It is shown that for each point $p$ in the interior of the polynomial hull of a disc fibration $X$ over the unit sphere $\partial B^n$ there exists an $H^\infty$ analytic disc with boundary in $X$ and passing through $p$.

1. INTRODUCTION

Let $B^n = \{ z \in \mathbb{C}^n ; |z| < 1 \}$ be the open unit ball centred at the point 0 in the $n$ dimensional complex space $\mathbb{C}^n$. Let $\varphi$ be a continuous function on the unit sphere $\partial B^n$. In this short note we investigate the presence of analytic discs in the polynomial hull of the compact set

$$X = \{(z, w) \in \partial B^n \times \mathbb{C}; |w| \leq e^{-\varphi(z)}\}$$

fibred over $\partial B^n$. Recall that the polynomial hull $\widetilde{K}$ of a compact set $K \subseteq \mathbb{C}^m$ is defined as

$$\widetilde{K} = \left\{ z \in \mathbb{C}^m ; |p(z)| \leq \max_{K} |p| \text{ for every polynomial } p \text{ in } m \text{ variables} \right\}$$

and that by the maximum principle the image $F(\Delta)$ of every $H^\infty$ holomorphic mapping $F : \Delta \to \mathbb{C}^m$ with boundary in $K$, that is, $F^*(e^{i\theta}) \in K$ for almost every $\theta$, belongs to the polynomial hull $\widetilde{K}$ of $K$. For a bounded holomorphic mapping $F$ on $\Delta$ the notation $F^*$ is used to denote its almost everywhere defined boundary values.

Let $\Phi$ be the maximal plurisubharmonic function on $\mathbb{B}^n$, continuous on $\overline{\mathbb{B}}^n$, such that $\Phi|_{\partial \mathbb{B}^n} = \varphi$, that is, $\Phi$ is the unique solution on $\mathbb{B}^n$ of the Dirichlet problem for the Monge-Ampère operator [5]

$$\Phi \in \text{PSH}(\mathbb{B}^n) \cap L^\infty_{\text{loc}}(\mathbb{B}^n)$$
$$\left(d \, d^c \Phi\right)^n = 0 \text{ on } \mathbb{B}^n$$
$$\Phi|_{\partial \mathbb{B}^n} = \varphi \text{ on } \partial \mathbb{B}^n.$$

(1.1)

It is a classical result [4, p.99] that the polynomial hull of the set $X$ is

$$\widetilde{X} = \{(z, w) \in \overline{\mathbb{B}}^n \times \mathbb{C}; |w| \leq e^{-\Phi(z)}\}.$$

In this note we show that the interior of the polynomial hull $\widetilde{X}$ contains a lot of analytic discs with boundaries in $X$. More precisely, we prove the following statement:

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**PROPOSITION 1.1.** For each point \((z_0,w_0) \in \text{Int}(\hat{X})\) there exists an \(H^\infty\) analytic disc \(F : \Delta \to \mathbb{C}^n \times \mathbb{C}\) with boundary in \(X\) such that \(F(0) = (z_0, w_0)\).

Some remarks are in order. In the case \(n = 1\), much more was proved in the series of papers [1, 3, 9, 10]. Using the graphs of analytic functions on \(\Delta = \mathbb{B}^1\) with boundaries in \(X\), a complete description of the polynomial hull of a fibration over the unit circle was given for geometrically much more complicated fibres, for example, in [10] it was only assumed that each fibre over the unit circle is a simply connected continuum.

In higher dimensions related results were proved by Whittlesey in [11] for disc fibrations over \(\partial \mathbb{B}^2\) of the form

\[
X = \left\{ (z,w) \in \partial \mathbb{B}^2 \times \mathbb{C} ; |w - \alpha(z)| \leq R(z) \right\},
\]

where \(\alpha\) is a continuous complex valued function on \(\partial \mathbb{B}^2\) and \(R \in C^2(\partial \mathbb{B}^2)\) a positive real function such that \(|\alpha(z)| \leq R(z), z \in \partial \mathbb{B}^2\). Working with the assumption that \((\mathbb{B}^2 \times \mathbb{C}) \setminus \hat{X}\) is a pseudoconvex domain, it was proved in [11] that the polynomial hull of \(X\) can be foliated by the graphs of analytic balls. On the other hand, there are examples of maximal plurisubharmonic functions on \(\mathbb{B}^n\) for which for certain points \(z \in \mathbb{B}^n\) there is no germ \(V\) of an analytic variety containing \(z\) and such that \(\Phi|_V\) is harmonic, for example, Sibony's example [2, p.73]. Therefore, in view of these examples and our result, one can not, in general, expect to get a foliation of the whole \(\hat{X}\) with analytic discs. Namely, if \((\text{in the case } \alpha = 0)\) there exists a nontrivial analytic disc \(F = (f, g) : \Delta \to \hat{X} \subseteq \mathbb{B}^n \times \mathbb{C}\) such that its image \(F(\Delta)\) touches \(\partial \hat{X} \cap (\mathbb{B}^n \times \mathbb{C})\), that is, if there exists \(z_0 \in \Delta\) such that \(F(z_0) \in \partial \hat{X} \cap (\mathbb{B}^n \times \mathbb{C})\), then, by the maximum principle for the subharmonic function

\[
\xi \longmapsto |g(\xi)|e^{\Phi(f(\xi))}
\]
on \(\Delta\), we actually have \(F(\Delta) \subseteq \partial \hat{X} \cap (\mathbb{B}^n \times \mathbb{C})\). Hence \(g(\xi) \neq 0\) and

\[
\Phi(f(\xi)) = -\log |g(\xi)|
\]
on \(\Delta\). Therefore \(\Phi|_{f(\Delta)}\) is harmonic.

We also observe that in the case \(\alpha = 0\) the complement \((\mathbb{B}^n \times \mathbb{C}) \setminus \hat{X}\) is pseudoconvex if and only if the function \(\Phi\) is pluriharmonic on \(\mathbb{B}^n\). In this case a foliation of \(\hat{X}\) by the graphs of analytic balls is obvious: since \(\Phi\) is pluriharmonic on \(\mathbb{B}^n\), there exists an analytic function \(H : \mathbb{B}^n \to \mathbb{C}\) such that \(\Phi = \text{Re}(H)\). Then the graphs of the family of analytic functions on the ball \(H\xi(z) := \xi e^{H(z)}\), \(\xi \in \Delta\), form a foliation of \(\hat{X}\).

2. PROOF OF THE PROPOSITION

The proof of the proposition uses Poletsky's characterisation of the maximal plurisubharmonic function with the given continuous boundary data. It was proved in [6, 7]
that for \( z \in \mathbb{B}^n \) (one may replace \( \mathbb{B}^n \) by any smoothly bounded strongly pseudoconvex domain \( \Omega \subseteq \mathbb{C}^n \)) the value \( \Phi(z) \) of the solution \( \Phi \) of the problem \((1.1)\) is given by

\[
(2.1) \quad \Phi(z) = \inf_{f} \frac{1}{2\pi} \int_{0}^{2\pi} \varphi(f^*(e^{i\theta})) \, d\theta,
\]

where the infimum is taken over all holomorphic mappings of the unit disc \( f : \Delta \to \mathbb{B}^n \) with \( f(0) = z \) and whose boundary values \( f^* \) satisfy \( f^*(e^{i\theta}) \in \partial\mathbb{B}^n \) for almost every \( \theta \).

Recall that

\[
\hat{X} = \{(z, w) \in \overline{\mathbb{B}}^n \times \mathbb{C}; |w| \leq e^{-\Phi(z)}\}
\]

and let \((z_0, w_0) \in \text{Int}(\hat{X})\). Because each fibre

\[
\hat{X}_z = \{w \in \mathbb{C}; |w| \leq e^{-\Phi(z)}\}
\]

is a disc in the complex plane with centre at 0, we have that \( w_0 = \eta e^{-\Phi(z_0)} \) for some \( \eta \in \Delta \). By using a rotation in \( \mathbb{C} \), we may assume that

\[
(2.2) \quad w_0 = t e^{-\Phi(z_0)} \in \mathbb{R}
\]

for some \( t \in [0, 1) \).

Let \( \varepsilon > 0 \) be so small that

\[
t e^{-\varphi(z)} + |w_0|(1 - e^{-\varepsilon}) < e^{-\varphi(z)}
\]

for every \( z \in \partial\mathbb{B}^n \). By Poletsky’s theorem \((2.1)\) there exists a holomorphic mapping \( f : \Delta \to \mathbb{B}^n \) such that \( f(0) = z_0, f^*(e^{i\theta}) \in \partial\mathbb{B}^n \) for almost every \( \theta \) and

\[
(2.3) \quad \Phi(z_0) \leq \frac{1}{2\pi} \int_{0}^{2\pi} \varphi(f^*(e^{i\theta})) \, d\theta \leq \Phi(z_0) + \varepsilon.
\]

Let \( u(e^{i\theta}) := \varphi(f^*(e^{i\theta})) \) and let \( P[u] \) denote its Poisson integral

\[
P[u](\xi) = \frac{1}{2\pi} \int_{0}^{2\pi} \text{Re} \left( \frac{e^{i\theta} + \xi}{e^{i\theta} - \xi} \right) u(e^{i\theta}) \, d\theta.
\]

Since \( \varphi \) is a continuous function on \( \partial\mathbb{B}^n \), \( u \) is a bounded measurable function on \( \partial\Delta \) and hence \( P[u] \) has the nontangential limit \( u(e^{i\theta}) \) for almost every \( \theta \), \([8]\). Let \( H[u] \) be the harmonic conjugate of \( P[u] \) on \( \Delta \) such that \( H[u](0) = 0 \). Although the function \( H[u] \) is not necessarily bounded on \( \Delta \), this is the case for the function

\[
(2.4) \quad g(\xi) = t e^{-(P[u](\xi)+i H[u](\xi))},
\]

which is holomorphic on \( \Delta \). For this function we have \( g(0) = t e^{-P[u](0)} \) and the value \( P[u](0) \) is given as the integral average of \( u \) over \( \partial\Delta \). Thus

\[
P[u](0) = \frac{1}{2\pi} \int_{0}^{2\pi} u(e^{i\theta}) \, d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} \varphi(f^*(e^{i\theta})) \, d\theta.
\]
The inequalities (2.3) and assumption (2.2) imply
\[ w_0 e^{-\varepsilon} \leq g(0) \leq w_0 \]
and so
\[ |w_0 - g(0)| \leq |w_0|(1 - e^{-\varepsilon}). \]
Hence for the boundary values \( g^* \) we have
\[ |g^*(e^{i\theta}) + (w_0 - g(0))| \leq t e^{-\bar{u}(e^{i\theta})} + |w_0|(1 - e^{-\varepsilon}) \leq e^{-\sigma(f^*(e^{i\theta}))} \]
for almost every \( \theta \) and the holomorphic disc
\[ \xi \in \Delta \mapsto F(\xi) = \left( f(\xi), g(\xi) + (w_0 - g(0)) \right) \]
is such that \( F^*(e^{i\theta}) \in X \) for almost every \( \theta \) and \( F(0) = (z_0, w_0) \).

REFERENCES


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