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IMMERSIONS WITH SEMI-DEFINITE SECOND FUNDAMENTAL FORMS

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1. Introduction. Let M be a complete connected Riemannian manifold of dimension n and let $\xi: M \to \mathbf{R}^{n+k}$ be an isometric immersion into the Euclidean space $\mathbf{R}^{n+k}$. Let ∇ be the connection on M^n and let $\overline{\nabla}$ be the Euclidean connection on $\mathbf{R}^{n+k}$. Also let

$$B: T_p(M) \times T_p(M) \to N_p(M)$$

denote the second fundamental form $B(X, Y) = (\overline{\nabla}_X Y)^{\perp}$. Here $T_p(M)$ denotes the tangent space at p, $N_p(M)$ the normal space and $(\ldots)^{\perp}$ the normal component. To each normal vector ζ we associate a (real valued) second fundamental form

$$B_{\zeta}(X, Y) = \langle \overline{\nabla}_X Y, \zeta \rangle,$$

where $\langle \ldots, \ldots \rangle$ is the Euclidean inner product in $\mathbf{R}^{n+k}$. The Gauss curvature equation gives the sectional curvature $K_{\sigma(X,Y)}$ of the section $\sigma(X, Y)$ defined by a pair of orthonormal vectors $X, Y \in T_p(M)$:

$$K_{\sigma(X,Y)} = \langle B(X, X), B(Y, Y) \rangle - \langle B(X, Y), B(X, Y) \rangle.$$

We wish to study immersions $\xi: M^n \to \mathbf{R}^{n+k}$ with the following hypothesis:

(H) *At every $p \in M$ and for every $\zeta \in N_p(M)$ the second fundamental form B_{ζ} is semi-definite.*

These immersions with semi-definite second fundamental forms were studied by M. do Carmo and E. Lima [2] in the case where M is compact. Their method involves Morse theory and is not suited for non-compact manifolds. In this paper we present an approach that works even when M is non-compact. In particular we obtain the following:

THEOREM. *Let M be a complete Riemannian manifold. If $\xi: M^n \to \mathbf{R}^{n+k}$ is an isometric immersion with semi-definite second fundamental forms then one of the following two possibilities holds:*

(a) *There is a linear subvariety $\mathbf{R}^{n+1}$ of $\mathbf{R}^{n+k}$ such that $\xi: M^n \to \mathbf{R}^{n+1} \subset \mathbf{R}^{n+k}$, and $\xi(M^n)$ is the boundary of an open convex set in $\mathbf{R}^{n+1}$.*

(b) *The immersion $\xi: M^n \to \mathbf{R}^{n+k}$ is $(n-1)$-cylindrical.*

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An immersion $\xi: M^n \to \mathbf{R}^{n+k}$ is called $(n-1)$-cylindrical if M and ξ are expressible as Riemannian products, $M = Q \times \mathbf{R}^{n-1}$, $\xi = \gamma \times \mathrm{id}$, $\mathbf{R}^{n+k} = \mathbf{R}^{k+1} \times \mathbf{R}^{n-1}$, γ is a curve in $\mathbf{R}^{k+1}$ and id the identity on $\mathbf{R}^{n-1}$.

Since case (b) is clearly exluded for compact manifolds, the theorem of do Carmo and Lima [**2**] is a corollary of ours.

It is clear that conversely all immersions of type (a) and (b) satisfy hypothesis (H).

It is easy to see that (H) is equivalent to the following hypothesis:

(H′) *At each point $p \in M$ there is a ray $S_p \subset N_p(M)$ emanating from* $0 \in N_p(M)$ *such that $B(X, X) \in S_p$ for all $X \in T_p(M)$.*

Both (H) and (H′) imply that all sectional curvatures of M are non-negative.

The proof of our theorem may be outlined as follows:

(i) If M^n contains a submanifold W which is flatly embedded onto an affine space $\mathbf{R}^{n-1} \subset \mathbf{R}^{n+k}$ by ξ, then according to a theorem of Toponogov [**8**], M is isometric to $J \times \mathbf{R}^{n-1}$ for some 1-manifold J. If follows that all sectional curvatures vanish.

(ii) If all sectional curvatures on M vanish, it is not hard to see that under the assumptions of the theorem the relative nullity of ξ is $\geqq n - 1$. Then the paper of Hartman [**3**] shows that ξ is $(n-1)$-cylindrical.

(iii) Let M_0 be the subset of M of all points at which the second fundamental form vanishes. Then every connected component of M_0 is mapped isometrically onto a convex subset of $\mathbf{R}^{n+k}$. This will be proved in § 4 by a proposition borrowed from Sacksteder [**6**, Lemma 7]. It will follow in § 5 that if M_0 separates M, M contains a submanifold W with the properties described in (i), and consequently ξ will be $(n-1)$-cylindrical. The remaining case, where M_0 does not separate M is dealt with as follows:

(iv) Let $M_1 \supset M_0$ be the set of points of M at which all sectional curvatures vanish. A direct calculation will prove that each component of $M \backslash M_1$ is immersed in an $(n+1)$-dimensional linear subvariety of $\mathbf{R}^{n+k}$. This will be the subject matter of § 2.

(v) On $M_1 \backslash M_0$ the relative nullity is precisely $n - 1$. By studying the integral manifolds of the distribution of the $(n-1)$-dimensional spaces of relative nullity on the interior of $M_1 \backslash M_0$ we will find that each component of $M \backslash M_0$ is immersed in an $(n+1)$-dimensional linear subvariety of $\mathbf{R}^{n+k}$. This material will be dealt with in § 3.

(vi) Once the first part of conclusion (a) has been obtained, the second part, that $\xi(M)$ is the boundary of an open convex set, follows from Sacksteder [**6**].

Unless it is specifically indicated otherwise, M and ξ will always be assumed to satisfy the hypothesis of the theorem.

2. The case where the sectional curvature is nowhere identically zero.

In this section we deal with point (iv) of the outline given in the introduction. Specifically we will prove the following

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Proposition 1. (*M is not assumed to be complete in this proposition.*) *If at each point of M at least one sectional curvature does not vanish, then ξ immerses M in a linear subvariety $\mathbf{R}^{n+1}$ of $\mathbf{R}^{n+k}$.*

Proof. It is clear that under this hypothesis B is non-zero at each point. Let $(\ldots)^{\perp\perp}$ indicate the component orthogonal to the first osculating space $T_p + S_p = T_p + B(T_p, T_p) = K_p$ (say). Then define the third fundamental form

$$\beta(X, Y, Z) = (\bar{\nabla}_X \bar{\nabla}_Y Z)^{\perp\perp}.$$

It is easy to show that β is a symmetric trilinear vector-valued form on $T_p(M)$ satisfying the following property:

$$\sum_{i=1}^{r} \lambda_i B(Y_i, Z_i) = 0 \Rightarrow \sum_{i=1}^{r} \lambda_i \beta(X, Y_i, Z_i) = 0.$$

Under hypothesis (H) we have $\dim S_p(M) = 1$, and so at a fixed point p, B can be thought of as a positive semi-definite inner product on $T_p(M)$. The hypothesis of this proposition implies that

$$B(X, X)B(Y, Y) - B(X, Y)^2 > 0$$

for at least one pair of distinct vectors X, $Y \in T_p(M)$. Therefore the isotropy subspace of $T_p(M)$ associated with B has dimension at most $n - 2$. Let us choose a basis $X_1, \ldots, X_n$ of $T_p(M)$ such that

$$\begin{aligned} B(X_i, X_i) &= 1, \qquad i = 1, \ldots, r, \\ B(X_i, X_i) &= 0, \qquad i = r + 1, \ldots, n, \\ B(X_i, X_j) &= 0 \text{ if } i \neq j. \end{aligned}$$

Therefore $\beta(X_k, X_i, X_j) = 0$ if $i \neq j$ and if $i = j > r$. On the other hand, $B(X_i, X_i) = B(X_1, X_1)$ if $2 \leqq i \leqq r$, and so $\beta(X_i, X_i, X_i) = \beta(X_i, X_1, X_1) = 0$ if $2 \leqq i \leqq r$ and $\beta(X_1, X_1, X_1) = \beta(X_1, X_i, X_i) = 0$ if $2 \leqq i \leqq r$. This proves that $\beta = 0$ at p. From this it is clear that all derivatives $\bar{\nabla}_{Y_1} \bar{\nabla}_{Y_2} \ldots \bar{\nabla}_{Y_{m-1}} Y_m$ of all orders m with $Y_i \in T_p(M)$, $i = 1, \ldots, m$, lie in the first osculating space $K_p(M)$. Since p is arbitrary, this holds for each point in M.

Now for a sufficiently small open neighbourhood U of $p \in M$ we can find a fixed linear $(n + 1)$-variety L such that orthogonal projection $\pi : \mathbf{R}^{n+k} \to L$ is an isomorphism when restricted to $K_q(M)$, $q \in U$. Let L_1 be the orthogonal complement of L and $\rho : \mathbf{R}^{n+k} \to L_1$ the orthogonal projection. Let $V_1, \ldots, V_{n+1}$ be a fixed basis for L. Then $\{\rho\pi^{-1}(V_i)\}_{i=1}^{n+1}$ completely describe $K_q(M)$. Clearly $\{\pi^{-1}(V_i)\}_{i=1}^{n+1}$ is a basis for $K_q(M)$, and varies differentiably with q. Since all derivatives at q lie in $K_q(M)$ we have for $X \in T_q(M)$,

$$\nabla_X \pi^{-1}(V_i) = \sum \lambda_{ij} \pi^{-1}(V_j) \quad \text{for some } \lambda_{ij} \in \mathbf{R}.$$

But

$$0 = \nabla_X V_i = \nabla_X \pi\pi^{-1}(V_i) = \pi(\nabla_X \pi^{-1}(V_i))$$
$$= \pi\left(\sum_j \lambda_{ij}\pi^{-1}(V_j)\right) = \sum_j \lambda_{ij} V_j.$$

Thus $\lambda_{ij} = 0$ for all i, j. That is, $\nabla_X \pi^{-1}(V_i) = 0$. Therefore

$$\nabla_X \rho\pi^{-1}(V_i) = \rho\nabla_X \pi^{-1}(V_i) = 0.$$

This shows that the first osculating space $K_q(M) \equiv K$ (constant) on U. But then $K_p(M) \equiv K$ for all $p \in M$.

Now let p and q be two points on M and let K_0 be the linear subvariety $K + \xi(p)$ of $\mathbf{R}^{n+k}$.

Then if γ is a C^∞ curve joining p and q in M we see that $d\xi(\gamma'(t)) \in K$ for all t. Therefore $\xi(\gamma) \subset K_0$, and $\xi(q) \in K_0$. Thus $\xi(M) \subset K_0 = \mathbf{R}^{n+1}$.

3. Zero curvature points. This section is concerned with the points in $M \backslash M_0$ as outlined in part (v) of the introduction. Specifically, we will prove the following proposition:

PROPOSITION 2. *If M satisfies the hypothesis* (H) *and if U is any connected component of $M \backslash M_0$, then either ξ is $(n - 1)$-cylindrical or else ξ immerses U as a hypersurface in an $(n + 1)$-dimensional linear subvariety of* $\mathbf{R}^{n+k}$.

Proof. Let $V = U \cap \operatorname{int} M_1$. That is, on U we have $B \not\equiv 0$ while on V all sectional curvatures vanish.

On U the first osculating space $K_p(M)$, $p \in U$, is always $(n + 1)$-dimensional. We can think of $K_p(M)$ as a smooth mapping κ of U into $G_{n+1,n+k}$, the Grassman variety of $(n + 1)$-dimensional linear subspaces of $\mathbf{R}^{n+k}$.

$$\kappa(p) = K_p(M).$$

It follows from § 2 that $d\kappa = 0$ on $M \backslash M_1$. By continuity, it follows that $d\kappa = 0$ on $U \backslash V$. The relative nullity is equal to $n - 1$ on V. In [**1**, Lemma 5.1 and Theorem 6.2] it is shown that the distribution of $(n - 1)$-spaces 0_p of relative nullity on V is smooth and integrable, that each leaf is totally geodesic in $\mathbf{R}^{n+k}$, and that the tangent space $T_p(M)$ is constant along each leaf. Let $p \in V$ and let X be a tangent vector field, defined on a neighbourhood of p and such that $X_q \in 0_q$ for all q. Let Y be any other tangent field locally defined. Then, modulo the osculating space $K_p(M)$ we have

$$\bar{\nabla}_X B(Y, Y) = \bar{\nabla}_X(\bar{\nabla}_Y Y)^\perp = \bar{\nabla}_X \bar{\nabla}_Y Y$$
$$= \bar{\nabla}_Y \bar{\nabla}_X Y = \bar{\nabla}_Y(B(X, Y)) = 0,$$

since $X_q \in 0_q$, for all q. Thus $K_p(M)$ is constant along any leaf of the distribution of spaces of relative nullity in V.

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If there is a complete leaf in V, then by (i) and (ii) in the introduction we see that ξ is $(n - 1)$-cylindrical. If there is no complete leaf through p, then some geodesic $\gamma_p(t)$ emanating from p in a direction of relative nullity must meet ∂V. Let $X_p = \gamma'_p(0)$ and extend X_p to a neighbourhood of p so that $X_q \in 0_q$ for each q. Let t_0 be the first positive value of t such that $\gamma_p(t) \notin V$. Then by [**1**, Theorem 6.2], the relative nullity at $\gamma_p(t_0)$ is $n - 1$. Thus $\gamma_p(t_0) \in U \backslash V$ and so $d_K = 0$ at $\gamma_p(t_0)$. But $\gamma_q(t_0)$ varies smoothly with q. Thus also $K_{\gamma_q(t_0)}$ varies smoothly with q. Hence $d_K = 0$ at p. Since p is an arbitrary point of V we now see that the first osculating space $K_p(M)$ is constant for $p \in U$. The argument used at the end of § 2 now shows that ξ embeds U in an $(n + 1)$-dimensional linear subvariety of $\mathbf{R}^{n+k}$.

4. Convexity of components of $\mathbf{M}_0$. In this section we treat point (iii) of the introduction. We begin by proving:

LEMMA 1 (cf. [**8**, Lemma 6]). *Let C be a connected component of M_0. Then on C the tangent space $T_p(M)$ is a constant n-plane L and $\xi(C)$ lies in an n-dimensional linear subvariety parallel to L.*

Proof. Let $\tau: M \rightarrow G_{n,n+k}$ be the generalized Gauss map, $\tau(p) = T_p(M)$. It is not hard to see that on M_0 the rank of the map τ is zero. But then by a theorem of Sard [**7**, Theorem 6.1], $\tau(M_0)$ is a one-dimensional zero set. In particular, $\tau(M_0)$ is totally disconnected. This proves that $\tau(C)$ consists of a single point in $G_{n,n+k}$. Now let ζ be a fixed vector normal to $\tau(C)$ and let $f(p) = \langle \xi(p), \zeta \rangle$, $p \in M$. Then the smooth function f is singular on C. Thus, again by Sard's theorem $f(C)$ is a single point. That is, $\xi(C) \subset L_0$ where L_0 is a linear variety parallel to the fixed n-plane $\tau(C)$.

We would like to prove:

PROPOSITION 3. *Each connected component C of M_0 is embedded by ξ homeomorphically onto a closed convex subset $\xi(C)$ of a linear variety L_0 of n-dimensions.*

Let $\xi(C) \subset L_0$, L_0 an n-dimensional linear variety. For a point $p \in C$ we choose a neighbourhood U with the following properties: if $\pi: \mathbf{R}^{n+k} \rightarrow L_0$ is the orthogonal projection then $\pi \circ \xi$ maps U diffeomorphically onto an open ball in L_0. We need two lemmas for the proof of Proposition 3:

LEMMA 2 (cf. Sacksteder [**8**, Lemma 7]). *Suppose the codimension $k = 1$. Then $\xi | C \cap U$ is a homeomorphism onto a convex subset of L_0.*

Proof. Let A be a component of $U \backslash C$. Define another immersion η of U as follows:

$$\eta(x) = \xi(x) \quad \text{if } x \in A,$$

$$\eta(x) = \pi\xi(x) \quad \text{if } x \notin A.$$

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Since $\partial A \subset C$ and the second fundamental form vanishes on C this is a C^2 immersion.

$$\eta(x) = (\pi\xi(x), h(x)),$$

where $h(x)$ is the perpendicular distance from $\eta(x)$ to L_0. We may assume that $h(x)$ is then a convex function. Therefore, if $q_0, q_1 \in U\backslash A$, letting $q_t = (\pi\xi)^{-1}((1-t)\xi(q_0) + t\xi(q_1))$ we get that

$$\frac{d^2}{dt^2} h(q_t) \geqq 0, \quad h(q_0) = h(q_1) = 0,$$

and thus $h \equiv 0$ on the segment $\gamma = \{q_t | 0 \leqq t \leqq 1\}$. Now let v be any vector along L_0. Since U is open there is an $\epsilon > 0$ such that $q_t + sv \in \pi\xi(U)$ for all $t \in [0, 1]$ and $s \in [-\epsilon, \epsilon]$. Since h is a convex function,

$$g_t(s) = h(q_t + sv) - (1-t)h(q_0 + sv) - th(q_1 + sv) \leqq 0 \text{ for all } s \in [-\epsilon, \epsilon].$$

But $g_t(0) = 0$ and thus

$$\left.\frac{d}{ds} g_t(s)\right|_{s=0} = 0 \quad \text{and} \quad \left.\frac{d^2}{ds^2} g_t(s)\right|_{s=0} \leqq 0.$$

However

$$\left.\frac{d}{ds} h(q_0 + sv)\right|_{s=0} = \left.\frac{d^2}{ds^2} h(q_0 + sv)\right|_{s=0} = 0$$

since $q_0 \in U\backslash A$, and similar equations hold for q_1. Consequently,

$$\left.\frac{d}{ds} h(q_t + sv)\right|_{s=0} = 0$$

for all vectors v, so that $L = T_{q_t}(M)$. In the second place,

$$\left.\frac{d^2}{ds^2} h(q_t + sv)\right|_{s=0} \leqq 0.$$

On the other hand,

$$\left.\frac{d^2}{ds^2} h(q_t + sv)\right|_{s=0} \geqq 0$$

because h is convex. Thus

$$\left.\frac{d^2}{ds^2} h(q_t + sv)\right|_{s=0} = 0; \text{ i.e. } q_t \in M_0.$$

Since t is arbitrary it follows that $\gamma \subset C$.

This shows that $\xi(U\backslash A)$ is convex for each component A of $U\backslash C$. But $U \cap C$ is precisely the intersection of all these sets $U\backslash A$.

LEMMA 3. *For arbitrary codimension k, $\xi | C \cap U$ is a homeomorphism onto a convex subset of L_0.*

Proof. Let ρ be the orthogonal projection $\rho: \mathbf{R}^{n+k} \to K_0$ to an $(n+1)$-dimensional linear variety containing L_0. By the choice of U it is clear that $\rho \circ \xi$ immerses U in K_0. It is also easy to see that the projection $\rho \circ \xi$ satisfies the hypothesis (H). Let D_ρ be the set of points of U at which the second fundamental form of $\rho \circ \xi$ vanishes. Clearly $C \cap U \subset D_\rho$. Let C_ρ be the smallest union of connected components of D_ρ that still contains $C \cap U$. Then $\rho\xi(C_\rho) \subset L_0$. By Lemma 2, $\rho\xi(C_\rho)$ is convex. Therefore $\bigcap_\rho \rho\xi(C_\rho)$ is convex if the intersection is taken over all orthogonal projections to linear $(n+1)$-varieties containing L_0. But it is easy to see that $\bigcap_\rho C_\rho = C \cap U$. Let π denote orthogonal projection onto L_0. Then

$$\xi(C \cap U) = \pi\xi \bigcap_\rho C_\rho = \bigcap_\rho \pi\xi C_\rho = \bigcap_\rho \rho\xi(C_\rho).$$

Proof of Proposition 3. It only remains to show that if $\xi(C)$ is locally convex in the sense of Lemma 3 then it is convex. But that follows from a theorem of Tietze for which a simple proof appears in [**4**, p. 448].

5. Separation of M by convex sets. It is clear that there is no loss of generality by assuming the manifold M to be simply connected. It is shown in Wilder [**9**, Chapter VII, Corollary 9.3], that if M_0 separates M, then one of its components C separates M. In this section we will prove:

PROPOSITION 4. *If a connected component C of M_0 separates M then it contains a submanifold W that ξ maps isometrically onto a linear $(n-1)$-variety.*

Proof. Suppose $\xi(C)$ is a subset of the linear n-variety L_0. Let π be orthogonal projection onto L_0. For each $p \in C$ choose a neighbourhood U_p such that $\xi|U_p$ is a diffeomorphism onto an open ball in L_0. Let $U = \bigcup_{p \in C} U_p$. Then clearly $\xi|U$ is a diffeomorphism onto connected open neighbourhood of $\xi(C) \subset L_0$. Note that for homology with coefficients in $\mathbf{Z}$ we have

$$H_s(M, M\backslash C) \cong H_s(U, U\backslash C) \cong H_s(L_0, L_0\backslash\xi(C))$$

by the excision theorem. Consider the exact homology sequences

$$\begin{array}{ccccccc} & & H_1(M, M\backslash C) & \to & H_0(M\backslash C) & \to & H_0(M) \to 0 \\ & & \wr\| & & & & \\ H_1(L_0) & \to & H_1(L_0, L_0\backslash\xi(C)) & \to & H_0(L_0\backslash\xi(C)) & \to & H_0(L_0) \to 0 \end{array}$$

Since $H_1(L_0) = 0$ and $H_0(M) \cong H_0(\mathbf{R}^n) \cong \mathbf{Z}$, we conclude that $H_0(M\backslash C) \cong \mathbf{Z}$ if and only if $H_0(L_0\backslash\xi(C)) \cong \mathbf{Z}$. That is, C separates M if and only if $\xi(C)$ separates L_0. But by [**5**, Proposition (1.2)], if $\xi(C)$ separates L_0 it contains a $(n-1)$ dimensional linear variety.

6. Proof of the theorem. If M_0 separates M then by Proposition 3, § 5, and points (i) and (ii) in § 1 it follows that M is $(n-1)$-cylindrical. If M_0 does not separate M, then by Proposition 2 either M is $(n-1)$-cylindrical or else ξ immerses $M\backslash M_0$ as a hypersurface in an $(n+1)$ dimensional linear subvariety

K_0 of $\mathbf{R}^{n+k}$. But by Lemma 1 each component C of M_0 is embedded into an n-dimensional linear variety $L_0(C)$. By continuity of the tangent space $T_p(M)$ it follows that $L_0(C) \subset K_0$ for each component C. Thus ξ immerses M into K_0. But then if not all sectional curvatures vanish, $\xi(M)$ is the boundary of a convex body in K_0 [**6**]. If all sectional curvatures vanish it is $(n-1)$-cylindrical [**3**].

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