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THE PIERCE-BIRKHOFF CONJECTURE FOR CURVES

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ABSTRACT. The results obtained extend Madden's result for Dedekind domains to more general types of 1-dimensional Noetherian rings. In particular, these results apply to piecewise polynomial functions $t: C \longrightarrow R$ where R is a real closed field and $C \subseteq R^n$ is a closed 1-dimensional semi-algebraic set, and also to the associated "relative" case where t, C are defined over some subfield $K \subseteq R$.

The Pierce-Birkhoff Conjecture [2, 5] asserts that if R is a real closed field (for example, $R = \mathbb{R}$), then any (continuous) piecewise-polynomial function $t: R^n \longrightarrow R$ is expressible as $t = \sup_i(\inf_j f_{ij})$ where $\{f_{ij}\} \subseteq R[\mathbf{x}] = R[x_1, \ldots, x_n]$ is a finite set of polynomial functions.

In [10], Mahé proves the conjecture for $n \leq 2$ and, as well, proves that a weaker version of the conjecture holds for any n. In [4], Delzell looks at a certain "relative" version of the conjecture: If $t: R^n \longrightarrow R$ as above is defined over some subfield $K \subseteq R$ (in some suitable sense), is it true that $t = \sup_i(\inf_j f_{ij})$ with $f_{ij} \in K[\mathbf{x}]$? Delzell proves this if $n \leq 2$. Of course, Delzell's result generalizes the corresponding result of Mahé.

In [8], Madden introduces an abstract ring $\mathrm{PW}(A)$, defined for any ring A which is commutative with 1, and asks under what conditions a given $t \in \mathrm{PW}(A)$ is expressible as $t = \sup_i(\inf_j f_{ij})$ with $f_{ij} \in A$. If $A = R[\mathbf{x}]$, then $\mathrm{PW}(A)$ is just the ring of all piecewise polynomial functions $t: R^n \longrightarrow R$. Madden's work exploits the theory of the real spectrum of a ring introduced by Coste and Roy (e.g., see [1, 3, 7]). Working from an earlier result of Keimel [6], Madden gives local conditions on pairs of orderings which are equivalent to the condition that a given $t \in \mathrm{PW}(A)$ is sup-inf definable. Using this, he proves that if A is a field or Dedekind domain, then any $t \in \mathrm{PW}(A)$ is sup-inf definable.

In the present paper, we examine more closely Madden's local conditions. The new results we obtain allow extension of Madden's result on Dedekind domains to more general types of 1-dimensional Noetherian rings. In particular, these new results apply to piecewise polynomial functions $t: C \longrightarrow R$ in case $C \subseteq R^n$ is a curve (more generally, any closed 1-dimensional semi-algebraic set) as well as to the associated "relative" case where $C \subseteq R^n$ and $t: C \longrightarrow R$ are defined over some subfield $K \subseteq R$. Of course, the case where $C \subseteq R^n$ is a smooth curve is already covered by the result in [8].

Just after the initial version of this paper was completed, Madden announced a proof of the Pierce-Birkhoff conjecture for smooth algebraic surfaces [9].

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1. **The rings** $\mathrm{PW}_X(A)$ **and** $\mathrm{SI}_X(A)$**.** Let A be a commutative ring with 1 and let $\operatorname{Sper} A$ denote the real spectrum of A [1, 3, 7]. For $\alpha \in \operatorname{Sper} A$, the prime ideal $p_\alpha = \alpha \cap -\alpha$ is called the *support* of α. $R(\alpha)$ denotes the real closure of the quotient field $K(p_\alpha)$ of the domain A/p_α at the ordering induced by α. For $a \in A$, denote by $a(\alpha)$ the image of a in $R(\alpha)$ via the composite map $A \longrightarrow A/p_\alpha \subseteq K(p_\alpha) \subseteq R(\alpha)$.

Let $X \subseteq \operatorname{Sper} A$ be any subset. We say $t \in \prod_{\alpha \in X} R(\alpha)$ is *piecewise* A *on* X if there exist $a_1, \ldots, a_k \in A$ and sets $S_1, \ldots, S_k \subseteq X$ which are relatively closed and relatively constructible such that $X = S_1 \cup \cdots \cup S_k$ and $t = a_i$ on S_i, $i = 1, \ldots, k$. $\mathrm{PW}_X(A)$ denotes the subring of the product ring $\prod_{\alpha \in X} R(\alpha)$ consisting of all t which are piecewise A on X. In case $X = \operatorname{Sper} A$, $\mathrm{PW}_X(A)$ is denoted simply by $\mathrm{PW}(A)$.

$\mathrm{PW}_X(A)$ is actually subring of a big ring $C_X(A) \subseteq \prod_{\alpha \in X} R(\alpha)$ consisting of all constructible sections on X (e.g., see [1]), but we can get by without this here. All we need is that, for any $t \in \mathrm{PW}_X(A)$, the set $\{\alpha \in X : t(\alpha) \geq 0\}$ is closed and constructible in the relative topology on X, but this is clear from our definition.

There are natural lattice operations $\vee$, $\wedge$ on $\mathrm{PW}_X(A)$ defined by $t \vee u = \sup(t, u)$, $t \wedge u = \inf(t, u)$. $t \in \mathrm{PW}_X(A)$ is said to be sup-inf *definable on* X if $t = \bigvee_i \bigwedge_j a_{ij}$ for some (finite) set of elements $\{a_{ij}\} \subseteq A$. As is well-known [4, 5, 8, 10] (but non-trivial), the set of all sup-inf definable $t \in \mathrm{PW}_X(A)$ form a subring of $\mathrm{PW}_X(A)$ which we denote by $\mathrm{SI}_X(A)$. Again, if $X = \operatorname{Sper} A$, we denote $\mathrm{SI}_X(A)$ simply by $\mathrm{SI}(A)$.

Suppose $X \subseteq \operatorname{Sper} A$ and $I \subseteq A$ is an ideal such that $I \subseteq \bigcap_{\alpha \in X} p_\alpha$. Then, after identifying $X \subseteq \operatorname{Sper} A/I \subseteq \operatorname{Sper} A$ in the natural way, we have $\mathrm{PW}_X(A) \cong \mathrm{PW}_X(A/I)$, $\mathrm{SI}_X(A) \cong \mathrm{SI}_X(A/I)$.

More generally, suppose $p: A \longrightarrow B$ is any ring homomorphism and $X \subseteq \operatorname{Sper} A$, $Y \subseteq \operatorname{Sper} B$ satisfy $(*)$ $p^{-1}(\beta) \in X$ for all $\beta \in Y$. Since $R\big(p^{-1}(\beta)\big)$ is naturally identified with a subfield of $R(\beta)$, this yields, in a natural way, a ring homomorphism $p_{X,Y}: \prod_{\alpha \in X} R(\alpha) \longrightarrow \prod_{\beta \in Y} R(\beta)$. Moreover,

$$p_{X,Y}\big(\mathrm{PW}_X(A)\big) \subseteq \mathrm{PW}_Y(B) \text{ and } p_{X,Y}\big(\mathrm{SI}_X(A)\big) \subseteq \mathrm{SI}_Y(B).$$

EXAMPLE. If $B = A$ and $p =$ the identity, then the condition $(*)$ on X, Y just reads $Y \subseteq X$. In this case, $p_{X,Y}$ is just "restriction to Y".

We are interested in studying the relationship between $\mathrm{PW}_X(A)$ and $\mathrm{SI}_X(A)$ for arbitrary A, X. In particular, we look for conditions on A and X which insure $\mathrm{PW}_X(A) = \mathrm{SI}_X(A)$. Note: The restriction map $\mathrm{SI}(A) \longrightarrow \mathrm{SI}_X(A)$ is surjective. Thus, a necessary condition for $t \in \mathrm{PW}_X(A)$ to be in $\mathrm{SI}_X(A)$ is that t be in the image of $\mathrm{PW}(A) \longrightarrow \mathrm{PW}_X(A)$. As in [8], A is called *Pierce-Birkhoff* if $\mathrm{PW}(A) = \mathrm{SI}(A)$.

Let R be a real closed field and let $R[\mathbf{x}]$ denote the polynomial ring $R[x_1, \ldots, x_n]$ ($n \geq 1$ being fixed). For $C \subseteq R^n$ any semi-algebraic set, $\tilde{C} \subseteq \operatorname{Sper} R[\mathbf{x}]$ denotes the associated constructible set [1, 3, 7]. Since $C \hookrightarrow \tilde{C}$, it makes sense to consider the ring $\mathrm{PW}_C(R[\mathbf{x}])$. (Observe: $R(\alpha) = R$ for all $\alpha \in C$, so elements of $\mathrm{PW}_C(R[\mathbf{x}])$ are functions $t: C \longrightarrow R$.) The restriction mapping $\mathrm{PW}_{\tilde{C}}(R[\mathbf{x}]) \longrightarrow \mathrm{PW}_C(R[\mathbf{x}])$ is an isomorphism. The proof of this requires the Finiteness Theorem: One has to know that if $S \subseteq C$ is semi-algebraic and relatively closed in C, then $\tilde{S}$ is relatively closed in $\tilde{C}$.

If $C \subseteq R^n$ is an algebraic set, we denote by $R[C]$ the coordinate ring of C, i.e., $R[C] = R[\mathbf{x}]/\bigcap_{\alpha\in C} p_\alpha$. In this case, $\tilde{C}$ is naturally identified with $\operatorname{Sper} R[C]$ so we have canonical isomorphisms

$$\mathrm{PW}_C(R[\mathbf{x}]) \cong \mathrm{PW}_{\tilde{C}}(R[\mathbf{x}]) \cong \mathrm{PW}(R[C]).$$

Thus, for an algebraic set, $\mathrm{PW}_C(R[\mathbf{x}]) = \mathrm{SI}_C(R[\mathbf{x}])$ holds iff the ring $R[C]$ is Pierce-Birkhoff. In particular, the Pierce-Birkhoff Conjecture is equivalent to the conjecture that the polynomial ring $R[\mathbf{x}]$ is Pierce-Birkhoff.

2. **Separating ideals.** If $\gamma,\delta \in \operatorname{Sper} A$, the *separating ideal* of γ,δ is

$$\langle\gamma,\delta\rangle = (\gamma\cap-\delta) + (-\gamma\cap\delta).$$

THEOREM 2.1. *(1) $\langle\gamma,\delta\rangle$ is an ideal of A.*

(2) γ,δ have a common specialization in $\operatorname{Sper} A$ *iff $\langle\gamma,\delta\rangle \neq A$. In this case, $\alpha = \gamma + \sqrt{\langle\gamma,\delta\rangle} = \delta + \sqrt{\langle\gamma,\delta\rangle}$ is the least common specialization of γ,δ. Moreover, $p_\alpha = \sqrt{\langle\gamma,\delta\rangle}$ (so $\sqrt{\langle\gamma,\delta\rangle}$ is prime in this case).*

THEOREM 2.2. *Assume $X \subseteq$* $\operatorname{Sper} A$ *is Tychonoff closed and $t \in$* $\mathrm{PW}_X(A)$. *For each $\alpha \in X$, fix $t_\alpha \in A$ such that $t = t_\alpha$ at α. Then $t \in$* $\mathrm{SI}_X(A)$ *iff $t_\gamma - t_\delta \in \langle\gamma,\delta\rangle$ $\forall\gamma,\delta \in X$.*

For the proofs, see [8]. Note: Our definition of $\langle\gamma,\delta\rangle$ is different from that given in [8], but the *proof* in [8] shows both definitions are equivalent. Also, the statement of our Theorem 2.2 is slightly more general that the corresponding result in [8], but it is clear that the proof given in [8] goes through in this more general situation.

EXAMPLE. Suppose $C \subseteq R^n$ is s.a. and $t\colon C \longrightarrow R$ is piecewise polynomial. Thus $\exists f_1,\dots,f_k \in R[\mathbf{x}]$ and relatively closed s.a. sets $C_1,\dots,C_k \subseteq C$ such that $C = C_1 \cup \dots \cup C_k$ and $t = f_i$ on C_i, $i = 1,\dots,k$. Since constructibles are clopen in the Tychonoff topology, Theorem 2.2 applies to $\mathrm{PW}_C(R[\mathbf{x}]) \cong \mathrm{PW}_{\tilde{C}}(R[\mathbf{x}])$. Thus t is sup-inf definable on C iff $f_i - f_j \in \langle\gamma,\delta\rangle$ holds $\forall\gamma \in \tilde{C}_i$ and $\forall\delta \in \tilde{C}_j$, $i,j = 1,\dots,k$.

Recall that $X \subseteq \operatorname{Sper} A$ is closed iff X is Tychonoff closed and closed under specialization (e.g., see [1, Proposition 2.11]). Suppose X is closed, $t \in \mathrm{PW}_X(A)$. Suppose $\gamma,\delta \in X$, $\langle\gamma,\delta\rangle \neq A$, and let $\alpha \in \operatorname{Sper} A$ be the least common specialization of γ,δ. Thus $\alpha \in X$ so, at α, $t = t_\gamma = t_\delta = t_\alpha$. Since $p_\alpha = \sqrt{\langle\gamma,\delta\rangle}$, this means $t_\gamma - t_\delta \in \sqrt{\langle\gamma,\delta\rangle}$. Of course, this is also true if $\langle\gamma,\delta\rangle = A$. Thus

$$t \in \mathrm{PW}_X(A), X \text{ closed } \Rightarrow t_\gamma - t_\delta \in \sqrt{\langle\gamma,\delta\rangle}\ \forall\gamma,\delta \in X.$$

Thus $t_\gamma - t_\delta \in \langle\gamma,\delta\rangle$ is trivially true in cases where $\langle\gamma,\delta\rangle = \sqrt{\langle\gamma,\delta\rangle}$. This occurs, for example, if either (i) γ,δ have no common specialization or (ii) $\gamma \subseteq \delta$ or $\delta \subseteq \gamma$. Note: in regard to (ii), we always have $p_\gamma + p_\delta \subseteq \langle\gamma,\delta\rangle$.

For $\gamma \in \operatorname{Sper} A$, let B_γ be the convex hull of A/p_γ in $R(\gamma)$, i.e.,

$$B_\gamma = \{x \in R(\gamma) : \exists a \in A, |x| \leq \bar{a}\}.$$

(Here, $\bar{a} := a(\gamma)$.) B_γ is a valuation ring in $R(\gamma)$ with real closed residue field which we denote by $\bar{B}_\gamma$. Denote by β the ordering on A induced by the composite map $A \longrightarrow A/p_\gamma \subseteq B_\gamma \longrightarrow \bar{B}_\gamma$. Thus $A/p_\beta \subseteq R(\beta) \subseteq \bar{B}_\gamma$ and, by construction, $\bar{B}_\gamma$ is archimedian over A/p_β. As is well-known, β is the unique maximal specialization of γ. Actually, all our computations will take place in the quotient field $K(p_\gamma)$ of A/p_γ, so we could just as well work with the smaller valuation ring $B_\gamma \cap K(p_\gamma)$ in $K(p_\gamma)$.

Suppose now that γ is a proper generalization of some $\alpha \in \operatorname{Sper} A$ (so $\gamma \subset \alpha \subseteq \beta$). Assume the ideal $p_\alpha \subseteq A$ is finitely generated. Set

$$S_\gamma = \{a \in p_\alpha : v_\gamma(\bar{a}) \leq v_\gamma(\bar{b})\ \forall b \in p_\alpha\}.$$

Here, v_γ denotes the additive valuation on $R(\gamma)$ associated to B_γ. If δ is another proper generalization of α we say γ, δ *have the same direction at* α if $S_\gamma = S_\delta$ and all elements of $S_\gamma = S_\delta$ have the same sign at γ, δ. Otherwise, we say γ, δ *have different direction at* α. We say γ, δ *have opposite direction at* α if $S_\gamma = S_\delta$ and all elements of $S_\gamma = S_\delta$ have opposite sign at γ, δ.

THEOREM 2.3. *Hypothesis is as above. Then the following statements are equivalent:*
(1) $\langle \gamma, \delta \rangle \neq p_\alpha$.
(2) γ, δ *have the same direction at* α.

PROOF. (1) $\Rightarrow$ (2). For this it suffices to prove (a), (b):
(a) If $\exists x \in S_\gamma \cap S_\delta$ changing sign at γ, δ, then $\langle \gamma, \delta \rangle = p_\alpha$.
(b) If $S_\gamma \neq S_\delta$ then $\exists x \in S_\gamma \cap S_\delta$ changing sign at γ, δ.
To prove (a), suppose $y \in p_\alpha$. Let $z = ax - y$ where $a \in A$ is to be determined. In $R(\gamma)$ and $R(\delta)$, $\bar{z} = \bar{a}\bar{x} - \bar{y} = \bar{x}(\bar{a} - \bar{y}/\bar{x})$. Since both of $\bar{B}_\gamma$, $\bar{B}_\delta$ are archimedian over A/p_β, $\exists a \in A$ such that $\bar{a} - \bar{y}/\bar{x}$ is strictly positive both as an element of $R(\gamma)$ and as an element of $R(\delta)$. Thus $x, z = ax - y$ both change sign at γ, δ so $y = ax - (ax - y) \in \langle \gamma, \delta \rangle$. Thus $p_\alpha \subseteq \langle \gamma, \delta \rangle$ and consequently $\langle \gamma, \delta \rangle = p_\alpha$. To prove (b), suppose $y \in S_\gamma$, $y \notin S_\delta$. Pick any $z \in S_\delta$. We can assume $y \geq_\gamma 0$, $z \geq_\delta 0$. Let $x = ay - z$, $a \in A$. In $R(\gamma)$, $\bar{x} = \bar{y}(\bar{a} - \bar{z}/\bar{y})$ and, as above, we can choose $a \in A$ so that $\bar{a} - \bar{z}/\bar{y}$ is strictly positive in $R(\gamma)$ so $x \geq_\gamma 0$. In $R(\delta)$, $\bar{x} = -\bar{z}(1 - \bar{a}\bar{y}/\bar{z})$. Since $v_\delta(\bar{y}) > v_\delta(\bar{z})$, it is clear that $1 - \bar{a}\bar{y}/\bar{z}$ is strictly positive in $R(\delta)$ and consequently that $x \leq_\delta 0$.

(2) $\Rightarrow$ (1). Suppose (2) holds but (1) fails. Pick any $x \in S_\gamma$. Since $p_\alpha = \langle \gamma, \delta \rangle$, $x = y + z$ with y, z changing sign at γ, δ. In $R(\gamma)$, either $v_\gamma(\bar{y}) \leq v_\gamma(\bar{x})$ or $v_\gamma(\bar{z}) \leq v_\gamma(\bar{x})$. We may as well assume $v_\gamma(\bar{y}) \leq v_\gamma(\bar{x})$. Thus $y \in S_\gamma = S_\delta$. This contradicts the hypothesis that γ, δ have the same direction at α.

COROLLARY 2.4. *For A Noetherian of (Krull) dimension 1 and $X \subseteq \operatorname{Sper} A$ closed, the following are equivalent:*

(1) For each $\alpha \in X$ with p_α maximal, distinct proper generalizations of α in X have different directions at α.

(2) $\langle \gamma, \delta \rangle = \sqrt{\langle \gamma, \delta \rangle}\ \forall \gamma, \delta \in X$.

Also, if either (1) or (2) holds, then
(3) $\mathrm{PW}_X(A) = \mathrm{SI}_X(A)$.

EXAMPLE. Suppose A is a Dedekind domain with quotient field K. Then, for any $\alpha \in \mathrm{Sper}\, A$ with p_α maximal, the local ring A_{p_α} is a discrete valuation ring in K. Thus there are only two proper generalizations γ, δ of α, $K(p_\gamma) = K(p_\delta) = K$, $B_\gamma \cap K = B_\delta \cap K = A_{p_\alpha}$ and

$$S_\gamma = S_\delta = \{x \in A : x \text{ is a uniformizing parameter for } A_{p_\alpha}\}.$$

Clearly γ, δ have opposite direction at α.

3. **Application to curves.** Suppose $C \subseteq R^n$ is a closed 1-dimensional s.a. set, R real closed. Fix $\alpha \in C$ and denote by $\gamma \leftrightarrow b_\gamma$ the natural bijection between proper generalizations of α in $\tilde{C}$ and half-branches of C at α [3, Proposition 10.3.1]. To simplify notation, assume $\alpha = (0, \ldots, 0)$ so $p_\alpha = (x_1, \ldots, x_n)$. Suppose that $\gamma \in \tilde{C}$ generalizes α properly. Clearly $x_i \in S_\gamma$ for some i. Reindexing, we can assume $x_1 \in S_\gamma$. $K(p_\gamma)$ is a function field in one variable over R so $B_\gamma \cap K(p_\gamma)$ is a discrete valuation ring in $K(p_\gamma)$. Also, the residue field of $B_\gamma \cap K(p_\gamma)$ is a finite (ordered) extension of R so is equal to R. Thus, for each $i \in \{1, \ldots, n\}$, the image of $\bar{x}_i/\bar{x}_1 \in B_\gamma$ via the mapping $B_\gamma \longrightarrow \bar{B}_\gamma$ is some element $m_i \in R$. Either $x_1 >_\gamma 0$ or $x_1 <_\gamma 0$. Replacing x_1 by $-x_1$ if necessary, we can assume $x_1 >_\gamma 0$. Consider the half-line $T_\gamma \subseteq R^n$ defined by

$$T_\gamma = \{\mathbf{x} \in R^n : x_i = m_i x_1, i = 2, \ldots, n, x_1 > 0\}.$$

THEOREM 3.1. *Set-up as above. Then*
(1) T_γ *is the half-tangent to* b_γ *at* α.
(2) If $\delta \in \tilde{C}$ *is another proper generalization of* α, *then* γ, δ *have the same direction at* α *iff* $T_\gamma = T_\delta$.

PROOF. (1) Let $r \in R$, $r > 0$, and consider the cone

$$D_r = \{\mathbf{x} \in R^n : x_1 > 0 \text{ and } |x_i/x_1 - m_i| < r, i = 2, \ldots, n\}.$$

By definition of $m_2, \ldots, m_n$, $\gamma \in \tilde{D}_r$. On the other hand, by the correspondence between orderings and half-branches, γ is the unique ordering in $\bigcap\{\widetilde{b_\gamma(s)} : s \in R, s > 0\}$ where

$$b_\gamma(s) = \{\mathbf{x} \in R^n : \mathbf{x} \in b_\gamma \text{ and } 0 < \|\mathbf{x}\| < s\}.$$

If $b_\gamma(s) \backslash D_r \neq \emptyset$ for all $s > 0$ then, by compactness of $\mathrm{Sper}\, R[\mathbf{x}]$ in the Tychonoff topology, $\bigcap\{\widetilde{b_\gamma(s)} : s \in R, s > 0\} \backslash \tilde{D}_r \neq \emptyset$, contradicting the uniqueness of γ. Thus $\exists s \in R$, $s > 0$ such that $b_\gamma(s) \subseteq D_r$.

(2) Each $f \in p_\alpha = (x_1, \ldots, x_n)$ is expressible as

$$f = \sum_{i=1}^{n} a_i x_i + \text{ terms in } x_1, \ldots, x_n \text{ of degree } \geq 2,$$

$a_1, \ldots, a_n \in R$. Clearly $f \in S_\gamma \Leftrightarrow \sum_{i=1}^n a_i m_i \neq 0$ and, if $f \in S_\gamma$, then $f >_\gamma 0 \Leftrightarrow \sum_{i=1}^n a_i m_i > 0$. The result follows easily from these two facts.

COROLLARY 3.2. *Suppose C is a closed 1-dimensional semi-algebraic set in R^n. Then the following are equivalent:*

(1) For each point $\alpha \in C$, distinct half-branches of C at α have distinct half-tangents.

(2) $\langle \gamma, \delta \rangle = \sqrt{\langle \gamma, \delta \rangle}$ for all $\gamma, \delta \in \tilde{C}$.

(3) $\mathrm{PW}_C(R[\mathbf{x}]) = \mathrm{SI}_C(R[\mathbf{x}])$.

PROOF. (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) follows from Corollary 2.4 and Theorem 3.1. (3) $\Rightarrow$ (1): Suppose, at some point α on C we have half-branches $b_\gamma, b_\delta, \ldots$ with b_γ, b_δ sharing the same half-tangent at α. We can assume $\alpha = (0, \ldots, 0)$ so $p_\alpha = (x_1, \ldots, x_n)$ and that $x_1 \in S_\gamma = S_\delta$. Denote by $\bar{B}^n(\varepsilon)$ (resp. $S^{n-1}(\varepsilon)$) the closed ball (resp. sphere) in R^n with radius ε centered at $(0, \ldots, 0)$. Pick $\varepsilon > 0$ sufficiently small, let $\mathbf{a} = (a_1, \ldots, a_n)$ be the point of intersection of b_γ with $S^{n-1}(\varepsilon)$, pick $c \in R$, $0 < c < a_1$ and define $t \colon C \longrightarrow R$ by

$$t(\mathbf{x}) = \begin{cases} x_1 & \text{if } \mathbf{x} \in b_\gamma \cap \bar{B}^n(\varepsilon), \quad 0 \leq x_1 \leq c \\ \frac{c}{c-a_1}(x_1 - a_1) & \text{if } \mathbf{x} \in b_\gamma \cap \bar{B}^n(\varepsilon), \quad c \leq x_1 \leq a_1 \\ 0 & \text{elsewhere on } C. \end{cases}$$

Then $t \in \mathrm{PW}_C(R[\mathbf{x}])$ so t extends to $\tilde{t} \in \mathrm{PW}_{\tilde{C}}(R[\mathbf{x}])$. $\tilde{t}_\gamma - \tilde{t}_\delta = x_1 - 0 = x_1$. If $x_1 \in \langle \gamma, \delta \rangle$ then $x_1 = p - q$ where $p, q \in p_\alpha$ change sign at γ, δ. Since $x_1 \in S_\gamma$, one of p, q is in S_γ, say $p \in S_\gamma = S_\delta$. But this is impossible since γ, δ have the same direction at α. Thus $\tilde{t}_\gamma - \tilde{t}_\delta \notin \langle \gamma, \delta \rangle$ so $t \notin \mathrm{SI}_C(R[\mathbf{x}])$.

EXAMPLES. (1) If $\alpha \in C$ is a non-singular point (i.e., the local ring $R[C]_{p_\alpha}$ is a discrete valuation ring) then there are two half-branches at α and these have opposite direction.

(2) If $\alpha \in C$ is an isolated point, there are no half-branches at α so there is nothing to check.

(3) For each of the following plane curves, the origin is the only singular point. In (a), (b) distinct half-branches have distinct half-tangents, but this is not true in (c), (d), (e).

$(a)\quad y^3 = x^6 + x^8 \qquad (b)\quad y^2 = x^2 - x^3 \qquad (c)\quad y^2 = x^3$

$(d)\quad y^2 = x^4 + x^5 \qquad (e)\quad y^3 = x^5 - x^3 y.$

NOTE. If C is a curve, then by [3, Theorem 9.4.6], the number of half-branches at $\alpha \in C$ is always even. This is because there is natural pairing: Two proper generalizations $\gamma, \delta \in \tilde{C}$ of α, $\gamma \neq \delta$, are paired iff $p_\gamma = p_\delta$ (so $K(p_\gamma) = K(p_\delta)$) and $B_\gamma \cap K(p_\gamma) = B_\delta \cap K(p_\delta)$. Clearly, if γ, δ are paired, then γ, δ have the same (resp. opposite) direction at α iff elements of $S_\gamma = S_\delta$ have even (resp. odd) value.

4. **A relative version of the result.** Let K be a subfield of R and let $\bar{K} \subseteq R$ denote the real closure of K at the ordering α_0 of K induced by $K \subseteq R$. (e.g., we could take $R = \mathbb{R}$, $K = \mathbb{Q}$, so $\bar{K} = \mathbb{R}_{\mathrm{alg}}$.) Let $X = X(\alpha_0) \subseteq \operatorname{Sper} K[\mathbf{x}]$ denote the (closed) set of orderings on $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ extending α_0. Consider the restriction mappings

$$\psi \colon \operatorname{Sper} R[\mathbf{x}] \longrightarrow \operatorname{Sper} \bar{K}[\mathbf{x}], \qquad \rho \colon \operatorname{Sper} \bar{K}[\mathbf{x}] \longrightarrow X.$$

According to [1, Theorem 2.1], ρ is a homeomorphism. For any s.a. set $D \subseteq \bar{K}^n$, let $D(R) = \psi^{-1}(\tilde{D}) \cap R^n$. $D(R)$ is a s.a. set in R^n and $D(R) \cap \bar{K}^n = D$. A s.a. set $C \subseteq R^n$

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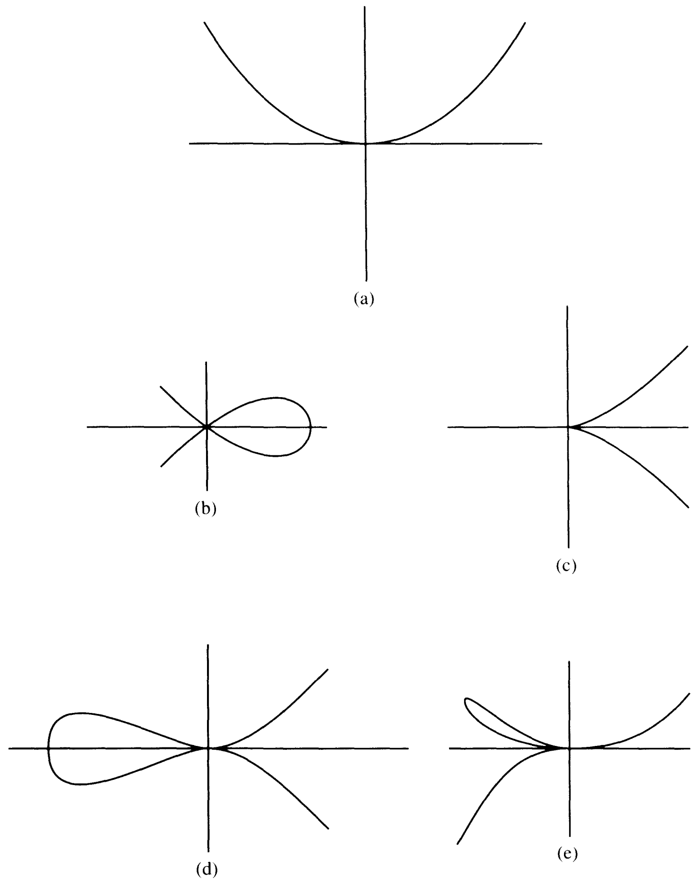

is said to be *defined over K* if $C = D(R)$ for some s.a. set $D \subseteq \bar{K}^n$. In this situation, if $E = \rho(\tilde{D})$, then the functorial mapping

$$p\colon \mathrm{PW}_E(K[\mathbf{x}]) \longrightarrow \mathrm{PW}_C(R[\mathbf{x}])$$

is injective. We say $t \in \mathrm{PW}_C(R[\mathbf{x}])$ is *defined over K* if t is in the image of p. This just means that $\exists$ relatively closed s.a. sets $C_1, \ldots, C_m \subseteq C$ defined over K and $g_1, \ldots, g_m \in K[\mathbf{x}]$ such that $C = \bigcup_{i=1}^m C_i$ and $t = g_i$ on C_i, $i = 1, \ldots, m$.

One can ask when each $t \in \mathrm{PW}_C(R[\mathbf{x}])$ defined over K is expressible as $t = \bigvee_i \bigwedge_j g_{ij}$ on C with $g_{ij} \in K[\mathbf{x}]$. Clearly this is the same as asking when the two rings $\mathrm{PW}_E(K[\mathbf{x}])$, $\mathrm{SI}_E(K[\mathbf{x}])$ are equal. Delzell considers this question in [4] in case $C = R^n$ (so $E = X$).

Here, we are interested in the case where C is closed and 1-dimensional. In this case, by the Transfer Principle, D is also closed and 1-dimensional. Moreover, if $b_1, \ldots, b_k$ are the half-branches of D at some point $\bar{\alpha} \in D$ and T_i is the half-tangent to b_i at $\bar{\alpha}$ then, by the Transfer Principle, $b_1(R),\ldots,b_k(R)$ are the half-branches of C at $\bar{\alpha}$ and $T_i(R)$ is the half-tangent to $b_i(R)$ at $\bar{\alpha}$.

Since $\bar{K}[\mathbf{x}]$ is integral over $K[\mathbf{x}]$ and D is 1-dimensional, $K[\mathbf{x}]/\bigcap_{\alpha\in E} p_\alpha$ is 1-dimensional so we can apply Corollary 2.4. But first we must determine the meaning of Condition (1) of Corollary 2.4. Suppose $\alpha \in E$, p_α is maximal, and $\gamma \in E$ is a proper generalization of α. Let $\bar{\alpha}$, $\bar{\gamma}$ be the corresponding orderings in $\tilde{D}$. Thus $\bar{\gamma}$ is a proper generalization of $\bar{\alpha}$ and $\bar{\alpha} \in D$, say $\bar{\alpha} = (a_1,\ldots,a_n)$. Thus $p_{\bar{\alpha}} = (x_1-a_1,\ldots,x_n-a_n)$. Since $\bar{K}[\mathbf{x}]$ is integral over $K[\mathbf{x}]$, $R(\bar{\gamma}) = R(\gamma)$ and $B_{\bar{\gamma}} = B_\gamma$. We can assume $v_\gamma(x_1-a_1) \leq v_\gamma(x_i-a_i)$ for $i \geq 2$ and that the half-tangent T_γ to $b_{\bar{\gamma}}$ at $\bar{\alpha}$ is given by $x_i - a_i = m_i(x_1-a_1)$, $i = 2,\ldots,n$, $x_1 - a_1 > 0$, $m_2,\ldots,m_n \in \bar{K}$. Take $f_1 \in K[x_1]$ to be the minimal polynomial of a_1. Then $f_1 = (x_1-a_1)\bar{f}_1$, $\bar{f}_1 \in \bar{K}[x_1]$, $\bar{f}_1(a_1) \neq 0$, so $v_\gamma(f_1) = v_\gamma(x_1-a_1)$. From this we see that

$$S_\gamma = \{f \in p_\alpha : v_\gamma(f) = v_\gamma(x_1-a_1)\}.$$

Also, any $f \in p_\alpha$ decomposes in $\bar{K}[\mathbf{x}]$ as $f = \sum_{i=1}^n D_{x_i}f(\mathbf{a})(x_i-a_i)$ + terms in $x_1 - a_1,\ldots,x_n-a_n$ of degree ≥ 2, so, dividing by x_1-a_1 and pushing down via $B_\gamma \longrightarrow \bar{B}_\gamma$, we see that

(a) $f \in S_\gamma$ iff $\sum_{i=1}^n D_{x_i}f(\mathbf{a})m_i \neq 0$ (taking $m_1 = 1$).

Moreover, since we are assuming $x_1 - a_1 > 0$ at $\bar{\gamma}$,

(b) if $f \in S_\gamma$, then $f >_\gamma 0$ iff $\sum_{i=1}^n D_{x_i}(\mathbf{a})m_i > 0$.

THEOREM 4.1. *Suppose $\delta \in E$ is another proper generalization of α. Then the following are equivalent:*

(1) γ,δ have different direction at α.

(2) There exists a hyperplane through $\bar{\alpha} = (a_1,\ldots,a_n)$ defined over $K(p_\alpha) = K(a_1,\ldots,a_n)$ separating T_γ and T_δ.

PROOF. ((1) $\Leftarrow$ (2)) Suppose γ,δ have the same direction at α. Then $v_\delta(x_i-a_i)$ is also minimal at $i = 1$ (since $f_1 \in S_\gamma = S_\delta$). Also, $x_1 - a_1$ has the same sign at $\bar{\gamma},\bar{\delta}$ (since f_1 has the same sign at γ,δ). Thus we can suppose T_δ is given by $x_i - a_i = n_i(x_1-a_1)$, $i = 2,\ldots,n$, $x_1 - a_1 > 0$. By (2) we have $g_1,\ldots,g_n \in K[\mathbf{x}]$ such that

(c) $\sum g_i(\mathbf{a})m_i > 0$, $\sum g_i(\mathbf{a})n_i < 0$.

Let $f_j \in K[x_j]$ be the minimal polynomial of a_j and let $h_j \in K[\mathbf{x}]$ be such that $h_j(\mathbf{a}) = g_j(\mathbf{a})/f_j'(a_j)$, and take $f = \sum_{j=1}^n h_jf_j$. Then $f \in p_\alpha$ and one checks easily that $D_{x_i}f(\mathbf{a}) = g_i(\mathbf{a})$, $i = 1,\ldots,n$. Thus, from (b) and (c), we see that $f \in S_\gamma = S_\delta$, $f >_\gamma 0$, $f <_\delta 0$, a contradiction.

((1) $\Rightarrow$ (2)) Assume γ,δ have different directions at α. After disposing of trivial cases where (2) is clear, we are left with the case where $v_\gamma(x_i-a_i)$, $v_\delta(x_i-a_i)$ both achieve their minimum at the same i (say $i = 1$) and $x_1 - a_1$ does not change sign at $\bar{\gamma},\bar{\delta}$, (say $x_1 - a_1 > 0$ at $\bar{\gamma},\bar{\delta}$). Now pick $f \in S_\gamma \cap S_\delta$, $f >_\gamma 0$, $f <_\delta 0$. Such f exists by the proof of Theorem 2.3. Then $\sum D_{x_i}f(\mathbf{a})m_i > 0$, $\sum D_{x_i}f(\mathbf{a})n_i < 0$ using (b). This proves (2).

We also have the following analogue of Corollary 3.2:

COROLLARY 4.2. *Suppose $C \subseteq R^n$ is a closed 1-dimensional s.a. set which is defined over K. Then the following are equivalent:*

(1) At each point $\mathbf{a} \in C \cap \bar{K}^n$, distinct half-branches of C have (distinct) half-tangents which are separated by a hyperplane through $\mathbf{a}$ defined over $K(\mathbf{a})$.

(2) $\langle \gamma, \delta \rangle = \sqrt{\langle \gamma, \delta \rangle}$ for all $\gamma, \delta \in E$.

(3) $\mathrm{PW}_E(K[\mathbf{x}]) = \mathrm{SI}_E(K[\mathbf{x}])$, i.e., for each $t \in \mathrm{PW}_C(R[\mathbf{x}])$, if t is defined over K, then $\exists g_{ij} \in K[\mathbf{x}]$ so that $t = \bigwedge_i \bigvee_j g_{ij}$ on C.

PROOF. $(1) \Leftrightarrow (2) \Rightarrow (3)$ follows from Corollary 2.4 and Theorem 4.1

$(3) \Rightarrow (1)$. If (1) fails, then by Theorem 4.1, we have $\alpha \in E$ with p_α maximal and proper generalizations $\gamma \neq \delta$ of α in E having the same direction at α. We can suppose $v_\gamma(x_i - a_i)$, $v_\delta(x_i - a_i)$ are both minimal at $i = 1$ and that $x_1 - a_1 > 0$ at $\bar{\gamma}, \bar{\delta}$. Let $\bar{\alpha} = (a_1, \ldots, a_n)$ and let $f_1 \in K[x_1]$ denote the minimal polynomial of a_1. Denote by $\bar{B}^n(\varepsilon)$ (resp. $S^{n-1}(\varepsilon)$) the closed ball (resp. sphere) in R^n with radius ε centered at $\bar{\alpha}$. Pick $\varepsilon > 0$ in $\bar{K}$ sufficiently small, and let $\mathbf{b} = (b_1, \ldots, b_n) \in \bar{K}^n$ be the point of intersection of $b_{\bar{\gamma}}$ with $S^{n-1}(\varepsilon)$. Let $p \in K[x_1]$ be the minimal polynomial of b_1 and choose $c \in \bar{K}$, $a_1 < c < b_1, p(c) \neq 0$, and let $q \in K[x_1]$ be the minimal polynomial of c. Let $r, s \in K[x_1]$ be such that $rp + sq = 1$. Define $t\colon C \longrightarrow R$ by

$$t(\mathbf{x}) = \begin{cases} f_1(x_1) & \text{if } \mathbf{x} \in b_{\bar{\gamma}} \cap \bar{B}^n(\varepsilon),\ a_1 \leq x_1 \leq c \\ f_1(x_1)r(x_1)p(x_1) & \text{if } \mathbf{x} \in b_{\bar{\gamma}} \cap \bar{B}^n(\varepsilon),\ c \leq x_1 \leq b_1 \\ 0 \text{ elsewhere on } C. & \end{cases}$$

Then $t \in \mathrm{PW}_E(K[\mathbf{x}])$. Also, $f_1 \in S_\gamma = S_\delta$ so, as in the proof of Corollary 3.2, $t \notin \mathrm{SI}_E(K[\mathbf{x}])$.

Condition (1) of Corollary 4.2 requires further comment. It clearly holds at non-singular points $\mathbf{a} \in C \cap \bar{K}^n$ (since then the half-tangent have opposite direction so are separated by some hyperplane $x_i - a_i = 0$). Also, if K is dense in $\bar{K}$, then the statement of Corollary 4.2 can be simplified:

THEOREM 4.3. *If K is dense in $\bar{K}$ then Condition (1) of Corollary 4.2 is equivalent to the simpler condition:*

(1′) At each point $\mathbf{a} \in C \cap \bar{K}^n$, distinct half-branches of C have distinct half-tangents.

PROOF. Ignoring trivial cases, we can suppose the half-tangents T_j, $j = 1, 2$ are given by $x_i - a_i = m_{ij}(x_1 - a_1)$, $i = 2, \ldots, n$, $x_1 - a_1 > 0$. Then, picking i minimal with $m_{i1} \neq m_{i2}$, T_1, T_2 are separated by any hyperplane of the form $x_i - a_i = b(x_1 - a_1)$ where $b \in K$ is between m_{i1} and m_{i2}.

We conclude by giving an example to show the density assumption in Theorem 4.3 cannot be deleted.

EXAMPLE. Suppose K is not dense in $\bar{K}$. Thus $\exists m_1, m_2 \in \bar{K}$, $m_1 < m_2$ such that the interval (m_1, m_2) has empty intersection with K. Let $f, g \in K[x]$ be the minimal polynomials of m_1, m_2 respectively. We can assume $f \neq g$. (Just replace m_2 by some $m \in \bar{K}$, $m_1 < m < m_2$ which is not a root of f.) Consider $h(x, y) \in K[x, y]$ defined by

$$h(x, y) = x^d f(y/x) x^e g(y/x) + (y - nx)^{d+e+1}$$

where $d = \deg f$, $e = \deg g$, and $n \in K$ is any fixed element which is not a root of fg. Take $C \subseteq R^2$ to be the curve defined by $h(x,y) = 0$. C is irreducible and (0,0) is its only singularity. Moreover, (0,0) is an ordinary multiple point. Thus, the branches of C at (0,0) correspond bijectively to the roots in $\bar{K}$ of fg in such a way that if $m \in \bar{K}$ is a root of fg then the associated branch passes through (0,0) with tangent $y = mx$. In particular, we have two half-branches with half-tangents $y = m_1 x$, $x > 0$ and $y = m_2 x$, $x > 0$ respectively. Since $(m_1, m_2) \cap K = \emptyset$, it is clear that these half-tangents are not separated by any hyperplane through $(0, 0)$ defined over K.

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