EXISTENCE OF SOLUTIONS OF OSEEN TYPE INTEGRAL EQUATIONS

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Integral equations of Oseen type are first kind Fredholm
equations whose kernels have a logarithmic singularity.
They arise in exterior boundary value problems in fluid
flow and heat transfer. Subject to the assumption of
uniqueness of solutions of the parent exterior boundary
value problem, solutions of the Oseen type integral
equations are shown to exist.

1. Introduction

Integral equation methods are being used with increasing
frequency to solve problems in applied mathematics. Given some initial/
boundary value problem, an integral equation is set up whose solution
can be used to generate the solution of the parent problem. Unfortunately
there are examples (Ursell [9], Jaswon and Symm [5]) where an integral
equation may fail to have a solution even though there is no difficulty
with the underlying boundary value problem. It is therefore desirable
to have available an existence theorem, particularly when an exact
solution cannot be determined so that asymptotic or numerical techniques
are needed.

In this note a class of problems is considered which can be
reduced to Fredholm integral equations of the first kind. The kernels
of the integral equations are weakly singular with a logarithmic
singularity; the integral equations arise in exterior boundary value
problems in which Oseen's approximation (described in Section 2) is

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used. It is shown that provided the boundary value problem has no more
than one solution, the corresponding integral equation always has a
solution (within some suitable Hilbert space, for a class of prescribed
functions which includes the physically reasonable cases; existence of
solutions of the boundary value problem then follows).

The results are proved by expressing the integral operator as
the sum of a singular part plus the regular part. An explicit
representation of the inverse operator for the singular part is used
to obtain an equivalent Fredholm equation of the second kind. The new
integral operator is shown to be compact and the Fredholm alternative
theorem, combined with the uniqueness theorem for the boundary value
problem are used to deduce the existence of solutions of the original
integral equation.

2.   Oseen type integral equations and
some related boundary value problems

There are many problems in viscous flow, heat transfer, chemical
reaction theory and so on which can be classified as steady diffusion-
convection problems. A field quantity such as heat or vorticity is
produced at internal surfaces and is distributed around the region
exterior to the boundaries by the processes of diffusion and forced
convection. Certain conservation principles may be used to construct
a system of partial differential equations satisfied by the field
quantities. Often the field quantity of primary interest affects the
externally driven flow causing a coupling of the partial differential
equations; the coupling may occur non-linearly rendering the equations
intractable in many important situations (non-linearities can also arise
through the influence of field quantities on the diffusion coefficients,
but these are not considered here). An important linearisation known in
viscous flow as Oseen's approximation [8], can be applied to problems
in which the forced flow field is unidirectional with constant velocity.
In this approximation the perturbation to the external flow field is
ignored in certain of the convection terms; in two-dimensional heat
problems one obtains (see [1])

(2.1) $\operatorname{div}(-\kappa \underline{\nabla} T + \underline{q} T) = 0$, $\underline{q} = U\hat{\underline{\imath}}$

which imply that the temperature T satisfies

(2.2)
$$-\nabla^2 T + \frac{U}{\kappa} \frac{\partial T}{\partial x} = 0 ,$$

while in two-dimensional viscous flow the vorticity ω and stream function ψ satisfy

(2.3)
$$\text{div}(-\nu\underline{\nabla}\omega + \underline{q}\omega) = 0 , \quad -(\nabla^2\psi)\underline{\hat{k}} = \omega\underline{\hat{k}} , \quad \underline{q} = \text{curl}(\psi\underline{\hat{k}})$$

so with $\underline{q} = U\underline{\hat{i}}$ in (2.3a) the stream function satisfies

(2.4)
$$\nabla^2\left(\nabla^2\psi - \frac{U}{\nu} \frac{\partial\psi}{\partial x}\right) = 0 .$$

(κ, ν and U are respectively, thermal diffusivity, viscosity, and free stream velocity.)

Problems which involve a flat plate set tangential to a uniform stream are particularly appropriate to this approximation. Green's functions which take account of the far field boundary conditions may be determined for these problems enabling the temperature field and velocity fields corresponding to equations (2.2) and (2.4) to be represented as follows,

(2.5)
$$T(x,y) = \int_{-1}^{1} f(t)e^{\lambda(x-t)}K_0\left(\lambda[(x-t)^2 + y^2]^{\frac{1}{2}}\right)dt ,$$

and

(2.6) $\displaystyle u - iv = \frac{\partial\psi}{\partial y} + i\frac{\partial\psi}{\partial x} = \int_{-1}^{1} f(t)\left[\lambda e^{\lambda(x-t)}\{K_0(\lambda r)+K_1(\lambda r)e^{-i\theta}\} - \frac{e^{-i\theta}}{r} \right]dt ,$

where $r\cos\theta = x-t$, and $r\sin\theta = y$.

[The coordinates have been scaled so that the flat plate, of length 2ℓ in the unscaled system, is placed at $y = 0$, $-1 \leq x \leq 1$. The constant λ is a non-dimensional quantity given by

$$\lambda = \frac{U\ell}{\kappa} , \quad \text{or} \quad \frac{U\ell}{\nu}$$

which is commonly called the Peclet number in heat transfer or Reynold's number in viscous flow; it characterises the ratio of the convective flux to diffusive flux, of heat, or circulation.]

Applying the boundary conditions on $[-1,1]$, (prescribed temperature or velocity) we obtain the following well known integral equations,

(2.7)
$$\int_{-1}^{1} f(t)e^{\lambda(x-t)}K_0(\lambda|x-t|)dt = g(x) \; ,$$

and

(2.8)
$$\int_{-1}^{1} f(t)\left[\lambda e^{x-t}(K_0(\lambda|x-t|) + \frac{|x-t|}{(x-t)} K_1(\lambda|x-t|)) - \frac{1}{x-t}\right]dt = g(x) \; .$$

Further problems in which the Oseen approximation has been used have been discussed recently ([4] and [7]) with a derivation of the concomitant integral equations given in each case. Both integral equations are Fredholm first kind with difference kernels which have a logarithmic singularity and decay to zero for large values of their arguments. For convenience these will be referred to as 'Oseen type' integral equations.

Fredholm equations of the first kind on a finite interval are difficult to solve and it appears unlikely that closed form solutions of equations (2.7) or (2.8) can be found. Recourse must be made to numerical and asymptotic techniques therefore (see [6]). Before embarking on such procedures an existence theory is desirable. There are circumstances in which integral equations derived from exterior boundary value problems fail to have solutions, although the boundary value problems do have solutions. This is true of the first kind equation on (-2,2) having $\log|x-t|$ as its kernel (see [5]). Similarly integral equations which arise from exterior problems in acoustics (see [9]) also fail to have solutions. In the following sections it is shown that integral equations of Oseen type do not suffer from this deficiency.

3. Existence of solutions of Oseen type integral equations

It will now be shown that integral equations of Oseen type always have a unique solution provided the corresponding exterior boundary value problem has no more than one solution.

The essence of the proof is to express the kernel as the sum of a logarithmic part plus a remainder whose derivative is square integrable.

The inverse operator for the logarithmic part is applied to obtain an equivalent second kind equation. Once the operator in the second kind equation has been shown to be compact the conclusions of the Fredholm alternative theorem together with the uniqueness properties of the exterior boundary value problem ensure the existence of a unique solution of the original equation.

These steps can be carried through only in the correct function space setting. In addition a suitable representation of the inverse operator for the logarithmic part of the kernel is needed. The definitions and representation given here can be found in Wolfe [10] in the context of an integral equation arising from solutions of the reduced wave equation.

Define

(3.1)
$$L_{2(p)} \equiv \{f : \int_{-1}^{1} |f(t)|^2 (1-t^2)^{-\frac{1}{2}} dt \quad \text{exists}\},$$

and

(3.2)
$$W_{2(q)}^1 \equiv \{f : f \text{ is absolutely continuous and } \int_{-1}^{1} |f'(t)|^2 (1-t^2)^{\frac{1}{2}} dt \text{ exists}\}.$$

Under the norms

(3.3)
$$\|f\|_{L_{2(p)}}^2 = \int_{-1}^{1} |f(t)|^2 (1-t^2)^{-\frac{1}{2}} dt ,$$

and

(3.4)
$$\|f\|_{W_{2(q)}^1}^2 = \int_{-1}^{1} |f(t)|^2 (1-t^2)^{\frac{1}{2}} dt + \int_{-1}^{1} |f'(t)|^2 (1-t^2)^{\frac{1}{2}} dt ,$$

these are (complete) Hilbert spaces. On $L_{2(p)}$ the inner product is defined by

(3.5)
$$(\phi_1, \phi_2)_{L_{2(p)}} = \int_{-1}^{1} \frac{\phi_1(t) \overline{\phi_2(t)}}{\sqrt{1-t^2}} \ dt .$$

Integral equations of Oseen type have difference kernels $k(x-t)$ which are expressible as

$$(3.6) \quad k(u) = c\log|u| + \sum_{n=0}^{\infty} a_n u^n + \log|u| \sum_{n=1}^{\infty} b_n u^n = k_S(u) + k_R(u) ,$$

$$\text{with } k_S(u) = c\log|u| ,$$

where the sums are analytic for all $|u| \geq 0$. The integral operators K, K_S and K_R are defined by

$$(3.7) \quad Kf(x) \equiv \int_{-1}^{1} \frac{f(t)}{(1-t^2)^{\frac{1}{2}}} k(x-t)dt, \quad K_S f(x) \equiv \int_{-1}^{1} \frac{cf(t)}{(1-t^2)^{\frac{1}{2}}} \log|x-t|dt \text{ and } K_R = K - K_S.$$

The operator K_S maps $L_{2(p)}$ onto $W^1_{2(q)}$, the latter space being characterised by the fact that (Wolfe, [10]) g belongs to $W^1_{2(q)}$ if and only if

$$(3.8) \quad \sum_{n=0}^{\infty} n^2 \left| (g, T_n)_{L_{2(p)}} \right|^2 < \infty .$$

T_n is the Chebyshev polynomial of degree n; Chebyshev polynomials are orthogonal on $L_{2(p)}$. In view of the identities

$$(3.9) \quad \frac{1}{\pi} \int_{-1}^{1} T_n(t)(1-t^2)^{-\frac{1}{2}} \log(|x-t|)dt = \begin{cases} -\log 2, & n = 0 , \\ -\dfrac{T_n(x)}{n}, & n \geq 1 , \end{cases}$$

it follows from (3.8) that K_S^{-1} maps $W^1_{2(q)}$ onto $L_{2(p)}$; it is defined explicitly by

$$(3.10) \quad K_S^{-1} g \equiv \frac{-(g, T_0)T_0}{c\pi\log 2} + \sum_{n=1}^{\infty} \left(\frac{-n}{c\pi}\right)(g, T_n)T_n .$$

[There are closed form expressions for K_S^{-1}, see Carrier et al [2], for example, but they are too cumbersome to permit the analysis given here.]

Let g belong to $W^1_{2(q)}$ and denote an integral equation of Oseen type such as (2.7) or (2.8) by

$$(3.11) \qquad Kf = g \ .$$

Express the equation as

$$(3.12) \qquad (K_S + K_R)f = g$$

and apply K^{-1}_S to obtain

$$(3.13) \qquad f + K^{-1}_S K_R f = K^{-1}_S g \ .$$

Now $k'_R(u)$ has a logarithmic singularity at $u = 0$ and is otherwise analytic. This is enough to ensure that K_R maps $L_{2(p)}$ into $W^1_{2(q)}$; thus for any g in $W^1_{2(q)}$, equation (3.13) is an equation from $L_{2(p)}$ to $L_{2(p)}$ and is equivalent to (3.11).

Now it will be shown that $K^{-1}_S K_R$ is a compact operator from $L_{2(p)}$ to itself. Note that

$$(3.14) \qquad K^{-1}_S = \lim_{n \to \infty} K^{-1}_{Sn} \ , \quad K^{-1}_{Sn} \equiv (-\pi \log 2)^{-1} (\cdot, T_0) T_0 + \sum_{i=1}^{n} \left(\frac{-i}{\pi}\right) (\cdot, T_i) T_i$$

where a pointwise limit is meant in (3.14). It will be shown that

$$(3.15) \qquad K^{-1}_{nS} K_R \to K^{-1}_S K_R \ ,$$

in the $L_{2(p)}$ norm, from which the compactness of $K^{-1}_S K_R$ follows, since each $K^{-1}_{nS} K_R$ is of finite rank, and therefore compact.

Denote by σ_{mn} the expression

$$(3.16) \qquad \sigma_{mn} = \| K^{-1}_{Sn} K_R f - K^{-1}_{Sm} K_R f \|^2_{L_{2(p)}} \qquad .$$

Suppressing the $L_{2(p)}$ suffix, we have

$$(3.17) \quad \pi^2 \sigma_{mn} = \left\| \sum_{i=m+1}^{n} i(K_R f, T_i) T_i \right\|^2 = \sum_{m+1}^{n} \left| i(K_R f, T_i) \right|^2 = \sum_{m+1}^{n} \left| (f, i K_R^* T_i) \right|^2 .$$

The properties of the Chebyshev polynomials enable $K_R^* T_i$ to be integrated by parts so that

$$(3.18)$$
$$K_R^* T_i = \frac{1}{2i} \left[(K_R^{*\prime}) T_{i-1} - (K_R^{*\prime}) T_{i+1} \right], \text{ where } K_R^{*\prime} f \equiv \int_{-1}^{1} \frac{f(t)}{(1-t^2)^{\frac{1}{2}}} \left(\frac{\partial}{\partial t} k(t-x) \right) dt .$$

Thus applying this to (3.17) and applying the Cauchy-Schwarz inequality we have

$$(3.19) \quad \pi^2 \sigma_{mn} \leq \frac{1}{4} \|f\|^2 \left(\sum_{m+1}^{n} \left(\| K_R^{*\prime} T_{i-1} \|^2 + \| K_R^{*\prime} T_{i+1} \|^2 \right) \right) .$$

Now if the $K_R^{*\prime}$ operator is written out explicitly, in view of (3.6) it is easily observed that $K_R^{*\prime}$ can be similarly expressed; that is,

$$(3.20) \quad K_R^{*\prime} = c_1 K_S + K_T$$

and K_S is the same operator as defined at (3.7). Thus

$$(3.21) \quad (K_R^{*\prime}) T_i = c_1 K_S T_i + K_T T_i$$

and so applying the relations (3.9) to the first term and the equation (3.18) to the second term on the right hand side of (3.21) the estimates

$$(3.22) \quad \| K_R^{*\prime} T_{i\pm1} \|^2 \leq \frac{c_1^2 \pi^3}{i^2} + \frac{\pi}{2i^2} \| (K_T^{*\prime}) \|^2$$

are obtained. These ensure that for any preassigned ε, and m and n large enough,

$$(3.23) \quad \sigma_{mn} \leq \|f\|^2 \varepsilon ,$$

and the validity of (3.15) in the $L_{2(p)}$ norm is demonstrated.

Now the Fredholm alternative may be applied to (3.13). In the event of (3.13) having a solution for zero right hand side, then (3.11) would have the same solution. This would generate a non-trivial solution to the exterior boundary value problem for prescribed data equal to zero. Hence, provided the boundary value problem has suitable uniqueness properties which can be deduced independently of the arguments presented here, equation (3.13) could not have a non-trivial solution for zero data, and thus for any $K_S^{-1}g$ in $L_{2(p)}$, i.e. any g in $W_{2(q)}^1$, a unique solution of (3.13), and therefore (3.11) also, exists.

4. Concluding remarks

The existence of solutions of equations (2.7) and (2.8), for any g in $W_{2(q)}^1$, follows from the results of Section 3, since the corresponding heat transfer and viscous flow problems can be shown to have at the most one solution. In the temperature problem a maximum principle can be proved for the operator $(\nabla^2 - \lambda^2)$ which is obtained after removal of the first derivative term by expressing the solutions in terms of $e^{-\lambda x}T(x,y)$. In both problems energy arguments can also be given (see Finn [3] for the viscous problem). In the viscosity problem the uniqueness results strictly apply to the perturbation in the velocity field from the free stream since any solution for ψ is only determined up to an additive constant.

The proof of existence of solutions of equation (2.7) could be simplified somewhat by combining the exponential terms in the kernel with f and g. In this new form the kernel has an expansion with the term $u\log|u|$ (indeed all terms $u^{2n+1}\log|u|$) absent; this allows two integration by parts, whereas only one was used to obtain (3.19), and thereby shortens the argument leading to (3.23).

A key factor in proving these results is the decay rate at infinity of the field variables. Note that the corresponding potential problems do not have these properties; this perhaps explains the non-existence for the logarithmic kernel on $[-2,2]$.

As far as the examples from [4] and [7] are concerned, it is conjectured that uniqueness theorems for the exterior boundary value problems can be proved but this has yet to be attempted.

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