Criteria for an analytic function $f$ defined in $|z| < 1$ to belong to $B_0$, the class of Bloch functions satisfying

$$\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0,$$

and criteria for a meromorphic function $g$ defined in $|z| < 1$ to belong to $N_0$, namely, to satisfy

$$\lim_{|z|\to 1} \frac{|g'(z)|}{1 + |g(z)|^2} = 0,$$

are obtained in terms of the area and the length of the images of hyperbolic disks and hyperbolic circles, respectively.

§1.

Let $f$ be a holomorphic function in the unit disk $D = \{z| |z| < 1\}$ of the complex plane $\mathbb{C} = \{z| |z| < \infty\}$. Let $B$ be the family of holomorphic functions $f$ in $D$ such that

$$\sup_{z \in D} (1 - |z|^2)|f'(z)| < \infty$$

and $B_0$ the family of holomorphic functions $f$ in $D$ such that

$$\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0.$$

If $f \in B$, then $f$ is said to be a Bloch function. In Theorem 1 we shall propose some criteria for $f$ to belong to $B_0$. These criteria are

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immediate consequences of Yamashita's Theorem in [4].

Let
\[ d(z,w) = \frac{1}{2} \log \frac{|1 - zw| + |z - w|}{|1 - \overline{z}w| - |z - w|} \]
be the hyperbolic distance between \( z \) and \( w \) in \( D \). For \( 0 < r < \infty \) and for \( z \in D \), we set
\[ U(z,r) = \{ w \in D | \ d(w,z) < r \} \]
and
\[ \Gamma(z,r) = \{ w \in D | \ d(w,z) = r \}. \]

Let \( A_f(z,r) \) be the euclidean area of the Riemannian image \( F(z,r) \) of \( U(z,r) \) by \( f \), and let \( A_f(z,r) \) be the euclidean area of the image \( F(z,r) \) of \( U(z,r) \) by \( f \); we note that \( F(z,r) \) is the projection of \( F(z,r) \) to \( \mathbb{C} \). Let \( L_f(z,r) \) be the euclidean length of the Riemannian image of \( \Gamma(z,r) \) by \( f \), and \( L_f(z,r) \) the euclidean length of the outer boundary of \( F(z,r) \). The outer boundary of a bounded domain \( G \) in \( \mathbb{C} \) means the boundary of \( \mathbb{C} \setminus E \), where \( E \) is the unbounded component of the complement \( \mathbb{C} \setminus G \) of \( G \). The inequalities
\[ A_f(z,r) \geq A_f(z,r) \quad \text{and} \quad L_f(z,r) \geq L_f(z,r) \]
hold for each \( 0 < r < \infty \) and each \( z \in D \).

Yamashita proved the following:

**THEOREM A.** Let \( f \) be non-constant and holomorphic in \( D \). Then the following are mutually equivalent:

1. \( f \in B; \)
2. there exists \( 0 < r < \infty \) such that \( \sup_{z \in D} A_f(z,r) < \infty; \)
3. there exists \( 0 < r < \infty \) such that \( \sup_{z \in D} A_f(z,r) < \infty; \)
4. there exists \( 0 < r < \infty \) such that \( \sup_{z \in D} L_f(z,r) < \infty; \)

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(V) there exists \( 0 < r < \infty \) such that \( \sup_{z \in D} L_f(z,r) < \infty \).

From this theorem we obtain

**THEOREM 1.** Let \( f \) be non-constant and holomorphic in \( D \). Then the following are mutually equivalent:

(I) \( f \in B_0 \);

(II) there exists \( 0 < r < \infty \) such that \( \lim_{|z| \to 1} A_f(z,r) = 0 \);

(III) there exists \( 0 < r < \infty \) such that \( \lim_{|z| \to 1} A_f(z,r) = 0 \);

(IV) there exists \( 0 < r < \infty \) such that \( \lim_{|z| \to 1} L_f(z,r) = 0 \);

(V) there exists \( 0 < r < \infty \) such that \( \lim_{|z| \to 1} L_f(z,r) = 0 \).

**Proof.** The assertions follow immediately from the proof of Theorem A in [4] by replacing the bounded term by a sequence of terms converging to zero.

§2.

The meromorphic analogue of a Bloch function is a normal meromorphic function. A function \( f \), meromorphic in \( D \), is said to be normal in \( D \) if \( \sup_{z \in D} (1 - |z|^2)f^*(z) < \infty \) where \( f^*(z) = |f'(z)|/(1 + |f(z)|^2) \) is the spherical derivative of \( f \) (cf. [3]). We denote by \( N \) the family of all normal meromorphic functions in \( D \). Further, let \( N_0 \) be the family of meromorphic functions \( f \) in \( D \) such that

\[
\lim_{|z| \to 1} (1 - |z|^2)f^*(z) = 0 .
\]

The euclidean area and length used in the above theorems will be replaced by the spherical area and spherical length. We shall denote the spherical area of \( F(z,r) \) by \( B_f(z,r) \) and the spherical area of \( F(z,r) \) by \( B_f(z,r) \). Let \( M_f(z,r) \) be the spherical length of the
Riemannian image of $\Gamma(z,r)$ by $f$, and let $M_f(z,r)$ be the length of the boundary of $F(z,r)$. The corresponding inequalities as above are valid, that is,

1. $B_f(z,r) \geq B_f(z,r)$ and $M_f(z,r) \geq M_f(z,r)$.

For normal meromorphic functions we cannot obtain results corresponding to those in Theorem A, as shown by Yamashita in [4]. For example, implication (III) $\Rightarrow$ (I) does not hold as Lappan has shown in [2] and the implication (V) $\Rightarrow$ (I) is still open. Therefore it is interesting to notice that the meromorphic analogue of Theorem 1 for the functions of $N_0$ is valid.

For the proof of our theorem we shall make use of the following lemma [1, Lemma II]:

**LEMMA.** For the function $g$ meromorphic in $D$ suppose that the spherical area $B_g(0,r)$ is strictly less than $\pi$. Then,

$$g^*(0) \leq \frac{B_g(0,r)}{\arctan(1 - \frac{B_g(0,r)}{\pi})},$$

where $x = (e^{2r} - 1)/(e^{2r} + 1)$.

**THEOREM 2.** Let $f$ be non-constant and meromorphic in $D$. Then the following are mutually equivalent:

1. $f \in N_0$;
2. There exists $0 < r < \infty$ such that $\lim_{|z| \to 1} B_f(z,r) = 0$;
3. There exists $0 < r < \infty$ such that $\lim_{|z| \to 1} B_f(z,r) = 0$;
4. There exists $0 < r < \infty$ such that $\lim_{|z| \to 1} M_f(z,r) = 0$ and $B_f(z,r) \leq \alpha < \pi$ for all $z$, $r_0 < |z| < 1$;
5. There exists $0 < r < \infty$ such that $\lim_{|z| \to 1} M_f(z,r) = 0$ and $B_f(z,r) \leq \alpha < \pi$ for all $z$, $r_0 < |z| < 1$. 


Proof. We prove first (III) ⇒ (I); let

$$g(ω) = f\left(\frac{ω + z}{1 + 2ω}\right).$$

By the assumption there is a $r_0 > 0$ such that $B_f(z, r) < π$ for all $z, r_0 < |z| < 1$. Let $|z| > r_0$. Then by a simple calculation and the Lemma we have

$$(1 - |z|^2) f^*(z) = g^*(0) \leq \left\{ \frac{B_f(0, r)}{\pi^2 (1 - \frac{B_f(0, r)}{π})} \right\}^{1/2} \leq \left\{ \frac{B_f(z, r)}{\pi^2 (1 - \frac{B_f(z, r)}{π})} \right\}^{1/2},$$

where $x = (e^{2r} - 1)/(e^{2r} + 1)$. Hence (III) ⇒ (I). (I) ⇔ (II) Yamashita has proved this result in [5]. (II) ⇒ (IV) By the above equivalence it is sufficient to prove that (I) ⇒ (IV). We choose a sequence of points $(z_n)$ for which $|z_n| \to 1$ as $n \to \infty$. Let $r > 0$.

We take the sequence of hyperbolic disks $(U(z_n^r, r))$ and form the functions

$$f_n(ξ) = f\left(\frac{ξ + z_n}{1 + \overline{z_n} ξ}\right).$$

Let $ξ_0 \in Γ(0, r)$ and let $z'_n = (ξ_0 + z_n)/(1 + \overline{z_n} ξ_0)$. The radius of $D$ going through $z_n$ intersects $Γ(z_n^r, r)$ in two points. We denote by $z_n''$ the point for which $|z_n''| < |z_n|$. Then we obtain for the spherical derivative

$$f_n^*(ξ_0') = \frac{1}{1 - \delta(z_n, z_n')^2} \cdot (1 - |z_n'|^2) f^*(z_n') \leq \frac{1}{\alpha (1 - |z_n''|^2)} \max_{z \in Γ(z_n^r, r)} f^*(z) = \frac{1}{\alpha (1 - |z_n'|^2)} f^*(z_n'').$$
where $z''_n \in \Gamma(z_n^r)$ and $1 - \delta(z_n^r, z'_n)^2 = 1 - \left| \frac{z_n^r - z'_n}{1 - \overline{z}_n^r z'_n} \right|^2 \geq \alpha > 0$, since
\[
d(z_n^r, z'_n) = d(0, z_0') = r.
\]
Now
\[
M_j(z_n^r, z'_n) = \int_{\Gamma(z_n^r, z'_n)} f^*_n(z) |dz| = \int_{\Gamma(0, r)} f^*_n(\zeta) |d\zeta|
\]
\[
= \frac{1}{\alpha} (1 - |z''_n|^2) f^*_n(z''_n) \int_{\Gamma(0, r)} |d\zeta|
\]
\[
= \frac{\pi}{\alpha} \log \frac{1 + r}{1 - r} \frac{1 - |z''_n|^2}{1 - |z''_n|^2} (1 - |z''_n|^2) f^*_n(z''_n) \to 0 ,
\]
since $|z''_n| \to 1$ and $\frac{1 - |z''_n|^2}{1 - |z''_n|^2} \to 1$.

The latter part of the assertion follows from the assumption (II).

(IV) $\Rightarrow$ (V) : This follows trivially from (1). (V) $\Rightarrow$ (III) : Let $(z'_n)$ be any sequence of points for which $|z'_n| \to 1$ as $n \to \infty$. Then, for sufficiently large $n$, either the diameter of $F(z_n^r, r)$

(2)
\[
diam F(z_n^r, r) \leq M_j(z_n^r, r)
\]
or the complement $\hat{\zeta} \setminus F(z_n^r, r)$ is divided into the components $E_i(z_n^r, r), i \in I$ ($I$ an index set) for which
\[
\sum_{i \in I} diam E_i(z_n^r, r) \leq M_j(z_n^r, r).
\]
When $n$ is large enough, the latter alternative is not possible by the assumption $B_j(z, r) \leq \alpha < \pi$. The assertion follows by (2) and thus the theorem is proved.

References


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