ denote the convex hull of $S[\Phi]$. For any point $p \in H(S)$, let $O(p)$ be the set of points of Φ whose osculating plane at each of these points passes through p.

1. THEOREM ([**8**]). *If* Φ *has no multiple points and* $p \in H(\Phi)$, *then* $|O(p)| \geqq 3[4]$ *when* p *is* [*is not*] *a vertex of* Φ.

2. THEOREM ([**9**]). a) *If the only self intersection point of* Φ *is a double point and* $p \in H(\Phi)$ *is not a vertex of* Φ, *then* $|O(p)| \geqq 2$.

b) *Let* Φ *possess exactly* n *vertices. Then*

(1) $|O(p)| \leqq n$ *for* $p \in H(S)$ *and*

(2) *if the osculating plane at each vertex of* Φ *meets* Φ *at exactly one point,* $|O(p)| = n$ *if and only if* $p \in H(\Phi)$ *is not vertex.*

It should be noted that Segre's proof of 1 required that Φ be C^3 and Weiner presented a simpler proof of this theorem in [**9**] with the assumption that Φ is C^2. Both proofs used the methods of classical differential geometry.

In 1979, P. Scherk conjectured that Φ need not be spherical in 1 and 2 as long as Φ was contained in the boundary of its convex hull. With the restrictions that Φ meets any plane in a finite number of points and any line in at most two points, we obtain such a generalization of 1 and 2 a) in Theorem 21 and of 2 b) in Theorem 26.

We remark that 1 and 2 imply that if Φ has no multiple points then Φ possesses at least four vertices. Similarly, 21 and 26 yield the Four-vertex Theorem 27.

Finally, the central idea of the proofs of 21 and 26 is the projection of a space curve onto a particular plane curve. This technique of proving a Four-vertex theorem for non-spherical space curves is also to be found in [**1**] and [**6**]. However Mohrmann considered curves lying on an ovaloid (closed convex surface met by any line in at most two points) and Barner examined curves which are "streng-konvex" (through every pair of distinct points of the curve there is a plane not meeting the curve elsewhere).

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1. Spherical curves. As a justification of our assumptions, we examine the purely geometric properties of a spherical curve Φ.

Firstly, Φ does lie on the boundary of $H(\Phi)$ and any line meets Φ in at most two points. A space curve with these properties, we call *convex*.

Let Φ be parametrized by a circle T; that is, $\Phi : T \rightarrow S$ is a regular C^2 function and Φ is identified with $\Phi(T)$. For $p = \Phi(t)$, let $\Phi_1(t)$ denote the line through p in the direction of the tangent vector $\Phi'(t)$ and let $\Phi_2(t)$ denote the osculating plane of Φ at p.

Let α be a plane through p. If $\alpha \cap \Phi_1(t) = \{p\}$ then $\Phi'(t) \neq 0$ and the continuity of Φ imply that α *cuts* Φ *at* p; that is, Φ does not lie on one side of α near p. If $\alpha \cap \Phi_2(t) = \Phi_1(t)$ then the order of contact of α and Φ at p is strictly less than the order of contact of $\Phi_2(t)$ and Φ at p. Thus Φ is closer to $\Phi_2(t)$ than α near p and it follows that α *supports* Φ *at* p; that is, Φ lies on one side of α near p. Finally, if $\alpha = \Phi_2(t)$ and Φ is not contained in α near p then by definition, α supports Φ at p if and only if p is a vertex of Φ.

We call a point of a curve, with the plane intersection property of a vertex of Φ, an *inflection point* and a curve, with the plane intersection property of Φ, an *inflectional curve*. Thus we extend 1 and 2 to inflectional convex space curves. We use the methods of order or direct differential geometry (cf. [7]) and for conformity with that theory, we argue in a real projective three-space P^3.

2. Directly differentiable curves. Let $p, q, \ldots, L, M, \ldots$, and $\alpha, \beta, \ldots$ denote the points, lines and planes of P^3 respectively. Let $\langle p, L, \alpha, \ldots \rangle$ denote the flat of P^3 spanned by $p, L, \alpha, \ldots$. We assume that P^3 is topologized in the standard way.

Let $T \subset P^3$ be an oriented line. For $t_0 \neq t_1$ in T, let $[t_0, t_1]$ and (t_0, t_1) denote respectively the closed and open oriented segments of T with initial point t_0 and terminal point t_1. We put

$$[t_0, t_1) = [t_0, t_1] \backslash \{t_1\} \quad \text{and} \quad (t_0, t_1] = [t_0, t_1] \backslash \{t_0\}.$$

Let $U(t) = (t_0, t_1)$ be a neighbourhood of t in T. We set

$$U^-(t) = (t_0, t),\ U^+(t) = (t, t_1) \text{ and } U'(t) = U^-(t) \cup U^+(t).$$

A *curve* Γ is a continuous map from T into P^3. A line, denoted by $\Gamma_1(t)$, is the *tangent* of Γ at t if

$$\Gamma_1(t) = \lim_{t' \neq t \rightarrow t} \langle \Gamma(t), \Gamma(t') \rangle$$

and a plane, denoted by $\Gamma_2(t)$, is the *osculating plane* of Γ at t if

$$\Gamma_2(t) = \lim_{t' \neq t \rightarrow t} \langle \Gamma_1(t), \Gamma(t') \rangle.$$

For $t \in T$, we also use the notation $\Gamma_0(t) = \Gamma(t)$ and $\Gamma_3(t) = P^3$. If $\mathcal{M} \subseteq T$ is a segment, we call $\Gamma/_{\mathcal{M}}$ a *subarc* of Γ. For convenience, we identify $\Gamma(T)$ with Γ and $\Gamma(\mathcal{M})$ with $\Gamma/_{\mathcal{M}}$.

We note that contrary to the terminology in [**9**]; a point $\Gamma(t)$ is *simple* if $\Gamma(t) \neq \Gamma(s)$ for $s \in T\backslash\{t\}$ and a curve Γ is *simple* if it has no multiple (self-intersection) points.

A (*directly differentiable*) *space curve* is a curve Γ with the property that $\Gamma_1(t)$ and $\Gamma_2(t)$ exist for each $t \in T$ and any plane meets Γ at a finite number of points. A (*directly differentiable*) *plane curve* is a curve Γ with the property that $\Gamma(T)$ is contained in a plane, $\Gamma_1(t)$ exists for each $t \in T$ and any line meets Γ at a finite number of points.

3. Space curves. Let $\Gamma(\mathcal{M})$ be a subarc of a space curve $\Gamma:T \to P^3$. If

$$k = \sup_{\alpha \subset P^3} |\alpha \cap \Gamma(\mathcal{M})|,$$

we say that the *order* of $\Gamma(\mathcal{M})$ is k and write $k = \operatorname{ord} \Gamma(\mathcal{M})$. Then the *order of a point* $\Gamma(t)$, ord $\Gamma(t)$, is the minimum order that a neighbourhood of $\Gamma(t)$ may possess. Clearly ord $\Gamma(t) \geqq 3$. We say that $\Gamma(t)$ is *ordinary* if ord $\Gamma(t) = 3$, otherwise, $\Gamma(t)$ is *singular*. We say that $\Gamma(t)$ is *elementary* if there exist $\Gamma(U^-(t))$ and $\Gamma(U^+(t))$, both of order three. Finally, $\Gamma(\mathcal{M})$ is *ordinary* [*elementary*] if each of its points is ordinary [elementary].

As a plane α meets Γ at a finite number of points, we note as in Section 1 that α supports or cuts Γ at t. For $t \in T$, let

$$S_i(t) = \{\alpha \subset P^3 | \alpha \cap \Gamma_{i+1}(t) = \Gamma_i(t)\},\ i = 0, 1, 2.$$

It is known that either all $\alpha \in S_i(t)$ support Γ at t or all $\alpha \in S_i(t)$ cut Γ at t. We thus assign a *characteristic* $(a_0(t), a_1(t), a_2(t))$ to $\Gamma(t)$ [denoted by $\Gamma(t) \equiv (a_0(t), a_1(t), a_2(t))$] by taking $a_i(t) = 1$ or 2 and requiring that $a_0(t) + \ldots + a_i(t)$ be even if and only if $\alpha \in S_i(t)$ supports Γ at t; $i = 0$, 1, 2. If $\Gamma(t) \equiv (1, 1, 1)$ [$\Gamma(t) \equiv (1, 1, 2)$], we say that $\Gamma(t)$ is a *regular* [*inflection*] point. Then a subarc is *regular* [*inflectional*] if each of its points is a regular [regular or inflection] point.

Finally, Γ is an *even* [*odd*] curve if any plane of P^3 cuts Γ at an even [odd] number of points. Since Γ is closed in P^3, Γ is trivially odd or even.

Let $\Gamma(\mathcal{M})$ be an open subarc of order three. It is well known that $\Gamma(\mathcal{M})$ is regular, $\Gamma_2(t) \cap \Gamma(\mathcal{M}) = \{\Gamma(t)\}$ for $t \in \mathcal{M}$ and both $\Gamma_1(t)$ and $\Gamma_2(t)$ depend continuously on $t \in \mathcal{M}$. Hence an elementary space curve possesses continuous tangents and osculating planes. We also note the following properties of an elementary space curve Γ; cf. [**7**]:

3. A regular point is ordinary.

4. For any $p \in P^3$ and $\Gamma(t) \in \Gamma$, there is an $U'(t)$ such that $p \notin \Gamma_2(s)$ for $s \in U'(t)$.

5. If Γ is inflectional, then Γ possesses an even [odd] number of inflections if Γ is even [odd].

4. Plane curves. Let $\Gamma: T \to \beta$ be a plane curve. By replacing "planes in P^3" with "lines in β" in Section 3; we define the *order of a subarc (point), ordinary and elementary points (subarcs) and even (odd) curves*. (A line $L \subset \beta$ supports [cuts] Γ at t if Γ lies [does not lie] on one side of L near $\Gamma(t)$). Thus a point $\Gamma(t)$ is ordinary if ord $\Gamma(t) = 2$ and elementary if there exist $\Gamma(U^-(t))$ and $\Gamma(U^+(t))$, both of order two.)

Let

$$S(t) = \{L \subset \beta | L \cap \Gamma_1(t) = \{\Gamma(t)\}\}.$$

Again either all $L \in S(t)$ support Γ at t or all $L \in S(t)$ cut Γ at t and

$$\Gamma(t) \equiv (a_0(t), a_1(t))$$

where $a_i(t) = 1$ or 2 and $a_0(t)$ $[a_0(t) + a_1(t)]$ is even if and only if $L \in S(t)$ $[\Gamma_1(t)]$ supports Γ at t. Thus Γ possesses four types of points: (1, 1), *regular*; (1, 2), *inflection*; (2, 1) *cusp*; and (2, 2), *beak*. We define regular and inflectional subarcs as in Section 3 and again note that an ordinary point is regular.

The *index*, ind $\Gamma(\mathcal{M})$, of a subarc $\Gamma(\mathcal{M})$ is the minimum number of points of $\Gamma(\mathcal{M})$ which can lie on any line of β. Thus ind $\Gamma > 0$ if Γ is odd. A point p is *strong* if there exist $s \neq t$ in T such that $p = \Gamma(s) = \Gamma(t)$ and ind $\Gamma[s, t] = 0$; in addition, p is *doubly strong* if $p = \Gamma(s) = \Gamma(t)$, ind $\Gamma[s, t] = 0$ and $p \in \Gamma(t, s)$ imply that ind $\Gamma[t, s] > 0$.

Let $n_1(\Gamma)$, $n_2(\Gamma)$, $n_3(\Gamma)$ and $s(\Gamma)$ be the number of inflections, cusps, beaks and strong points of Γ respectively. We note the following properties of a plane curve Γ.

6. If $\Gamma(s, t)$ is regular and simple then ind $\Gamma[s, t] = 0$. ([**2**], 3.14)

7. Let $\Gamma(s, t)$ be regular and simple. If $\Gamma(s) \neq \Gamma(t)$ and

$$\langle \Gamma(s), \Gamma(t) \rangle \cap \Gamma(s, t) = \Phi,$$

then ord $\Gamma(s, t) = 2$. ([**2**], 3.13)

8. If Γ is odd then $n(\Gamma) = n_1(\Gamma) + n_2(\Gamma) + n_3(\Gamma) \geqq 1$. ([**4**], pp. 1-7)

9. If Γ is a simple odd inflectional curve then $n_1(\Gamma) \geqq 3$. ([**5**])

10. Let Γ be an even elementary curve such that ind $\Gamma > 0$ and every strong point is doubly strong. Then

$$n_1(\Gamma) + 2n_2(\Gamma) + n_3(\Gamma) + 2s(\Gamma) \geqq 4.$$

([**2**], 3.2 and [**3**], 3)

5. Projection. Let $\Gamma: T \to P^3$ be a space curve, b a point and β a plane; $b \notin \beta$. For $t \in T$, let

$$11.\quad \Gamma_i^b(t) = \begin{cases} \langle b, \Gamma_i(t)\rangle \cap \beta & \text{, if } b \notin \Gamma_i(t) \\ \Gamma_{i+1}(t) \cap \beta & \text{, if } b \in \Gamma_i(t);\ i = 0, 1. \end{cases}$$

Then (cf. [7]) the map $\Gamma^b: T \to \beta$ such that $\Gamma^b(t) = \Gamma_0^b(t)$, $t \in T$, is a plane curve with $\Gamma_1^b(t)$, the tangent of Γ^b at t. We call Γ^b the *projection of* Γ *from* b *on* β. Furthermore if

$$\Gamma(t) \equiv (a_0(t), a_1(t), a_2(t)) \quad \text{and } \Gamma^b(t) \equiv (a_0^b(t), a_1^b(t)) \text{ for } t \in T,$$

then mod 2

$$12.\quad (a_0^b(t), a_1^b(t)) = \begin{cases} (a_0(t), a_1(t)) & \text{if } b \notin \Gamma_2(t) \\ (a_0(t), a_1(t) + a_2(t)) & \text{if } b \in \Gamma_2(t)\backslash\Gamma_1(t) \\ (a_0(t) + a_1(t), a_2(t)) & \text{if } b \in \Gamma_1(t)\backslash\Gamma(t) \\ (a_1(t), a_2(t)) & \text{if } b = \Gamma(t). \end{cases}$$

13. If Γ is inflectional and $b \notin \Gamma_1(t)\backslash\Gamma(t)$ for $t \in T$ then Γ^b is inflectional.

14. If Γ is inflectional then $\Gamma^b(t)$ is non-regular only if $b \in \Gamma_2(t)$.

15. If $\Gamma(t)$ is elementary then $\Gamma^b(t)$ is elementary. ([7], 5.2.2)

We note that though in the preceding we assumed that $b \notin \beta$, the results 12 to 15 are in fact independent of β.

6. Inflectional convex space curves. Let $\mathscr{R}$ be a compact subset of P^3 disjoint from a plane β in P^3. Then $\mathscr{R}$ is a bounded subset of the affine space $A^3 = P^3\backslash\beta$ and we denote by $H(\mathscr{R})$, the convex hull of $\mathscr{R}$ in A^3.

Let $\beta \subset P^3$ and $\Gamma: T \to A^3$ be a space curve with $B = H(\Gamma)$ in A^3. Then Γ is *convex* if Γ lies on the boundary $\partial(B)$ of B and $|L \cap \Gamma| \leqq 2$ for any line L. In this section, we assume that

16. Γ is an inflectional convex space curve with continuous $\Gamma_i (i = 1, 2)$ and possessing at most one double point as a multiple point.

We note that $\beta \cap \Gamma = \Phi$ implies that Γ is even. Let $\Gamma(t) \in \Gamma$. Since $\Gamma(t) \in \partial(B)$, there is a supporting plane $\pi(t)$ of B through $\Gamma(t)$. Since Γ is inflectional and $\pi(t)$ also supports Γ at t, we have $\Gamma_1(t) \subset \pi(t)$. Clearly B is contained in one of the two closed half-spaces of P^3 bounded by β and $\pi(t)$.

17. LEMMA. *Let* $\Gamma(t) \in \Gamma$. *Then*

1. $H(\pi(t) \cap \Gamma) = \pi(t) \cap B$,
2. $\Gamma(t) \notin \mathrm{int}_{\pi(t)}(\pi(t) \cap B)$,
3. $\langle \Gamma(r), \Gamma(s)\rangle \cap (\pi(t) \cap B) = \Phi$ *for* $\Gamma(r) \neq \Gamma(s)$ *in* $\Gamma\backslash\pi(t)$

and

4. $\Gamma_1(t)$ *supports* $\pi(t) \cap B$ *in* $\pi(t)$.

Proof. 1. Immediate since $B = H(\Gamma)$.

2. Immediate since $\Gamma \cap \text{int } B = \Phi$ and B is a convex body.

3. Let the line $L = \langle \Gamma(r), \Gamma(s) \rangle$ meet $\pi(t) \cap B$ at the point p and set $L^* = L \cap B$. Since one of p, $\Gamma(r)$ and $\Gamma(s)$ lies in $\text{int}_{L^*} L^*$ and $\pi(t)$, $\pi(r)$ and $\pi(s)$ are supporting planes of B; $\pi(t) \cap L^* = \{p\}$ implies that say $\Gamma(s) \in \text{int}_{L^*} L^*$ and hence $L \subset \pi(s)$.

Since $|L \cap \Gamma| \leqq 2$, $p \notin \Gamma$. Then

$$|\pi(s) \cap \Gamma| < \infty \quad \text{and } H(\pi(s) \cap \Gamma) = \pi(s) \cap B$$

imply that p lies in the relative interior of a line segment in $\pi(s) \cap B$. But then

$$\Gamma(s) \in \text{int}_{\pi(s)}(\pi(s) \cap B);$$

a contradiction.

4. Suppose that

$$\Gamma_1(t) \cap \text{int}_{\pi(t)}(\pi(t) \cap B) \neq \Phi$$

and choose a point $b \in \Gamma \backslash (\pi(t) \cup \Gamma_2(t))$. Then clearly $b \neq \Gamma(s)$ for s near t in T. Let Γ^b be the projection of Γ from b on $\pi(t)$. Then

$$\Gamma_1(t) = \Gamma_1^b(t) = \lim_{s \neq t \to s} \langle \Gamma^b(t), \Gamma^b(s) \rangle$$

and by 14, $\Gamma^b(t)$ is regular. Since $\pi(t) \cap B$ is a polygon,

$$\Gamma_1^b(t) \cap \text{int}_{\pi(t)}(\pi(t) \cap B) \neq \Phi$$

clearly implies that

$$\Gamma^b(s) \in \text{int}_{\pi(t)}(\pi(t) \cap B)$$

for s near t. But then $\Gamma^b(s) = \langle b, \Gamma(s) \rangle \cap \pi(t)$ and $b \in \Gamma \backslash \pi(t)$ contradict 17.3.

18. LEMMA. *The set* ext B *of the extreme points of* B *is equal to* Γ.

Proof. Since $p \in \text{ext } B$ provided p does not lie in the relative interior of any line segment contained in B and Γ is convex, the claim follows by 17.1 and 17.2.

Let Γ^b be the projection of Γ from b on β, $b \in B$.

19. LEMMA. *Let* $b \in B \backslash \Gamma$.

1. *Then* Γ^b *is even,* ind $\Gamma^b > 0$, $s(\Gamma^b) \leqq s(\Gamma) \leqq 1$ *and every strong point of* Γ^b *is doubly strong.*

2. *If* $n(\Gamma^b) < \infty$ (*cf.* 8) *then* Γ^b *is elementary.*

Proof. 1. Since $b \notin \Gamma$, every intersection of Γ with a plane α through b is projected into an intersection of Γ^b with $\alpha \cap \beta$. As Γ is even and $b \in H(\Gamma)$, this implies that Γ^b is even and ind $\Gamma^b > 0$.

Let $\Gamma^b(r) = \Gamma^b(s)$ for $r \neq s$ in T. Then $b \notin \Gamma$ implies that

$$\langle b, \Gamma(r) \rangle = \langle b, \Gamma(s) \rangle.$$

If $\Gamma(r) \neq \Gamma(s)$ then $\Gamma = \text{ext } B$ yields that b lies in the relative interior of $\langle \Gamma(r), \Gamma(s) \rangle \cap B$, and thus any plane through b meets both $\Gamma[r, s]$ and $\Gamma[s, r]$. Therefore

$$\text{ind } \Gamma^b[r, s] \cdot \text{ind } \Gamma^b[s, r] > 0$$

and $\Gamma^b(r)$ is not strong. If $p = \Gamma(r) = \Gamma(s)$ then $p \neq \Gamma(t)$ for $t \in T\backslash\{r, s\}$ from 16. Thus $\Gamma^b(r)$ is the only possible strong point of Γ^b and the preceding readily yields that $\Gamma^b(r)$ is then doubly strong.

2. Let $t \in T$. Since $b \notin \Gamma$, there is an α through b and an $U(t)$ such that

$$\alpha \cap \Gamma(U(t)) = \Phi.$$

By 16, we may assume that $\Gamma(U(t))$ is simple. Then $\alpha \cap \Gamma(U(t)) = \Phi$ and $\Gamma = \text{ext } B$ imply that

$$b \notin \langle \Gamma(r), \Gamma(s) \rangle \quad \text{for } \{r, s\} \subset U(t)$$

and thus $\Gamma^b(U(t))$ is simple.

Let $n(\Gamma^b) < \infty$. Then we may assume that $\Gamma^b(U^-(t)) \cup \Gamma^b(U^+(t))$ is regular. Let $r \in U^-(t)$ and $L = \langle \Gamma^b(t), \Gamma^b(r) \rangle$. Since

$$|\langle b, \Gamma(t), \Gamma(r) \rangle \cap \Gamma| < \infty,$$

$|L \cap \Gamma^b| < \infty$ and there is an $r' \in [r, t)$ such that

$$\Gamma^b(r') \in L \text{ and } L \cap \Gamma^b(r', t) = \Phi.$$

Thus $\text{ord } \Gamma^b(r', t) = 2$ by 7. Similarly, there is an $s' \in U^+(t)$ such that

$$\text{ord } \Gamma^b(t, s') = 2$$

and thus $\Gamma^b(t)$ is elementary.

20. Lemma. *Let* $b = \Gamma(t)$.

1. *Then* Γ^b *is odd* [*even*] *if* b *is a simple* [*double*] *point.*
2. *If* Γ *is simple and* $\Gamma_1(t) \cap \Gamma = \{\Gamma(t)\}$ *then* Γ^b *is simple.*

Proof. 1. Let $L \subset \beta$ cut Γ^b at n points of T. Again we note that if $b \neq \Gamma(r)$ then L cuts Γ^b at r if and only if $\langle b, L \rangle$ cuts Γ at r.

If $\Gamma(t)$ is simple, we choose L so that $\Gamma^b(t) \notin L$. Then $\langle b, L \rangle$ cuts Γ at n points of $T\backslash\{t\}$ and by 11,

$$\langle b, L \rangle \cap \Gamma_1(t) = \{\Gamma(t)\}.$$

Since $\Gamma(t) \equiv (1, 1, 1)$ or $\Gamma(t) \equiv (1, 1, 2)$, $\langle b, L \rangle$ cuts Γ at t. Thus $\langle b, L \rangle$ cuts Γ altogether at $n + 1$ points and since Γ is even, n is odd.

If $\Gamma(t) = \Gamma(t')$, $t \neq t'$, we choose L so that

$$L \cap \{\Gamma^b(t), \Gamma^b(t')\} = \Phi.$$

As in the preceding, $\langle b, L\rangle$ cuts Γ at n points of $T\backslash\{t, t'\}$ as well as at t and t'. Thus $n + 2$ is even.

2. Let Γ be simple and $\Gamma_1(t) \cap \Gamma = \{\Gamma(t)\}$. Then the convexity of Γ implies that $\Gamma^b(r) \neq \Gamma^b(s)$ for $r \neq s$ in $T\backslash\{t\}$ and 11 implies that $\Gamma^b(t) \neq \Gamma^b(r)$ for $r \in T\backslash\{t\}$.

21. THEOREM. *Let $\Gamma:T \rightarrow P^3$ be an inflectional convex space curve with continuous $\Gamma_i (i = 1, 2)$ and possessing at most one double point as a multiple point. Let*

$$b \in B = H(\Gamma),\ B^* = \bigcup_{t \in T} (\Gamma_1(t) \cap B) \quad \textit{and}$$

$$O(b) = \{t \in T | b \in \Gamma_2(t)\}.$$

1. *If $b \in B\backslash B^*$ then $|O(b)| \geqq 4[2]$ when Γ is [is not] simple.*
2. *If $b \in B^*$ then $|O(b)| \geqq 2[1]$ when Γ [is not] simple.*
3. *Let $b \in B^* = \Gamma$.*
   a) *If Γ is simple then $|0(b)| \geqq 3[4]$ when b is [is not] an inflection.*
   b) *If Γ is not simple and b is not an inflection then $|O(b)| \geqq 2$.*

Proof. Since Γ is inflectional, 14 implies that $\Gamma^b(t)$ is non-regular only if $b \in \Gamma_2(t)$. Hence $|O(b)| \geqq n(\Gamma^b)$ and we may assume that $n(\Gamma^b) < \infty$.

1. Let $b \in B\backslash B^*$. *Since* $\Gamma \subseteq B^*$, Γ^b is an even elementary curve such that ind $\Gamma^b > 0$, $s(\Gamma^b) \leqq s(\Gamma) \leqq 1$ and every strong point is doubly strong by 19. Since $b \notin \Gamma_1(t)$ for $t \in T$, Γ^b is inflectional by 13. Thus

$$n_1(\Gamma^b) + 2s(\Gamma^b) \geqq 4$$

by 10 and

$$|O(b)| \geqq 4 - 2s(\Gamma^b) \geqq 2$$

by the preceding. Since $s(\Gamma^b) = 1$ only if Γ is not simple, the claim follows.

2. Let $b \in B^*$. Then $|O(b)| \geqq 1$ and we may assume that Γ is simple. If $b \notin \Gamma$ then $s(\Gamma) = 0$, 19 and 10 imply that

$$n_1(\Gamma^b) + 2n_2(\Gamma^b) + n_3(\Gamma^b) \geqq 4.$$

Hence $|O(b)| \geqq n(\Gamma^b) \geqq 2$.

Let $b = \Gamma(t) \in \Gamma$. Then we may assume that $b \notin \Gamma_1(r)$ for $r \in T\backslash\{t\}$ and hence Γ^b is inflectional by 13. By 20.1, Γ^b is an odd curve. If $\Gamma_1(t) \cap \Gamma = \{\Gamma(t)\}$ then Γ^b is simple by 20.2 and

$$|O(b)| \geqq n(\Gamma^b) = n_1(\Gamma^b) \geqq 3$$

by 9. Let $|\Gamma_1(t) \cap \Gamma| \neq 1$. Then $\Gamma_1(t)$ meets Γ at exactly one point $\Gamma(r) \neq \Gamma(t)$ and by 11, $\Gamma^b(r) = \Gamma^b(t)$ is the only multiple point of Γ^b. Since Γ^b is odd, one of the subarcs $\Gamma^b[r, t]$ and $\Gamma^b[t, r]$, say $\Gamma^b[r, t]$, must also be odd. Hence

$$\text{ind } \Gamma^b[\text{r}, t] > 0.$$

Since $\Gamma^b(r, t)$ is simple, 6 implies that $\Gamma^b(r, t)$ contains a non-regular point $\Gamma^b(s)$. Thus $b = \Gamma(t) \in \Gamma_2(s)$ by 14 and $|O(b)| \geqq 2$.

3. We note that $B^* = \Gamma$ implies that

$$|\Gamma_2(r) \cap \Gamma| = 1 \text{ for } r \in T.$$

Hence $b = \Gamma(t)$ and 13 imply that Γ^b is inflectional. If Γ is simple then Γ^b is simple and odd by 20. Hence

$$|O(b)| \geqq n(\Gamma^b) = n_1(\Gamma^b) \geqq 3$$

by 9. If in addition, $b = \Gamma(t) \equiv (1, 1, 1)$, then $\Gamma^b(t) \equiv (1, 1)$ by 12 and

$$|O(b)| \geqq 1 + n_1(\Gamma) \geqq 4.$$

If Γ is not simple then $|O(b)| \geqq 2$ when b is a double point. If $b = \Gamma(t)$ is simple and regular then again

$$\Gamma^b(t) \equiv (1, 1) \text{ and } |O(b)| \geqq 1 + n_1(\Gamma^b).$$

Since Γ^b is odd by 20.1, $n_1(\Gamma^b) \geqq 1$ by 8 and $|O(b)| \geqq 2$.

We observe that 21 is a generalization of 1 and 2a) since $B^* = \Gamma$ when Γ is spherical. It is also clear that Γ need not be spherical. For example: let C be a non-degenerate quadric cone with vertex v and let $\Gamma \subset C\setminus\{v\}$ be a space curve meeting any line through v in at most two points.

7. A four-vertex theorem. Unless stated otherwise, we assume that $\Gamma:T \to P^3$ is an elementary inflectional space curve with exactly n inflections, $\beta \cap \Gamma = \Phi$ and $B = H(\Gamma)$. Then n is even by 5.

Let $t \in T$. Since $|\Gamma_2(t) \cap \Gamma| < \infty$, there is an $U'(t) = U^-(t) \cup U^+(t)$ such that

$$\Gamma_2(t) \cap \Gamma(U'(t)) = \Phi.$$

Let B_t^- $[B_t^+]$ be the connected component of $B\setminus\Gamma_2(t)$ which contains $\Gamma(U^-(t))[\Gamma(U^+(t))]$. Thus

$$B = B_t^- \cup B_t^+ \cup (\Gamma_2(t) \cap B) \quad \text{and}$$

$$B_t^- \cap B_t^+ = \Phi \quad \text{if } \Gamma(t) \equiv (1, 1, 1) \quad \text{and}$$

$$B_t^- = B_t^+ \text{ if } \Gamma(t) \equiv (1, 1, 2).$$

Let $b \in B$ and $t \in T$. Set

$$T_b^0 = \{t \in T \mid b \in \Gamma_2(t) \text{ or } \Gamma(t) \equiv (1, 1, 2)\},$$

$$T_b^- = \{t \in T \backslash T_b^0 \mid b \in B_t^-\} \quad \text{and} \quad T_b^+ = \{t \in T \backslash T_b^0 \mid b \in B_t^+\}.$$

We call an element of T_b^0, T_b^- and T_b^+ a b^0 *point*, b^- *point* and b^+ *point* respectively. Clearly the three sets are mutually disjoint and

$$T = T_b^0 \cup T_b^- \cup T_b^+.$$

22. LEMMA. *For $b \in B$, T_b^- and T_b^+ are open in T, T_b^0 is closed in T,*

$$\text{cl } T_b^- = T_b^- \cup T_b^0 \quad \textit{and} \quad \text{cl } T_b^+ = T_b^+ \cup T_b^0.$$

Proof. Since Γ contains only n inflections and Γ_2 is continuous, T_b^0 is closed in T.

If $t \notin T_b^0$ then $\Gamma(t)$ is ordinary by 3 and there exists an $U(t)$ such that

$$\text{ord } \Gamma(U(t)) = 3.$$

By 4, we may assume that $b \notin \Gamma_2(s)$ for $s \in U'(t)$ and thus

$$U(t) \subset T_b^- \cup T_b^+.$$

Since $\text{ord } \Gamma(U(t)) = 3$, $\Gamma_2(s)$ meets $\Gamma(U(t))$ exactly at $\Gamma(s)$ for $s \in U(t)$ and therefore B_s^- and B_s^+ depend continuously on $s \in U(t)$. But then $b \in B_t^-[B_t^+]$ clearly implies that $b \in B_s^-[B_s^+]$ for s near t.

COROLLARY. *If $\Gamma(r, s)$ s regular and $b \notin \Gamma_2(t)$ for $t \in (r, s)$ then either $(r, s) \subset T_b^-$ or $(r, s) \subset T_b^+$.*

Proof. This is immediate since (r, s) is connected, T_b^- and T_b^+ are open in T and $(r, s) \subset T_b^- \cup T_b^+$.

23. LEMMA. *Let $b \in \Gamma_2(t) \cap B$; $\Gamma(t) \equiv (1, 1, 1)$. Then there exists a $U(t)$ such that*

$$U^-(t) \subset T_b^+ \quad \textit{and} \quad U^+(t) \subset T_b^-.$$

Proof. Since $\Gamma(t)$ is elementary, there is a $U(t)$ such that

$$\Gamma_2(t) \cap \Gamma(U'(t)) = \Phi$$

and

$$\text{ord } \Gamma(U^-(t)) = \text{ord } \Gamma(U^+(t)) = 3.$$

By 4, we may assume that neither b nor $\Gamma(t)$ lie on $\Gamma_2(s)$ for $s \in U'(t)$. Let Γ^b be the projection of Γ from b on β. By 15, $\Gamma^b(t)$ is elementary and hence we may also assume that

$$\text{ord } \Gamma^b(U^-(t)) = \text{ord } \Gamma^b(U^+(t)) = 2.$$

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Let $s \in U^-(t) = (r, t)$. Since ord $\Gamma(r, t) = 3$ and $\Gamma(t) \notin \Gamma_2(s)$, we obtain that

$$\Gamma_2(s) \cap \Gamma(r, t] = \{\Gamma(s)\}.$$

Hence $\Gamma(r, s) \subset B_s^-$, $\Gamma(s, t) \subset B_s^+$ and $s \in T^+_{\Gamma(t)}$. By 22 Corollary, $U^-(t) \subset T^+_{\Gamma(t)}$ and hence we may assume that $b \neq \Gamma(t)$.

Since $b \notin \Gamma_2(s)$ and $\Gamma(s) \equiv (1, 1, 1)$, Γ is supported by the plane

$$\alpha_s = \langle b, \Gamma_1(s) \rangle \neq \Gamma_2(s)$$

at s. Hence Γ^b is supported by the line $\Gamma_1^b(s) = \beta \cap \alpha_s$ at s. Since

$$\text{ord } \Gamma^b(U^-(t)) = 2,$$

this implies that

$$\Gamma_1{}^b(s) \cap \Gamma^b(U^-(t)) = \{\Gamma^b(s)\}$$

and therefore

$$\alpha_s \cap \Gamma(U^-(t)) = \{\Gamma(s)\}.$$

Let $\widetilde{B}_s$ denote the connected component of $B \backslash \alpha_s$ containing $\Gamma(U^-(t)) \backslash \Gamma(s)$. Then we observe that

$$\Gamma(r, s) \subset B_s^- \cap \widetilde{B}_s \text{ and } \Gamma(s, t) \subset B_s^+ \cap \widetilde{B}_s.$$

Suppose that s, and hence $U^-(t)$, is contained in T_b^-. Let $\pi(s)$ be the supporting plane of B at s. Then $\Gamma_1(s) \subset \pi(s)$ and since $\Gamma(s) \equiv (1, 1, 1)$, $\pi(s) \neq \Gamma_2(s)$. The convex set B_s^+ lies in the closed half-space of P^3 bounded by $\Gamma_2(s)$ and $\pi(s)$ which contains $\Gamma(s, t)$. If α_s is also a supporting plane of B at $\Gamma(s)$, then clearly

$$B_s^+ \subset \widetilde{B}_s = B \backslash \alpha_s.$$

Otherwise, $b \in B_s^-$ implies that the preceding half-space is contained in the closed half-space of P^3 bounded by α_s and $\pi(s)$ which contains $\Gamma(s, t)$. But then again $B_s^+ \subset \widetilde{B}_s$.

Let s tend to t in $U^-(t)$. Since $\Gamma_2(s)$, $\Gamma_1(s)$ and $\Gamma_1{}^b(s)$ all depend continuously on s, $b \in \Gamma_2(t)$ implies that both $\Gamma_2(s)$ and α_s tend to $\Gamma_2(t)$. Then the definition of $\widetilde{B}_s$ and $\Gamma(r, s) \subset \widetilde{B}_s$ yield that $\widetilde{B}_s$ tends to B_t^-. Since $\Gamma(t) \equiv (1, 1, 1)$, we note that B_s^+ tends to B_t^+, $B_t^+ \neq \Phi$ and $B_t^+ \cap B_t^- = \Phi$. But then $B_s^+ \subset \widetilde{B}_s$ and the preceding imply that

$$B_t^+ \cap B_t^- = B_t^+ \neq \Phi;$$

a contradiction.

Therefore $U^-(t) \subset T_b^+$ and by a similar argument, $U^+(t) \subset T_b^-$.

24. LEMMA. *Let $\Gamma(t_0, t_1)$ be regular. Then*

$$\Gamma_2(r) \cap \Gamma_2(s) \cap B = \Phi \textit{ for } r \neq s \textit{ in } [t_0, t_1].$$

Proof. Suppose that there exist $r < s$ (r preceding s) in (t_0, t_1) such that there is a point

$$b \in \Gamma_2(r) \cap \Gamma_2(s) \cap B.$$

By 4, it follows that there are only a finite number of points $t \in (t_0, t_1)$ such that $b \in \Gamma_2(t)$ and hence we may assume that $b \notin \Gamma_2(t)$ for $t \in (r, s)$. Then $(r, s) \subset T_b^-$ or $(r, s) \subset T_b^+$ by 22 Corollary. But 23 implies that there exist

$$U^+(r) \subset (r, s) \cap T_b^- \quad \text{and} \quad U^-(s) \subset (r, s) \cap T_b^+;$$

a contradiction.

The lemma now readily follows by the preceding and the continuity of Γ_2 if $\Gamma(t_0)$ or $\Gamma(t_1)$ are inflections.

25. LEMMA. *Let $\Gamma(t_0, t_1)$ be regular such that $\Gamma(t_i)$ is an inflection and*

$$\Gamma_2(t_i) \cap \Gamma = \{\Gamma(t_i)\}, i = 0, 1.$$

Let $b \in B \backslash \{\Gamma(t_0), \Gamma(t_1)\}$. Then there is exactly one $s \in (t_0, t_1)$ such that $b \in \Gamma_2(s)$.

Proof. By 24, there is at most one $s \in (t_0, t_1)$ such that $b \in \Gamma_2(s)$.

Since $B = H(\Gamma)$, $|\Gamma_2(t_i) \cap \Gamma| = 1$ clearly implies that $\Gamma_2(t_i)$ is a supporting plane of B and hence

$$B_{t_i}^- = B_{t_i}^+ = B \backslash \Gamma_2(t_i); \; i = 0, 1.$$

Let $s \in (t_0, t_1)$. Since $\Gamma_2(s) \cap \Gamma[t_0, t_1] = \{\Gamma(s)\}$ by 24, we obtain that

$$\Gamma[t_0, s) \subset B_s^- \quad \text{and} \quad \Gamma(s, t_1] \subset B_s^+.$$

By the continuity of $\Gamma_2(s)$, it follows that

$$B_s^+ \text{ tends to } B_{t_0}^-$$

as s tends to t_0 and

$$B_s^- \text{ tends to } B_{s_1}^+$$

as s tends to t_1. Thus $b \notin \Gamma_2(t_0) \cup \Gamma_2(t_1)$ yields that $b \in b_s^+ \; [B_s^-]$ for s near $t_0[t_1]$ in (t_0, t_1). But then

$$(t_0, t_1) \subset T_b^-, \quad (t_0, t_1) \subset T_b^+$$

and 22 Corollary imply that $b \in \Gamma_2(s)$ for some $s \in (t_0, t_1)$.

26. THEOREM. *Let $\Gamma : T \to P^3$ be an elementary convex space curve with exactly n inflections. Let $b \in B = H(\Gamma)$ and $O(b) = \{t \in T | b \in \Gamma_2(t)\}$. Then*

1. $|O(b)| \leqq n$ *and*

2. *if the osculating plane at each inflection point does not meet* Γ *elsewhere*,

$$|O(b)| = \begin{cases} n & \text{if } b \text{ is not an inflection} \\ n - 1 & \text{if } b \text{ is a simple inflection.} \end{cases}$$

Proof. Let $\Gamma(t_1)$, $\Gamma(t_2), \ldots, \Gamma(t_n)$ be the inflection points of Γ; $t_1 < t_2 < \ldots < t_n < t_1$. Then $\Gamma(t_i, t_{t+1})$ is regular,

$$\Gamma = \bigcup_{i=1}^{n} \Gamma(t_i, t_{i+1})$$

and by 24, $|O(b)| \leqq n$.

Let $|\Gamma_2(t_i) \cap \Gamma| = 1$ for each i. If b is not an inflection then $|O(b)| = n$ by 25. If $b = \Gamma(t_i)$ is simple then

$$O(b) \cap [t_{i-1}, t_{i+1}] = \{t_i\}$$

by 24 and

$$O(b) \cap (t_{i+1}, t_{i-1}) = n - 2$$

by 25.

THEOREM. *A simple elementary inflectional convex space curve possesses at least four inflections.*

Proof. Apply 21 and 26.

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