PROPERTIES OF RELATIVISTIC LINEAR STELLAR MODELS

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1. INTRODUCTION

In the late stages of stellar evolution, relativistic objects are formed, such as neutron stars or black holes. These relativistic stars possess a strong gravitational field, therefore their structure and their space-time geometry can be described only in the frame of GRT (Zeldovich and Novikov, 1971; Misner, Thorne and Wheeler, 1973). For this purpose, the following four-dimensional interval is used (spherical gravitational field), (Zeldovich and Novikov, 1971).

$$ds^2 = e^{\nu(r)}\, c^2 dt^2 - e^{\lambda(r)} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (1)$$

If we take the energy-momentum tensor in the form

$$T^{ik} = (\rho c^2 + P) u^i u^k - P\, g^{ik}, \qquad (2)$$

then for a relativistic star in equilibrium, with spherically symmetrical gravitational field, from Einstein's field equations (in the cold matter approximation) we obtain the following equations (Zeldovich and Novikov, 1971; Weinberg, 1975; Hawking and Ellis, 1977)

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \,, \quad M(0) = 0$$

$$\frac{dP(r)}{dr} = -\frac{G(\rho + P/c^2)\,(M(r) + 4\pi r^3 P/c^2)}{r^2(1 - 2GM(r)/c^2 r)} \,, \quad P(R) = 0 \qquad (3)$$

$$P = P(\rho) \,, \qquad \rho(R) = 0$$

where the notations are usual. For a given state equation $P = P(\rho)$ the system of equations (3) allows the determination of the functions

Z. Kopal and J. Rahe (eds.), Binary and Multiple Stars as Tracers of Stellar Evolution, 73–78.

$\varrho(r)$, $P(r)$ and $M(r)$ which describe the structure of the relativistic stars.

From Einstein's field equations we also obtain the following equations (Tooper, 1964)

$$e^{-\lambda} = \begin{cases} 1 - 2GM(r)/c^2 r, & \text{for } r < R \\ 1 - 2GM/c^2 r, & \text{for } r \geq R \end{cases}$$

$$e^{\nu} = e^{-\lambda} = 1 - 2GM/c^2 r, \quad \text{for } r \geq R \qquad (4)$$

$$\frac{1}{2}(\varrho c^2 + P)\frac{d\nu}{dr} + \frac{dP}{dr} = 0, \quad \text{for } r < R$$

$$\lim_{r \nearrow R} \nu(r) = \nu(R)$$

where M is the total mass of the star. The system of equations (4) allows the determination of the functions $\nu(r)$ and $\lambda(r)$, which describe the geometry of the space-time continuum inside and outside the relativistic star.

2. NON-DIMENSIONAL FORM OF EQUATIONS

For the structural and geometrical research of some concrete models of relativistic stars it is convenient to put the equations (3) and (4) in a non-dimensional form, through the transformations (Ureche, 1980 a,b, 1981)

$$r = a\eta, \quad \varrho = \varrho_c, \quad P = \varrho_c c^2 p, \quad M(r) = M^* m. \qquad (5)$$

Taking

$$M^* = 4\pi a^3 \varrho_c, \quad a^2 = c^2/4\pi G \varrho_c, \qquad (6)$$

from (3), (4) and (5) we obtain

$$\frac{dm}{d\eta} = \eta^2 \psi, \quad p = p(\psi), \quad m(0) = 0, \quad \psi(\eta_s) = 0$$

$$\frac{dp}{d\eta} = -\frac{(\psi + p)(m + \eta^3 p)}{\eta^2(1 - 2m/\eta)}, \quad p(\eta_s) = 0 \qquad (7)$$

$$e^{-\lambda} = \begin{cases} 1 - 2m/\eta, & \text{for } \eta < \eta_s \\ 1 - 2m_s/\eta, & \text{for } \eta \geq \eta_s, \quad m_s = m(\eta_s) \end{cases}$$

$$e^{\nu} = e^{-\lambda} = 1 - 2m_s/\eta, \quad \text{for } \eta \geq \eta_s \qquad (8)$$

https://doi.org/10.1017/S0252921100100375 Published online by Cambridge University Press

$$\frac{1}{2}(\psi + p)\frac{d\nu}{d\eta} + \frac{dp}{d\eta} = 0 \quad , \quad \text{for } \eta < \eta_s$$
$$\lim_{\eta \nearrow \eta_s} \nu(\eta) = \nu(\eta_s) \equiv \nu_s \qquad (8)$$

where $\eta_s = R/a$ is the value of the non-dimensional coordinate η at the surface of the star. The system of equations (7) will determine the physical structure of the relativistic star, while the system (8) will describe the geometry of the space-time continuum inside and outside the relativistic star.

3. RELATIVISTIC LINEAR STELLAR MODEL

In the study of the newtonian stars, Stein (1966) has used the linear stellar model, showing that this model is useful for the determination of some representative values of the stellar characteristics, as well as for the construction of some non-homogeneous stellar models (with envelopes).

We have studied the properties of the linear stellar model in the frame of GRT (Ureche, 1980 a). In this case the density distribution is given by the law

$$\rho = \rho_c(1) - r/R) \qquad (9)$$

Using the non-dimensional variables, the equation (9) becomes

$$\psi = 1 - \eta / \eta_s \qquad (10)$$

From the equations (7) and (10) we obtain the differential system

$$\frac{dm}{d\eta} = \eta^2(1 - \eta/\eta_s) \, , \quad m(0) = 0$$
$$\frac{dp}{d\eta} = - \frac{(1 - \eta/\eta_s + p)(m + \eta^3 p)}{\eta^2(1 - 2m/\eta)} \; ; \; p(\eta_s) = 0 \qquad (11)$$

where the quantities a and M^* have the expressions

$$a = R\sqrt{R/6R_g} \quad , \quad M^* = 2M\sqrt{R^3/6R_g^3} \qquad (12)$$

The first equation in (11) is immediately integrated, obtaining

$$m(\eta) = \frac{\eta^3}{3}(1 - \frac{3}{4}\frac{\eta}{\eta_s}) \, , \qquad (13)$$

that is formally the same expression as in the case of the newtonian stars.

For the equation of the hydrostatic equilibrium it cannot be obtained an exact solution as for the newtonian case. In order to integrate numerically this second equation from (11), we put it in another form by the change of variable $\eta = \eta_s y$. So, the non-dimensional density ψ will be written $\psi = 1 - y$, while the equation of the hydrostatic equilibrium becomes

$$\frac{dp}{dy} = -\frac{\eta_s^2}{2}\,\frac{y(p - y + 1)(12p - 3y + 4)}{6 - \eta_s^2 y^2(4 - 3y)}\ , \quad p(1) = 0 \qquad (14)$$

It is easy to verify that the differential equation (14) has an unique solution, if η_s^2 fulfills the restriction $\eta_s^2 < 729/128$. This solution was effectively obtained by numerical integration, using the Runge-Kutta algorithm (Gill's variant). The obtained results are given in tables and graphs (Ureche, 1980 a).

For the maximum mass of linear neutron stars we have obtained $3.4 M_\odot$, if $P \leq (1/3)\rho c^2$ and $4.7 M_\odot$ if $P \leq \rho c^2$.

From (8) and (13), with the change of variable $\eta = \eta_s y$, it results

$$e^{\lambda} = \begin{cases} \left(1 - \frac{1}{6}\eta_s^2 y^2(4 - 3y)\right)^{-1}, & \text{for } 0 \leq y < 1 \\ \left(1 - \frac{1}{6}\frac{\eta_s^2}{y}\right)^{-1}, & \text{for } y \geq 1 \end{cases} \qquad (15)$$

$$e^{\nu} = \begin{cases} e^{\nu}, & \text{where } \nu = \nu(y) \text{ is the solution of the differential equation (17) for } 0 \leq y < 1 \\ 1 - \frac{1}{6}\frac{\eta_s^2}{y}, & \text{for } y \geq 1 \end{cases} \qquad (16)$$

$$\frac{d\nu}{dy} = \eta_s^2\,\frac{y(12p - 3y + 4)}{3\eta_s^2 y^3 - 4\eta_s^2 y^2 + 6}\ , \quad \text{for } 0 \leq y < 1$$

$$\nu_s \equiv \nu(1) = \ln\left(1 - \eta_s^2/6\right) \qquad (17)$$

The determination of the function e^{ν} for $y \in (0, 1)$ requires the integration of the differential equation (17) which contains the function p(y). This function was tabulated (Ureche, 1980 a), but the use of these tables for the numerical integration of the differential equation (17) is not convenient. From a practical point of view it is more convenient to integrate simultaneously the

https://doi.org/10.1017/S0252921100100375 Published online by Cambridge University Press

differential equations (14) and (17).

So, we obtain the following system of differential equations

$$\frac{dp}{dy} = -\frac{\eta_s^2}{2}\frac{y(p - y + 1)(12p - 3y + 4)}{3\eta_s^2 y^3 - 4\eta_s^2 y^2 + 6}, \quad p(1) = 0$$

$$\frac{d\nu}{dy} = \eta_s^2 \frac{y(12p - 3y + 4)}{3\eta_s^2 y^3 - 4\eta_s^2 y^2 + 6}, \quad \nu(1) = \ln\left(1 - \frac{1}{6}\eta_s^2\right) \tag{18}$$

which has an unique solution, if $\eta_s^2 < 729/128$.

Using the numerical methods, the functions describing the geometry of the space-time continuum are determined. These functions are given in tables and graphs (Ureche, 1981).

For the relativistic linear stellar model an absolute minimum radius (for which $P_c \to \infty$) was determined, namely: $R_{min} = 1.335\ R_g$. The corresponding absolute maximum mass is (Brecher and Caporaso, 1977) $M_{max} = 6.22\ M_o$. This value can be compared with the value of $8M_o$ for the homogeneous model.

The coefficient of gravitational packing (Zeldovich and Novikov, 1971) has the expression

$$\alpha_1 = 12\sqrt{6}\int_0^1 \frac{y^2(1 - y)\,dy}{\sqrt{3\eta_s^2 y^3 - 4\eta_s^2 y^2 + 6}} - 1 \tag{19}$$

For the typical values of the parameter η_s^2 ($\eta_s^2 \simeq 3 \div 4$), the values of the coefficient α_1 are of the order of $0.3 \div 0.5$. This means that, for relativistic linear stars, the gravitational energy can reach 50% from total energy.

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